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A METHOD OF UPPER AND LOWER SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL INCLUSIONS

MOUFFAK BENCHOHRA
University of Sidi Bel Abbés
Department of Mathematics
BP 89, 22000 Sidi Bel Abbés, Algeria
E-mail: benchohra@yahoo.com

SOTIRIS K. NTOUYAS
University of Ioannina
Department of Mathematics
451 10 Ioannina, Greece
E-mail: sntouyas@cc.uoi.gr

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In this paper, a fixed point theorem for condensing maps combined with upper and lower solutions are used to investigate the existence of solutions for first order functional differential inclusions.

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1. Introduction

This paper is concerned with the existence of solutions for the initial multivalued problem:

$$y' \in F(t, y_t), \text{ for a.e. } t \in J = [0, T] \tag{1}$$

$$y_0 = \phi, \tag{2}$$

where $F: J \times C(J_0, \mathbb{R}) \to 2^{\mathbb{R}}(J_0 = [-r, 0])$ is a nonempty compact and convex valued multivalued map, $\phi \in C(J_0, \mathbb{R})$.

For any continuous function y defined on the interval $J_1 = [-r, T]$ and any $t \in J$, we denote by y_t the element of $C(J_0, \mathbb{R})$ defined by

$$y_t(\theta) = y(t + \theta) \quad \theta \in J_0.$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

The method of upper and lower solutions has been successfully applied to study the existence of multiple solutions for initial and boundary value problems of first order functional differential equations.

This method has been used only in the context of single-valued, functional differential equations. We refer to the papers of Haddock and Nkashama [6], Hristova and Bainov [10], Liz and Nieto [13], and Nieto, Jiang and Jurang [15]. For other results on functional differential equations using other methods, we refer to the books of Erbe, Qingai and Zhang [5], Hale [7], Henderson [9], and the survey paper of Ntouyas [16]. Notice that very recently this method has been used for initial and boundary value problems for differential inclusions in the papers of Benchohra and Boucherif [2], Benchohra and Ntouyas [3], and Halidias and Papageorgiou [8].

In this paper we establish an existence result for the problem (1)-(2). Our approach is based on the existence of upper and lower solutions and a fixed point theorem for condensing maps developed by Martelli [14].

2. Preliminaries

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel.

$AC(J,\mathbb{R})$ is the space of all absolutely continuous functions $y\colon J \to \mathbb{R}$.

$C(J_1,\mathbb{R})$ is the Banach space of all continuous functions $y\colon J_1 \to \mathbb{R}$ normed by

$$\| y \|_\infty = \sup\{ \mid y(t) \mid : t \in J_1\} \text{ for each } y \in C(J_1,\mathbb{R}).$$

Set $E\colon = C(J_1,\mathbb{R}) \cap AC(J,\mathbb{R})$. For $y, \overline{y} \in E_0\colon = \{y \in E\colon y(t) = \phi(t), \forall t \in J_0\}$ condition

$$y \leq \overline{y} \text{ if and only if } y(t) \leq \overline{y}\ (t) \text{ for all } t \in J_1$$

defines a partial ordering in E_0. If $\alpha, \beta \in E_0$ and $\alpha \leq \beta$, we denote

$$[\alpha, \beta] = \{y \in E_0\colon \alpha \leq y \leq \beta\}.$$

Let $(X, \| \cdot \|)$ be a normed space. A multivalued map $G\colon X \to 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in X for all bounded subsets B of X (i.e., $\sup_{x\in B}\{\sup\{ \| y \| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and, if for each open set N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subset X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$).

G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following, $CC(X)$ denotes the set of all nonempty compact and convex subsets of X.

An upper semi-continuous map $G\colon X \to 2^X$ is said to be condensing [14] if for any bounded subset $N \subseteq X$ with $\mu(N) \neq 0$, we have $\mu(G(N)) < \mu(N)$, where μ denotes the Kuratowski measure of noncompactness [1]. We remark that a compact map is the easiest example of a condensing map. For more details on multivalued maps, see the books of Deimling [4] and Hu and Papageorgiou [11].

The multivalued map $F\colon J \to CC(\mathbb{R})$ is said to be measurable if, for every $y \in \mathbb{R}$, the function $t \mapsto d(y, F(t)) = \inf\{ \| y - z \| : z \in F(t)\}$ is measurable.

