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PLANAR RANDOM MOTIONS WITH DRIFT

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In this paper we consider planar random motions with four directions and four different speeds, switching at Poisson paced times. We are able to obtain, in some cases, the explicit distribution of the position $(X(t), Y(t))$, $t > 0$ in all its components (the discrete one, lying on the edge ∂Q_t of the probability support Q_t, as well as the absolutely continuous one, concentrated inside Q_t).

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1. Introduction

Many concrete situations suggest the idea of planar random motions with drift.

The usual description of such motions based on Wiener process has some basic drawbacks. For example, the velocity of a Wiener particle is infinite and its sample paths have a fractal structure. We suggest a more realistic model where the trajectories are composed by finite-length segments and are run at finite velocity (changing with directions).

Furthermore, we observe that under a suitable rescaling, these motions can be approximated by the usual planar Brownian motion.

In principle, the number of possible directions of motion, as well as the angle formed by each segment of the trajectories, should be arbitrary. However it can be easily realized that the mathematical difficulties implied by the treatment of motions with an arbitrary number of directions cannot be overcome and we are obliged to consider cases which represent reasonable approximations of the reality with an acceptable level of mathematical difficulty.It seems to us that a balanced compromise is obtained by considering a planar motion with four possible directions D_k with four different velocities c_k, $k = 0, 1, 2, 3$.

The minimal number of directions of a non-trivial planar motion is three (changing cyclically or randomly) but the probabilistic results which can be obtained are not completely satisfactory (see [2] and [5]).

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The case of planar motions with four possible orthogonal directions and coinciding velocities (i.e. without drift) has been analyzed in two papers [3, 4]. The models considered in these papers differ because the chance mechanism governing the changes of direction is different.

In this paper we assume that the directions D_k are run at velocities c_k and that the changes among them are governed by a homogeneous Poisson process with rate $\lambda > 0$. More precisely, if the particle is moving with direction D_k and a Poisson event occurs, then it will move either with direction D_{k+1} or D_{k-1} (with probability $1/2$).

Therefore reflection and continuation on the same direction are excluded. The assumption that no reflection is possible after each change of direction plays a central role in our analysis.

For the probabilistic description of the random motion, we consider the position of the particle $(X(t), Y(t)),\ t > 0$ and the related distribution

$$\Pr\{X(t) \in dx, Y(t) \in dy\}. \tag{1.1}$$

The motion with four orthogonal directions and coinciding velocities $c_k = c$, $k = 0, 1, 2, 3,$ and orthogonal deviations at Poisson times is examined in [4]. In this case all components of the distribution, in the set Q_t of possible positions (having the form of a rotated square), have been obtained. In particular, the singular component of (1.1) for $(x, y) \in \partial Q_t$ and above all, the joint distribution (1.1) inside Q_t have been derived explicitly (this is so far the unique case).

In the present paper, the assumption that the four velocities c_k differ implies that the set of possible positions of the randomly moving particle is an irregular quadrangle.

The particle can be located either inside Q_t (when at least two speed changes have been recorded) or on the edge ∂Q_t (if the particle always chooses the outward pointing direction).

In Section 2 we derive the fourth-order hyperbolic equation governing the distribution (1.1) (see formula (2.5)) and despite all our efforts could not be further investigated. In the symmetric case, this coincides with equation (2.6) and has been extensively examined in [4].

The difficulties connected with the analysis of this type of planar motion with drift (with four different orthogonal velocities) are somehow circumvented in Section 3, where three directions D_k, $k = 0, 1, 2$ are assumed collinear with the axes and run with velocities c_k and the fourth one has components $D_4 = (c_0 - c_2, -c_1)$ (see Figure 2). The basic advantage of this assumption is that the random position $(X(t), Y(t))$ can be expressed as a suitable linear combination of two independent, one-dimensional, telegrapher's processes with drift.

Recently, the explicit distribution of the telegrapher's process (depending on two different velocities and two different rates) has been obtained (see [1]). This permits us to derive the joint distribution of the particle's position inside Q_t (formula (3.11)) and the distribution on the edge ∂Q_t (see Remark 3.4).

All these results generalize those of [4] and permit us to give a mathematical description of planar motion with a drift, roughly coinciding with the diagonal of the set Q_t depicted in Figure 2.

In Section 3 we present an analytical approach to the derivation of the distribution of the particle, by deriving and solving its governing equation. By means of the solutions of this equation, we are able to construct the singular components of the distribution (on the edge ∂Q_t) and the absolutely continuous part (lying inside Q_t).

We also verify that these distributions (obtained by means of analytical tools) coincide with those obtained directly from the representation (2.9) of the planar motion in terms of one-dimensional telegraph processes with drift.

2. Description of the Model and Derivation of the Equations Governing the Distributions

We assume that a particle (initially in the origin) can move according to the four orthogonal directions

$$D_k = \left\{c_k \cos \tfrac{k\pi}{2}, c_k \sin \tfrac{k\pi}{2}\right\}, k = 0, 1, 2, 3. \tag{2.1}$$

The changes of direction are governed by a homogeneous Poisson process (with rate $\lambda > 0$) and we suppose that at each Poisson event ($N(t)$ is the cumulative number of events up to time t) the particle can move from D_k to $D_{k\pm1}$ (i.e. only on the line orthogonal to which it was moving on) and the two possible directions are chosen with equal probability $1/2$. We point out that reflecting backwards on the same line considerably complicates the analysis, even in the case where $c_k = c$, $k = 0, 1, 2, 3$.