Definition 2.1: A multivalued map $F: J \times C(J_0, \mathbb{R}) \to 2^{\mathbb{R}}$ is said to be an L^1-Carathéodory if

(i) $t \mapsto F(t,u)$ is measurable for each $u \in C(J_0, \mathbb{R})$;

(ii) $u \mapsto F(t,u)$ is upper semicontinuous for almost all $t \in J$;

(iii) For each $k > 0$, there exists $\varphi_k \in L^1(J, \mathbb{R}_+)$ such that

$$\| F(t,u) \| = \sup[\, | v | : v \in F(t,u)] \leq \varphi_k(t) \text{ for all}$$

$$\| u \| \leq k \text{ and for almost all } t \in J.$$

Let us start b defining what we mean by a solution of the problem (1)-(2).

Definition 2.2: A function $y \in E$ is said to be a solution of (1)-(2) if there exists a function $v \in L^1(J, \mathbb{R})$ such that $v(t) \in F(t, y_t)$ a.e. on J, $y'(t) = v(t)$ a.e. on J and $y_0 = \phi$.

The following concept of lower and upper solutions for (1)-(2) was introduced by Halidias and Papageorgiou in [8] for second order multivalued boundary value problems. It will be the basic tool in the approach that follows.

Definition 2.3: A function $\alpha \in E_0$ is said to be a lower solution of (1)-(2) if there exists $v_1 \in L^1(J, \mathbb{R})$ such that $v_1(t) \in F(t, \alpha_t)$ a.e. on J, $\alpha'(t) \leq v_1(t)$ a.e. on J. Similarly, a function $\beta \in E_0$ is said to be an upper solution of (1)-(2) if there exists $v_2 \in L^1(J, \mathbb{R})$ such that $v_2(t) \in F(t, \beta_t)$ a.e. on J, $\beta'(t) \geq v_2(t)$ a.e. on J.

For the multivalued map F and for each $y \in C(J_1, \mathbb{R})$ we define $S^1_{F,y}$ by

$$S^1_{F,y} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y_t) \text{ for a.e. } t \in J\}.$$

Our main result is based on the following:

Lemma 2.4: [12] *Let X be a Banach space and J a real compact interval. Let $F: J \times X \to CC(X)$ be an L^1-Carathéodory multivalued map with $S^1_F \neq \emptyset$ and let Γ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$, then the operator*

$$\Gamma \circ S^1_F : C(J, X) \to CC(C(J, X)),\ y \mapsto (\Gamma \circ S^1_F)(y) := \Gamma(S^1_{F,y})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 2.5: [14] *Let $G: X \to CC(X)$ be an u.s.c. and condensing map. If the set*

$$M := \{v \in X : \lambda v \in G(v) \text{ for some } \lambda > 1\}$$

is bounded, then G has a fixed point.

3. Main Result

We are now in a position to state and prove our result for the problem (1)-(2).

Theorem 3.1: *Suppose $F: J \times C(J_0, \mathbb{R}) \to CC(\mathbb{R})$ is an L^1-Carathéordory multivalued map which satisfies the condition*

(H) *there exist α and β in E_0 lower and upper solutions, respectively, for the problem (1)-(2) such that $\alpha \leq \beta$.*

Then the problem (1)-(2) has at least one solution $y \in E$ such that

$$\alpha(t) \leq y(t) \leq \beta(t) \text{ for all } t \in J_1.$$

Proof: Set

$$C_0(J_1,\mathbb{R}): = \{y \in C(J_1,\mathbb{R}): y(t) = \phi(t), t \in J_0\}.$$

Transform the problem into a fixed point problem. Consider the following modified problem

$$y'(t) \in F(t,(\tau y)_t), \text{ a.e. } t \in J, \tag{3}$$

$$y_0 = \phi, \tag{4}$$

where $\tau: C_0(J_1,\mathbb{R}) \to C_0(J_1,\mathbb{R})$ is the truncation operator defined by

$$(\tau y)(t) = \begin{cases} \alpha(t), & \text{if } y(t) < \alpha(t); \\ y(t), & \text{if } \alpha(t) \le y \le \beta(t); \\ \beta(t), & \text{if } \beta(t) < y(t). \end{cases}$$