We note that at time t, the set Q_t of possible positions is the quadrangle depicted in Figure 1 (where two sample paths are sketched, one ending on the edge ∂Q_t and one inside Q_t).

Figure 1

We remark that, with probability $e^{-\lambda t}$, the particle lies, at time t, on one of the vertices of Q_t, while it is located on the outer boundary (excluding the corners) with probability

$$\Pr\{(X(t),Y(t))\in\partial Q_t-\text{corners}\}=\sum_{k=1}^{\infty}\Pr\{N(t)=k\}\tfrac{1}{2^{k-1}}$$

$$=2(e^{-\lambda t/2}-e^{-\lambda t}). \tag{2.2}$$

Thus the total amount of probability splits up into three parts, i.e. on the corners, on the edge ∂Q_t and inside Q_t (as in the no-drift case, see Orsingher [4]). In this case the distribution substantially differs on the two latest sets.

It is a simple matter to realize that the density functions

$$f_k(x,y,t)dxdy=\Pr\{X(t)\in dx, Y(t)\in dy, D(t)=D_k\}, k=0,1,2,3 \tag{2.3}$$

($D(t)$ being the current direction of motion at time t) are solutions of the following differential system:

$$\begin{cases}\frac{\partial f_0}{\partial t}=-c_0\frac{\partial f_0}{\partial x}+\frac{\lambda}{2}\{f_1+f_3\}-\lambda f_0\\ \frac{\partial f_1}{\partial t}=-c_1\frac{\partial f_1}{\partial y}+\frac{\lambda}{2}\{f_2+f_0\}-\lambda f_1\\ \frac{\partial f_2}{\partial t}=c_2\frac{\partial f_2}{\partial x}+\frac{\lambda}{2}\{f_1+f_3\}-\lambda f_2\\ \frac{\partial f_3}{\partial t}=c_3\frac{\partial f_3}{\partial y}+\frac{\lambda}{2}\{f_2+f_0\}-\lambda f_3.\end{cases} \tag{2.4}$$

With some calculations, it easy to check that the functions f_j, $j=0,1,2,3$, as well as $p=\sum_{j=0}^{3}f_j$ satisfy the fourth-order equation

$$\left[\left(\frac{\partial}{\partial t}+\lambda\right)^2+\left(\frac{\partial}{\partial t}+\lambda\right)(c_0-c_2)\frac{\partial}{\partial x}-c_0c_2\frac{\partial^2}{\partial x^2}\right]$$

$$\times\left[\left(\frac{\partial}{\partial t}+\lambda\right)^2+\left(\frac{\partial}{\partial t}+\lambda\right)(c_1-c_3)\frac{\partial}{\partial y}-c_1c_3\frac{\partial^2}{\partial y^2}\right]p \tag{2.5}$$

$$-\lambda^2\left\{\left(\frac{\partial}{\partial t}+\lambda\right)+\frac{c_0-c_2}{2}\frac{\partial}{\partial x}\right\}\left\{\left(\frac{\partial}{\partial t}+\lambda\right)+\frac{c_1-c_3}{2}\frac{\partial}{\partial y}\right\}p=0.$$

In the no-drift case, where $c_j=c$, $j=0,1,2,3$ equation (2.5) coincides with

$$\left(\frac{\partial}{\partial t}+\lambda\right)^2\left[\frac{\partial^2}{\partial t^2}+2\lambda\frac{\partial}{\partial t}-c^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]p+c^4\frac{\partial^4 p}{\partial x^2\partial y^2}=0 \tag{2.6}$$

(see formula (3.9) of [4]).

We observe that the transformation

$$p=e^{-\lambda t}w$$

converts (2.5) into

$$\left[\frac{\partial^2}{\partial t^2} + (c_0 - c_2)\frac{\partial^2}{\partial t \partial x} - c_0 c_2 \frac{\partial^2}{\partial x^2}\right] \left[\frac{\partial^2}{\partial t^2} + (c_1 - c_3)\frac{\partial^2}{\partial t \partial y} - c_1 c_3 \frac{\partial^2}{\partial y^2}\right] w$$
$$- \lambda^2 \Big\{\frac{\partial}{\partial t} + \frac{c_0 - c_2}{2}\frac{\partial}{\partial x}\Big\} \Big\{\frac{\partial}{\partial t} + \frac{c_1 - c_3}{2}\frac{\partial}{\partial y}\Big\} w = 0 \tag{2.7}$$

and thus the special case (2.6) into

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2}{\partial t^2} - \lambda^2 - c^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right] w + c^4 \frac{\partial^4 w}{\partial x^2 \partial y^2} = 0. \tag{2.8}$$

The presence of the drift considerably complicates the situation. We do not think that it is possible to obtain solutions to equation (2.7) in a closed form, expressed by means of well-known functions with which one can construct the distribution of $(X(t), Y(t))$.

Our idea is to study a less general case where the position vector $(X(t), Y(t))$, $t > 0$ can be expressed in terms of a suitable combination of two independent one-dimensional telegraph processes with drift, say U and V (the complete distribution of which is well-known, see [1]).

We assume that $U(t)$ and $V(t)$ have parameters $\lambda/2$ and velocities $(c_0, -c_2)$ and $(c_1, -c_3)$, respectively.

Our process is defined as

$$\begin{cases} X(t) = \frac{c_0}{c_1}\left\{\frac{c_1 U(t) + c_2 V(t)}{c_0 + c_2}\right\} \\ Y(t) = -\left\{\frac{c_1 U(t) - c_0 V(t)}{c_0 + c_2}\right\}. \end{cases} \tag{2.9}$$

The set of possible values of the random vector defined in (2.9) is the quadrangle Q_t with vertices

$$A \equiv \{c_0 t, 0\}, \quad B \equiv \{0, c_1 t\},$$
$$C \equiv \left\{-\frac{c_0 c_2}{c_1}\frac{c_1 + c_3}{c_0 + c_2} t,\ \frac{c_1 c_2 - c_0 c_3}{c_0 + c_2} t\right\} \tag{2.10}$$
$$D \equiv \left\{\frac{c_0}{c_1}\frac{c_1 c_0 - c_2 c_3}{c_0 + c_2} t,\ -c_0 \frac{c_1 + c_3}{c_0 + c_2} t\right\}.$$

The joint distribution of $(U(t), V(t))$, $t > 0$, is

$$p(u, v, t) = p(u, t) p(v, t)$$

where $(u, v) \in R_t = \{(u, v)\colon -c_2 t \le u \le c_0 t,\ -c_3 t \le v \le c_1 t\}$ and (consult [1], formula (4.3))

$$p(u,t) \tag{2.11}$$
$$= \frac{e^{-\lambda t/2}}{c_1+c_2}\left[\frac{\lambda}{2} I_0\left(\frac{\lambda}{c_0+c_2}\sqrt{(u+c_2t)(c_0t-u)}\right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c_0+c_2}\sqrt{(u+c_2t)(c_0t-u)}\right)\right.$$
$$\left. - \frac{(c_2-c_0)}{2}\frac{\partial}{\partial u} I_0\left(\frac{\lambda}{c_0+c_2}\sqrt{(u+c_2t)(c_0t-u)}\right)\right] + \frac{e^{-\lambda t/2}}{2}\{\delta(u+c_2t)+\delta(u-c_0t)\}$$

$$p(v,t) \tag{2.12}$$
$$= \frac{e^{-\lambda t/2}}{c_1+c_3}\left[\frac{\lambda}{2} I_0\left(\frac{\lambda}{c_1+c_3}\sqrt{(v+c_3t)(c_1t-v)}\right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c_1+c_3}\sqrt{(v+c_3t)(c_1t-v)}\right)\right.$$
$$\left. - \frac{(c_3-c_1)}{2}\frac{\partial}{\partial v} I_0\left(\frac{\lambda}{c_1+c_3}\sqrt{(v+c_3t)(c_1t-v)}\right)\right] + \frac{e^{-\lambda t/2}}{2}\{\delta(v+c_3t)+\delta(v-c_1t)\}.$$

We emphasize that, in the case of planar, symmetric motion, with four directions, that is $c_j = c,\ j = 0, 1, 2, 3,$ the set Q_t reduces to the square

$$S_t = \{(x,y)\colon\ |\ x+y\ |\ \le ct,\ |\ x-y\ |\ \le ct\} \tag{2.13}$$

and the representation (2.9) takes the form

$$\begin{cases} X(t) = \frac{1}{2}(U(t)+V(t)) \\ Y(t) = -\frac{1}{2}(U(t)-V(t)). \end{cases} \tag{2.14}$$

If $c_2 = c_0$, $c_3 = c_1$, the motion is obtained by composing two independent telegraph processes (each one being symmetric but with different velocities). In this case, Q_t reduces to a rhombus and the resulting asymmetry is reflected in the form of equation

$$\left(\frac{\partial}{\partial t}+\lambda\right)^2\left[\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} - \left\{c_0^2\frac{\partial^2}{\partial x^2} + c_1^2\frac{\partial^2}{\partial y^2}\right\}\right] p + c_0^2c_1^2\frac{\partial^4 p}{\partial x^2\partial y^2} = 0. \tag{2.15}$$

An intermediate situation occurs when only one direction is run with asymmetrically valued velocities (say the horizontal axis with velocities c_0, c_2 and the vertical one with $\pm c_1$).

In this case, in order to maintain the independence of the components of $(X(t), Y(t))$ and thus representation (2.9), the quadrangle Q_t must have vertices

$$A \equiv \{c_0t, 0\},\ B \equiv \{0, c_1t\},$$
$$C \equiv \left\{-\frac{2c_0c_2}{c_0+c_2}t, c_1\frac{c_2-c_0}{c_0+c_2}t\right\} \tag{2.16}$$
$$D \equiv \left\{c_0\frac{c_0-c_2}{c_0+c_2}t,\ -\frac{2c_0c_1}{c_0+c_2}t\right\}.$$

As with the general motion represented by (2.9), here the assumption of independence of the components U, V implies that two directions of motion are not collinear with the axes. In order to preserve representation (2.9) and approximate as much as possible the general situation described in the first part of this section, we assume that one velocity value, for example c_3, is a function of the other three as follows

$$c_3 = \frac{c_1 c_2}{c_0}.$$

The set Q_t of possible positions is represented in Figure 2, where two sample paths are also drawn.

Figure 2

The first three directions of motion coincide with $D_k,\ \ k = 0, 1, 2$ and $D_4 = (c_0 - c_2,\ -c_1)$ (one trajectory of Figure 2 twice chooses direction D_4).

It must be emphasized that, in the case of planar motion with drift, the orthogonality of directions, in general, excludes the possibility of representing $(X(t), Y(t))$ as the linear combination of independent, one dimensional, processes. On the other way independence excludes orthogonality of directions.