A solution to (3)-(4) is a fixed point of the operator $G: C_0(J_1,\mathbb{R}) \to 2^{C_0(J_1,\mathbb{R})}$ defined by

$$G(y) = \left\{ h \in C(J_1,\mathbb{R}): h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0; \\ \phi(0) + \int_0^t v(s)ds, & \text{if } t \in J \end{cases} \right\}$$

where $v \in \widetilde{S}^{\,1}_{F,\tau y}$ and

$$\widetilde{S}^{\,1}_{F,\tau y} = \{v \in S^1_{F,\tau y}: v(t) \ge v_1(t) \text{ a.e. on } A_1 \text{ and } v(t) \le v_2(t) \text{ a.e. on } A_2\},$$

$$S^1_{F,\tau y} = \{v \in L^1(J,\mathbb{R}): v(t) \in F(t,(\tau y)_t) \text{ for a.e. } t \in J\},$$

$$A_1 = \{t \in J: y(t) < \alpha(t) \le \beta(t)\},\ A_2 = \{t \in J: \alpha(t) \le \beta(t) < y(t)\}.$$

Remark 3.2: (i) For each $y \in C(J,\mathbb{R})$, the set $S^1_{F,y}$ is nonempty (see Lasota and Opial [1]).

(ii) For each $y \in C(J,\mathbb{R})$ the set $\widetilde{S}^{\,1}_{F,\tau y}$ is nonempty. Indeed, by (i) there exists $v \in S^1_{F,y}$. Set

$$w = v_1\chi_{A_1} + v_2\chi_{A_2} + v\chi_{A_3},$$

where

$$A_3 = \{t \in J: \alpha(t) \le y(t) \le \beta(t)\}.$$

Then by decomposability $w \in \widetilde{S}^{\,1}_{F,\tau y}$.

We shall show that G is a completely continuous multivalued map, u.s.c. with convex closed values. The proof will be given in several steps.

Step 1: $G(y)$ is convex for each $y \in C_0(J_1,\mathbb{R})$.

Indeed, if $h, \overline{h}$ belong to $G(y)$, then there exist $v \in \widetilde{S}^{\,1}_{F,\tau y}$ and $\overline{v} \in \widetilde{S}^{\,1}_{F,\tau y}$ such that

$$h(t) = \phi(0) + \int_0^t v(s)ds, \ \ t \in J$$

and

$$\overline{h}\ (t) = \phi(0) + \int_0^t \overline{v}\ (s)ds,\ \ t \in J.$$

Let $0 \leq k \leq 1$. Then for each $t \in J$ we have

$$[kh + (1-k)\overline{h}\ ](t) = \phi(0) + \int_0^t [kv(s) + (1-k)\overline{v}\ (s)]ds.$$

Since $\widetilde{S}^{\ 1}_{F,\tau y}$ is convex (because F has convex values) then

$$kh + (1-k)\overline{h}\ \in G(y).$$

Step 2: G sends bounded sets into bounded sets in $c_0(J,\mathbb{R})$.

Let $B_q\colon = \{y \in C_0(J,\mathbb{R})\colon \parallel y \parallel_{\ \infty} \leq q\}$ be a bounded set in $C_0(J,\mathbb{R})$ and $y \in B_q$, then for each $h \in G(y)$ there exists $v \in S^1_{F,\tau y}$ such that

$$h(t) = \phi(0) + \int_0^t v(s)ds,\ t \in J.$$

Thus, for each $t \in J$ we get

$$\mid h(t) \mid\ \leq\ \mid \phi(0) \mid\ + \int_0^t \mid v(s) \mid ds$$

$$\leq\ \parallel \phi \parallel\ +\ \parallel \varphi_q \parallel_{\ L^1}.$$

Step 3: G sends bounded sets in $C_0(J,\mathbb{R})$ into equicontinuous sets.

Let $u_1, u_2 \in J,\ \ u_1 < u_2,\ \ B_q\colon = \{y \in C_0(J,\mathbb{R})\colon \parallel y \parallel_{\ \infty} \leq q\}$ be a bounded set in $C_0(J,\mathbb{R})$ and $y \in B_q$. For each $h \in G(y)$ there exists $v \in \widetilde{S}^{\ 1}_{F,\tau y}$ such that

$$h(t) = \phi(0) + \int_0^t v(s)ds,\ t \in J.$$

We then have

$$\mid h(u_2) - h(u_1) \mid\ \leq \int_{u_1}^{u_2} \mid v(s) \mid ds$$
$$\leq \int_{u_1}^{u_2} \mid \varphi_q(s) \mid ds.$$

As a consequence of Step 2 and Step 3, together with the Ascoli-Arzela theorem,

we can conclude that $G: C_0(J,\mathbb{R}) \to 2^{C_0(J,\mathbb{R})}$ is a compact multivalued map and, therefore a condensing map.

Step 4: G has a closed graph.