3. Probabilistic Analysis of Motion

As stated above, we here examine in detail the case (related to Figure 2) where the possible directions of motion are represented by three vectors collinear to the axes and one has components $(c_0 - c_2, -c_1)$. This permits us to obtain the exact distribution of the position vector $(X(t), Y(t))$ in all its components and to obtain a probabilistic description of planar motions with drift. The differential system governing the distributions (2.3) reads

$$\begin{cases} \frac{\partial f_0}{\partial t} = -c_0 \frac{\partial f_0}{\partial x} + \frac{\lambda}{2}\{f_1 + f_3 - 2f_0\} \\ \frac{\partial f_1}{\partial t} = -c_1 \frac{\partial f_1}{\partial y} + \frac{\lambda}{2}\{f_2 + f_0 - 2f_1\} \\ \frac{\partial f_2}{\partial t} = c_2 \frac{\partial f_2}{\partial x} + \frac{\lambda}{2}\{f_1 + f_3 - 2f_2\} \\ \frac{\partial f_3}{\partial t} = -(c_0 - c_2) \frac{\partial f_3}{\partial x} + c_1 \frac{\partial f_3}{\partial y} + \frac{\lambda}{2}\{f_2 + f_0 - 2f_3\}. \end{cases} \tag{3.1}$$

The functions $w_j = e^{\lambda t} f_j$ as well as $w = \sum_{j=0}^{3} w_j$ satisfy the fourth-order equation

$$\left[\frac{\partial^2}{\partial t^2} + (c_0 - c_2)\frac{\partial^2}{\partial t \partial x} - c_0 c_2 \frac{\partial^2}{\partial x^2}\right]\left[\frac{\partial^2}{\partial t^2} + (c_0 - c_2)\frac{\partial^2}{\partial t \partial x} - c_1^2 \frac{\partial^2}{\partial y^2} + c_1(c_0 - c_2)\frac{\partial^2}{\partial x \partial y}\right] w$$
$$- \lambda^2 \left\{\frac{\partial}{\partial t} + \frac{(c_0 - c_2)}{2} \frac{\partial}{\partial x}\right\}^2 w = 0 \tag{3.2}$$

which coincides with (2.8) when $c_j = c,\ j = 0, 1, 2, 3$ and with (2.15) if $c_0 = c_2$ (after the introduction of exponential transformation).

Equation (3.2) can be further simplified by means of the Galilean transformation

$$\begin{cases} x' = x - \frac{c_0 - c_2}{2} t \\ y' = y \\ t' = t \end{cases} \tag{3.3}$$

which leads to

$$\left[\frac{\partial^2}{\partial t'^2} - \frac{(c_0 + c_2)^2}{4} \frac{\partial^2}{\partial x'^2}\right]\left[\frac{\partial^2}{\partial t'^2} - \frac{(c_0 - c_2)^2}{4} \frac{\partial^2}{\partial x'^2} - c_1^2 \frac{\partial^2}{\partial y'^2} + c_1(c_0 - c_2)\frac{\partial^2}{\partial x' \partial y'}\right] w$$
$$- \lambda^2 \frac{\partial^2 w}{\partial t'^2} = 0. \tag{3.4}$$

Equation (3.4) can be rewritten as follows (deleting the $'$)

$$\frac{\partial^4 w}{\partial t^4} - \frac{\partial^2}{\partial t^2}\Big\{\lambda^2 + \frac{c_0^2+c_2^2}{2}\frac{\partial^2}{\partial x^2} + c_1^2\frac{\partial^2}{\partial y^2} - c_1(c_0-c_2)\frac{\partial^2}{\partial x\partial y}\Big\}w$$
$$+ \frac{(c_0+c_2)^2}{4}\frac{\partial^2}{\partial x^2}\Big\{\frac{(c_0-c_2)^2}{4}\frac{\partial^2}{\partial x^2} + c_1^2\frac{\partial^2}{\partial y^2} - c_1(c_0-c_2)\frac{\partial^2}{\partial x\partial y}\Big\}w = 0. \tag{3.5}$$

From our point of view, equation (3.5) has the advantage that the odd order derivatives with respect to time t are absent. This makes the derivation of the explicit distribution of $(X(t), Y(t))$, $t > 0$, possible (despite the presence of drift).

We emphasize here that, in the case where the distribution is directed by Equation (2.5), all orders of the time derivatives appear and this makes the analytical approach prohibitively complex.

Clearly the Fourier transform

$$F(\alpha,\beta,t) = \int\int_{R^2} e^{i\alpha x+i\beta y} w(x,y,t)dxdy \tag{3.6}$$

is a solution to the biquadratic equation

$$\frac{d^4F}{dt^4} + \Big\{-\lambda^2 + \frac{c_0^2+c_2^2}{2}\alpha^2 + c_1^2\beta^2 - c_1(c_0-c_2)\alpha\beta\Big\}\frac{d^2F}{dt^2}$$
$$+ \frac{\alpha^2}{4}(c_0+c_2)^2\Big\{\frac{(c_0-c_2)^2}{4}\alpha^2 + c_1^2\beta^2 - c_1(c_0-c_2)\alpha\beta\Big\}F = 0. \tag{3.7}$$

The four roots of the algebraic equation associated to (3.7) are

$$r = \pm\frac{1}{2}\{\sqrt{\lambda^2-(c_1\beta+c_2\alpha)^2} \pm \sqrt{\lambda^2-(c_1\beta-c_0\alpha)^2}\} \tag{3.8}$$

and thus the general solution of equation (3.7) has the form

$$F(\alpha,\beta,t) =$$
$$Ae^{\frac{t}{2}\sqrt{\lambda^2-(c_1\beta+c_2\alpha)^2}+\frac{t}{2}\sqrt{\lambda^2-(c_1\beta-c_0\alpha)^2}}$$
$$+ Be^{\frac{t}{2}\sqrt{\lambda^2-(c_1\beta+c_2\alpha)^2}-\frac{t}{2}\sqrt{\lambda^2-(c_1\beta-c_0\alpha)^2}}$$
$$+ Ce^{-\frac{t}{2}\sqrt{\lambda^2-(c_1\beta+c_2\alpha)^2}+\frac{t}{2}\sqrt{\lambda^2-(c_1\beta-c_0\alpha)^2}} \tag{3.9}$$
$$+ De^{-\frac{t}{2}\sqrt{\lambda^2-(c_1\beta+c_2\alpha)^2}-\frac{t}{2}\sqrt{\lambda^2-(c_1\beta-c_0\alpha)^2}}$$

A, B, C, D being arbitrary constants.

To check this, we write equation (3.7) in the more convenient form

$$\frac{d^4F}{dt^4} - \frac{1}{2}\{\lambda^2-(c_1\beta+c_2\alpha)^2 + \lambda^2-(c_1\beta-c_0\alpha)^2\}\frac{d^2F}{dt^2}$$

$$+\ \tfrac{\alpha^2}{4}(c_0+c_2)^2\Big\{\tfrac{(c_0-c_2)^2}{4}\alpha^2+c_1^2\beta^2-c_1(c_0-c_2)\alpha\beta\Big\}F \tag{3.10}$$

$$=\tfrac{d^4F}{dt^4}\ -\ \tfrac{1}{2}\{H^2+K^2\}\tfrac{d^2F}{dt^2}$$

$$+\ \tfrac{\alpha^2}{4}(c_0+c_2)^2\Big\{\tfrac{(c_0-c_2)^2}{4}\alpha^2+c_1^2\beta^2-c_1(c_0-c_2)\alpha\beta\Big\}F=0.$$

where $H=\sqrt{\lambda^2-(c_1\beta+c_2\alpha)^2}$, $K=\sqrt{\lambda^2-(c_1\beta-c_0\alpha)^2}$.

The related algebraic equation is

$$r^4-\ \tfrac{1}{2}r^2\{H^2+K^2\}+\tfrac{\alpha^2}{4}(c_0+c_2)^2\Big\{\tfrac{(c_0-c_2)^2}{4}\alpha^2+c_1^2\beta^2-c_1(c_0-c_2)\alpha\beta\Big\}=0.$$

A routine calculation shows that

$$\begin{aligned}
r^4-\tfrac{1}{2}r^2\{H^2+K^2\}&\\
&=\text{(inserting (3.8))}\\
&=\tfrac{(H+K)^4}{2^4}\ -\ \tfrac{1}{2^3}(H+K)^2(H^2+K^2)\\
&=\tfrac{-H^4-K^4+2H^2K^2}{2^4}\\
&=\tfrac{-(c_1\beta+c_2\alpha)^4-(c_1\beta-c_0\alpha)^4+2(c_1\beta+c_2\alpha)^2(c_1\beta-c_0\alpha)^2}{2^4}\\
&=-\ \tfrac{\alpha^4(c_2^2-c_0^2)^2}{2^4}\ -\ \tfrac{\alpha^3\beta c_1}{2^2}(c_2^3-c_0^3+c_0c_2(c_2-c_0))-\ \tfrac{\alpha^2\beta^2c_1^2}{2^2}(c_0+c_2)^2\\
&=-\ \tfrac{\alpha^2}{4}(c_0+c_2)^2\Big\{\tfrac{(c_0-c_2)^2}{4}\alpha^2+c_1^2\beta^2-c_1(c_0-c_2)\alpha\beta\Big\},
\end{aligned}$$

which concludes our verification.

We are able to derive the distribution of $(X(t),Y(t))$ in two ways. The first approach is based on the representation (2.9) of motion (with the assumption that $c_3=c_1c_2/c_0$), while the second method is analytical and uses the solution (3.9) of the Fourier transform (3.6) of the governing equation (3.2).

We are now in a position to state the basic result of this section.

Theorem 3.1: *The absolutely continuous part of the distribution of* $(X(t),Y(t)),\ t>0$ *is*

$$p(x,y,t)=\tfrac{e^{-\lambda t}}{c_1(c_0+c_2)}\Big[\tfrac{\lambda}{2}I_0\Big(\tfrac{\lambda}{c_0+c_2}\sqrt{(x-\tfrac{c_2}{c_1}y+c_2t)(c_0t-x+\tfrac{c_2}{c_1y})}\Big)$$

$$+\ \tfrac{\partial}{\partial t}I_0\Big(\tfrac{\lambda}{c_0+c_2}\sqrt{(x-\tfrac{c_2}{c_1}y+c_2t)(c_0t-x+\tfrac{c_2}{c_1}y)}\Big) \tag{3.11}$$

$$- \frac{(c_2-c_0)}{2(c_2+c_0)}\left\{c_0\frac{\partial}{\partial x} - c_1\frac{\partial}{\partial y}\right\} I_0\left(\frac{\lambda}{c_0+c_2}\sqrt{(x - \frac{c_2}{c_1}y + c_2t)(c_0t - x + \frac{c_2}{c_1}y)}\right)\Bigg]$$

$$\left[\frac{\lambda}{2} I_0\left(\frac{\lambda}{c_0+c_2}\sqrt{(x + \frac{c_0}{c_1}y + c_2t)(c_0t - x - \frac{c_0}{c_1}y)}\right)\right.$$

$$+ \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c_0+c_2}\sqrt{(x + \frac{c_0}{c_1}y + c_2t)(c_0t - x - \frac{c_0}{c_1 y})}\right)$$

$$\left. - \frac{(c_2-c_0)}{2(c_2+c_0)}\left\{c_2\frac{\partial}{\partial x} + c_1\frac{\partial}{\partial y}\right\} I_0\left(\frac{\lambda}{c_0+c_2}\sqrt{(x + \frac{c_0}{c_1}y + c_2t)(c_0t - x - \frac{c_0}{c_1 y})}\right)\right].$$