Let $y_n \to y_0$, $h_n \in G(y_n)$ and $h_n \to h_0$. We shall prove that $h_0 \in G(y_0)$. $h_n \in G(y_n)$ means that there exists $v_n \in \widetilde{S}_{F,\tau y_n}$ such that

$$h_n(t) = \phi(0) + \int_0^t v_n(s)ds, \ \ t \in J.$$

We must prove that there exists $v_0 \in \widetilde{S}^1_{F,\tau y_0}$ such that

$$h_0(t) = \phi(0) + \int_0^t v_0(s)ds, \ t \in J.$$

Consider the linear continuous operator $\Gamma: L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by

$$(\Gamma v)(t) = \int_0^t v(s)ds.$$

We have

$$\| \ (h_n - \phi(0)) - (h_0 - \phi(0) \ \|_\infty \to 0.$$

From Lemma 2.4, it follows that $\Gamma \circ \widetilde{S}^1_F$ is a closed graph operator. Also, from the definition of Γ we have

$$h_n(t) - \phi(0) \in \Gamma(\widetilde{S}^1_{F,\tau y_n}).$$

Since $y_n \to y_0$, it follows from Lemma 2.4 that

$$h_0(t) = \phi(0) + \int_0^t v_0(s)ds, \quad t \in J$$

for some $v_0 \in \widetilde{S}^1_{F,\tau y_0}$.

Step 5: Now, we are going to show that the set

$$M: = \{v \in C_0(J,\mathbb{R}): \lambda v \in G(v) \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in M$ then $\lambda y \in G(y)$ for some $\lambda > 1$. Thus there exists $v \in \widetilde{S}^1_{F,\tau y}$ such that

$$y(t) = \lambda^{-1}\phi(0) + \lambda^{-1}\int_0^t v(s)ds, \ t \in J.$$

Thus

$$| y(t) | \ \leq \ \| \phi \| \ + \int_0^t | v(s) | \, ds, \ \ t \in J.$$

From the definition of τ there exists $\phi \in L^1(J, \mathbb{R}^+)$ such that

$$\| F(t, (\tau y)_t) \| \ = \sup\{ | v | : v \in F(t, (\tau y)_t)\} \leq \varphi(t) \text{ for each } y \in C(J, \mathbb{R}).$$

Thus we obtain

$$\| y \|_\infty = \sup_{t \in J_1} | y(t) | \ \leq \ \| \phi \| \ + \ \| \varphi \|_{L_1}.$$

This shows that M is bounded. Hence, Lemma 2.5 applies and G has a fixed point which is a solution to problem (3)-(4).

Step 6: We shall show that the solution y of (3)-(4) satisfies

$$\alpha(t) \leq y(t) \leq \beta(t) \text{ for all } t \in J_1.$$

Let y be a solution to (3)-(4). We prove that

$$\alpha(t) \leq y(t) \text{ for all } t \in J.$$

Suppose not. Then there are two cases:

(a) $\alpha(t) \geq y(t)$ for all $t \in J$, and there exists $t^* \in J$ such that $\alpha(t^*) > y(t^*)$ and

(b) there exists t^*, $t_* \in J$ such that $\alpha(t^*) > y(t^*)$ and $\alpha(t_*) < y(t_*)$.

In case (a), from the definition of τ one has

$$y'(t) \in F(t, \alpha_t) \text{ a.e. on } J.$$

Thus there exists $v(t) \in F(t, \alpha_t)$ a.e. on J with $v(t) \geq v_1(t)$ a.e. on J such that

$$y'(t) = v(t) \text{ a.e. on } J.$$

An integration from 0 to t^* yields

$$y(t^*) - \phi(0) = \int_0^{t^*} v(s) ds.$$

Since α is a lower solution to (1)-(2), then

$$\alpha(t^*) - \alpha(0) \leq \int_0^{t^*} v_1(s) ds.$$

It follows from the facts $\phi(0) = \alpha(0)$, $v(t) \geq v_1$ that $\alpha(t^*) < y(t^*)$ which is a contradiction, since $y(t^*) < \alpha(t^*)$.

The case (b) yields also to a contradiction. Consequently,

$$\alpha(t) \leq y(t) \text{ for all } t \in J.$$

Analogously, we can prove that

$$y(t) \leq \beta(t) \text{ for all } t \in J.$$

This shows that the problem (3)-(4) has a solution in the interval $[\alpha, \beta]$. Since $\tau(y) = y$ for all $y \in [\alpha, \beta]$, then y is a solution to (1)-(2).

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