Proof: We consider the transformation

$$\begin{cases} x = \frac{c_0}{c_1}\,\frac{c_1u+c_2v}{c_0+c_2} \\ y = -\,\frac{c_1u-c_0v}{c_0+c_2} \end{cases} \tag{3.12}$$

and its inverse

$$\begin{cases} u = x - \frac{c_2}{c_1}y \\ v = y + \frac{c_1}{c_0}x \end{cases} \tag{3.13}$$

with Jacobian

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{c_0+c_2}{c_0}.$$

The joint distribution of $(X(t), Y(t))$ becomes

$$p(x,y,t) = \frac{c_0+c_2}{c_0} p_U(x - \frac{c_2}{c_1}y, t) p_V(y + \frac{c_1}{c_0}x, t), \tag{3.14}$$

where p_U is given in (2.11) and p_V in (2.12), with $c_3 = c_1c_2/c_0$. In performing (3.14) it should be kept in mind that, by (3.12), we have that

$$\frac{\partial}{\partial u} = \frac{1}{c_0+c_2}\left\{c_0\frac{\partial}{\partial x} - c_1\frac{\partial}{\partial y}\right\}$$

$$\frac{\partial}{\partial v} = \frac{c_0}{c_0+c_2}\left\{\frac{c_2}{c_1}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right\}.$$

Remark 3.1: In the special case where $c_j = c$, $j = 0, 1, 2,$ the density (3.11) reduces to

$$p(x,y,t) = \frac{e^{-\lambda t}}{2c^2}\left[\frac{\lambda}{2} I_0\left(\frac{\lambda}{2c}\sqrt{c^2t^2-(x-y)^2}\right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{2c}\sqrt{c^2(t^2-(x-y)^2}\right)\right]$$

$$\left[\frac{\lambda}{2} I_0\left(\frac{\lambda}{2c}\sqrt{c^2t^2-(x+y)^2}\right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{2c}\sqrt{c^2t^2-(x+y)^2}\right)\right]$$

and coincides with (3.5) of [4].

If $c_0 = c_2 \neq c_1$, from (3.11) we also obtain

$$p(x,y,t)$$

$$= \frac{e^{-\lambda t}}{2c_0c_1}\left[\frac{\lambda}{2} I_0\left(\frac{\lambda}{2c_0c_1}\sqrt{c_0^2c_1^2t^2-(c_0y-c_1x)^2}\right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{2c_0c_1}\sqrt{c_0^2c_1^2t^2-(c_0y-c_1x)^2}\right)\right]$$

$$\left[\frac{\lambda}{2} I_0\left(\frac{\lambda}{2c_0c_1}\sqrt{c_0^2c_1^2t^2-(c_0y+c_1x)^2}\right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{2c_0c_1}\sqrt{c_0^2c_1^2t^2-(c_0y+c_1x)^2}\right)\right].$$

In view of constraint (2.2) it is easy to check that

$$Pr\{(X(t),Y(t)) \in Q_t - \partial Q_t\} = 1 - Pr\{(X(t),Y(t)) \in \partial Q_t\}$$

$$= 1 - 2e^{-\lambda t/2} + e^{-\lambda t}.$$

By considering (3.14) we have that

$$\iint\limits_{Q_t} p(x,y,t)dxdy = \frac{c_0+c_2}{c_0}\iint\limits_{Q_t} p_U(x - \tfrac{c_2}{c_1}y, t)p_V(y + \tfrac{c_1}{c_0}x, t)dxdy$$

$$= \text{[by (3.13)]}$$

$$= \iint\limits_{R_t} p_U(u,t)p_V(v,t)dudv$$

$$= \int_{-c_2t}^{c_0t} p_U(u,t)du \int_{-\frac{cc_1c_2}{c_0}t}^{c_1t} p_v(v,t)dv$$

$$= \text{[by Remark 5.1 of [1]]}$$

$$= \left(1 - e^{-\frac{\lambda t}{2}}\right)^2.$$

It is also very important to note that the bounds of the set Q_t in Figure 2 are implied by the analytical structure of the distribution (3.11).

Remark 3.2: The set Q_t, where the distribution (3.11) is concentrated, is

$$Q_t = \left\{(x,y)\colon -c_2t < x - \frac{c_2}{c_1}y < c_0t,\ -c_2t < x + \frac{c_0}{c_1}y < c_0t\right\},$$

represented in Figure 2.

By means of the Galilean transformation

$$\begin{cases} x' = x - \frac{c_0 - c_2}{2} t \\ y' = y \\ t' = t \end{cases} \tag{3.15}$$

the joint density (3.11) becomes

$$\begin{aligned} p(x', y', t') & \\ = \frac{e^{-\lambda t'}}{c_1(c_0+c_2)} & \Big[\frac{\lambda}{2} I_0 \Big(\frac{\lambda}{c_0+c_2} \sqrt{(\tfrac{c_0+c_2}{2} t')^2 - (x' - \tfrac{c_2}{c_1} y')^2} \Big) \\ & + \frac{\partial}{\partial t'} I_0 \Big(\frac{\lambda}{c_0+c_2} \sqrt{(\tfrac{c_0+c_2}{2} t')^2 - (x' - \tfrac{c_2}{c_1} y')^2} \Big) \\ & + \frac{(c_2 - c_0)}{2(c_2+c_0)} \Big\{ c_1 \frac{\partial}{\partial y'} + c_2 \frac{\partial}{\partial x'} \Big\} I_0 \Big(\frac{\lambda}{c_0+c_2} \sqrt{(\tfrac{c_0+c_2}{2} t')^2 - (x' - \tfrac{c_2}{c_1} y')^2} \Big) \Big] \\ & \Big[\frac{\lambda}{2} I_0 \Big(\frac{\lambda}{c_0+c_2} \sqrt{(\tfrac{c_0+c_2}{2} t')^2 - (y' + \tfrac{c_0}{c_1} x')^2} \Big) \\ & + \frac{\partial}{\partial t'} I_0 \Big(\frac{\lambda}{c_0+c_2} \sqrt{(\tfrac{c_0+c_2}{2} t')^2 - (y' + \tfrac{c_0}{c_1} x')^2} \Big) \\ & + \frac{(c_2 - c_0)}{2(c_2+c_0)} \Big\{ c_0 \frac{\partial}{\partial x'} - c_1 \frac{\partial}{\partial y'} \Big\} I_0 \Big(\frac{\lambda}{c_0+c_2} \sqrt{(\tfrac{c_0+c_2}{2} t')^2 - (y' + \tfrac{c_0}{c_1} x')^2} \Big) \Big] \end{aligned} \tag{3.16}$$

and is defined in the quadrangle

$$Q'_t = \Big\{ x', y' \colon \Big| x' - \tfrac{c_2}{c_1} y' \Big| \le \tfrac{c_0+c_2}{2} t', \Big| y' + \tfrac{c_0}{c_1} x' \Big| \le \tfrac{c_0+c_2}{2} t' \Big\}.$$

The transformation (3.15) eliminates the drift along the x axis due to different horizontal velocities.

In the next theorem we present the Fourier transform of the absolutely continuous component of the distribution (3.11).

Theorem 3.2: *The characteristic function of* $(X(t'), Y(t'))$ *is*

$$G(\alpha, \beta, t) = \iint_{R^2} e^{i\alpha x' + i\beta y'} p(x', y', t') dx' dy' \tag{3.17}$$

$$= \frac{1}{2^2} e^{-\lambda t} \Big[\Big(1 + \frac{\lambda}{\sqrt{\lambda^2 - (\alpha c_0 - \beta c_1)^2}} \Big) e^{t \sqrt{\lambda^2 - (\alpha c_0 - \beta c_1)^2}/2}$$

$$+\left(1-\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2}}\right)e^{-t\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2}/2}\Bigg]$$

$$\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_2+\beta c_1)^2}}\right)e^{t\sqrt{\lambda^2-(\alpha c_2+\beta c_1)^2}/2}\right.$$

$$\left.+\left(1-\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_2-\beta c_1)^2}}\right)e^{-t\sqrt{\lambda^2-(\alpha c_2-\beta c_1)^2}/2}\right].$$

Proof: For the convenience of the reader we report formula (5.3) of [1], suitably adapted:

$\int e^{iu\gamma}p_U(u,t)du$

$$=\frac{1}{2}e^{-\frac{1}{2}\{i\gamma(c_2-c_0)+\lambda\}t}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-\gamma^2(c_2+c_0)^2}}\right)e^{t\sqrt{\lambda^2-\gamma^2(c_2+c_0)^2}/2}\right.$$

$$\left.+\left(1-\frac{\lambda}{\sqrt{\lambda^2-\gamma^2(c_2+c_0)^2}}\right)e^{-t\sqrt{\lambda^2-\gamma^2(c_2+c_0)^2}/2}\right].$$

In view of formula (3.14) and of the transformation (3.15), we get

$$G(\alpha,\beta,t')=\frac{c_0+c_2}{c_0}\int\int_{R^2}e^{i\alpha x'+i\beta y'}p_U\left(x'-\frac{c_2}{c_1}y'+\frac{c_0-c_2}{2}t',t'\right)$$

$$p_V\left(y'+\frac{c_1}{c_0}x'+\frac{c_1(c_0-c_2)}{2c_0}t',t'\right)dx'dy'$$

$$\left[x'=\frac{c_0}{c_0+c_2}(u+\frac{c_2}{c_1}v)-\frac{c_0-c_2}{2}t,\ \ y'=\frac{c_0}{c_0+c_2}(v-\frac{c_1}{c_0}u),\ \ t'=t\right]$$

$$=e^{-i\alpha\frac{c_0-c_2}{2}t}\int_R e^{i\frac{\alpha c_0-\beta c_2}{c_0+c_2}u}p_U(u,t)du\int_R e^{i\frac{\alpha c_2c_0/c_1+\beta c_0}{c_0+c_2}v}p_V(v,t)dv$$

$$=e^{-i\alpha\frac{c_0-c_2}{2}t}\left\{\frac{1}{2}\exp\left\{\frac{i}{2}\left(\frac{\alpha c_0-\beta c_1}{c_0+c_2}\right)(c_0-c_2)t-\frac{\lambda t}{2}\right\}\right. \qquad (3.18)$$

$$\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2}}\right)e^{t\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2}/2}\right.$$

$$\left.\left.+\left(1-\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2}}\right)e^{-t\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2}/2}\right]\right\}$$

$$\left\{\frac{1}{2}\exp\left\{\frac{i}{2}\left[\frac{c_2 c_0}{c_1}\alpha+c_0\beta\right](c_1-\frac{c_1c_2}{c_0})\frac{t}{c_0+c_2}\ -\ \frac{\lambda t}{2}\right\}\right.$$

$$\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_2+\beta c_1)^2}}\right)e^{t\sqrt{\lambda^2-(\alpha c_2+\beta c_1)^2}/2}\right.$$

$$\left.\left.+\left(1-\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_2+\beta c_1)^2}}\right)e^{-t\sqrt{\lambda^2-(\alpha c_2+\beta c_1)^2}/2}\right]\right\}.$$

In the last step, formula (5.3) of [1] has been applied. Some simplifications yield (3.18).

Remark 3.3: If

$$A=\frac{1}{2^2}\left(1+\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_2+\beta c_1)^2}}\right)\left(1+\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2}}\right)$$

$$B=\frac{1}{2^2}\left(1+\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_2+\beta c_1)^2}}\right)\left(1-\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2}}\right)$$

$$C=\frac{1}{2^2}\left(1-\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_2+\beta c_1)^2}}\right)\left(1+\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2}}\right)$$

$$D=\frac{1}{2^2}\left(1-\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_2+\beta c_1)^2}}\right)\left(1-\frac{\lambda}{\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2}}\right)$$

the Fourier transform (3.17) can be written down as

$$G(\alpha,\beta,t)=e^{-\lambda t}\left[Ae^{t(\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2}+\sqrt{\lambda^2-(\alpha c_2+\beta c_1)^2})/2}\right.$$

$$+\,Be^{t(\sqrt{\lambda^2-(\alpha c_2-\beta c_1)^2}-\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2})/2}$$

$$+\,Ce^{-t(\sqrt{\lambda^2-(\alpha c_2-\beta c_1)^2}-\sqrt{\lambda^2-(\alpha c_0+\beta c_1)^2})/2}$$

$$+ De^{-t(\sqrt{\lambda^2-(\alpha c_2-\beta c_1)^2}+\sqrt{\lambda^2-(\alpha c_0-\beta c_1)^2})/2}\Big]$$

$$= e^{-\lambda t}F(\alpha,\beta,t).$$

This clearly shows that the function $e^{\lambda t'}p(x',y',t')$ (p is the joint distribution (3.16) in the frame (x',y',t')) has Fourier transform which satisfies equation (3.10). The Fourier transform of the distribution (3.11) in the frame (x,y,t), denoted by H, is then

$$H(\alpha,\beta,t) = G(\alpha,\beta,t)e^{-i\alpha(c_0-c_2)t/2}.$$

Remark 3.4: The components of the singular distribution on the edge ∂Q_t are

(i) on the lines $y + c_1x/c_0 - c_1t = 0$ and $y + c_1x/c_0 + c_1c_2t/c_0 = 0$

$$q(x,y,t) = \tfrac{e^{-\lambda t}}{2(c_0+c_2)}\Big[\tfrac{\lambda}{2}I_0\Big(\tfrac{\lambda}{c_0+c_2}\sqrt{(x-\tfrac{c_2}{c_1}y+c_2t)(c_0t-x+\tfrac{c_2}{c_1}y)}\Big)$$

$$+\tfrac{\partial}{\partial t}I_0\Big(\tfrac{\lambda}{c_0+c_2}\sqrt{(x-\tfrac{c_2}{c_1}y+c_2t)(c_0t-x+\tfrac{c_2}{c_1}y)}\Big) \tag{3.19}$$

$$-\tfrac{c_2-c_0}{2(c_2+c_0)}\Big\{c_0\tfrac{\partial}{\partial x}-c_1\tfrac{\partial}{\partial y}\Big\}I_0\Big(\tfrac{\lambda}{c_0+c_2}\sqrt{(x-\tfrac{c_2}{c_1}y+c_2t)(c_0t-x+\tfrac{c_2}{c_1}y)}\Big)\Big].$$

(ii) on $x - c_2y/c_1 - c_0t = 0$ and $x - c_2y/c_1 + c_0t = 0$

$$r(x,y,t) = \tfrac{e^{-\lambda t}}{2(c_0+c_2)}\Big[\tfrac{\lambda}{2}I_0\ \Big(\tfrac{\lambda}{c_0+c_2}\sqrt{(x+\tfrac{c_0}{c_1}y+c_2t)(c_0t-x-\tfrac{c_0}{c_1}y)}\Big)$$

$$+\tfrac{\partial}{\partial t}I_0\Big(\tfrac{\lambda}{c_0+c_2}\sqrt{(x+\tfrac{c_0}{c_1}y+c_2t)\ (c_0t-x-\tfrac{c_0}{c_1}y)}\Big) \tag{3.20}$$

$$-\tfrac{c_2-c_0}{2(c_2+c_0)}\Big\{c_2\tfrac{\partial}{\partial x}+c_1\tfrac{\partial}{\partial y}\Big\}I_0\Big(\tfrac{\lambda}{c_0+c_2}\sqrt{(x+\tfrac{c_0}{c_1}y+c_2t)(c_0t-x-\tfrac{c_0}{c_1}y)}\Big)\Big].$$

Formulas (3.19) and (3.20) can be easily derived from (2.9). Points (x,y) on the lines $y + c_1x/c_0 = \pm c_1t$ can be determined as intersections with $x - c_2y/c_1 = u$. Thus $X(t) - c_2Y(t)/c_1 = U(t)$ for all $(X(t),Y(t))$ and the distribution on this part of the edge ∂Q_t coincides with that of $U(t)$ given by (2.11), where $u = x - c_2y/c_1$.

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