A CLASSICAL APPROACH TO EIGENVALUE PROBLEMS ASSOCIATED WITH A PAIR OF MIXED REGULAR STURM-LIOUVILLE EQUATIONS II

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In the studies of acoustic waveguides in ocean, buckling of columns with variable cross sections in applied elasticity, transverse vibrations in nonhomogeneous strings, etc., we encounter a new class of problems of the type $L_1 y_1 \equiv -y_1'' + q_1(x)y_1 = \lambda y_1$ defined on an interval $[d_1, d_2]$ and $L_2 y_2 \equiv -y_2'' + q_2(x)y_2 = \lambda y_2$ defined on the interval $[d_2, d_3]$ satisfying certain matching conditions at the interface point $x = d_2$.

Earlier, in Part I, we constructed a fundamental system for (L_1, L_2) and derived certain estimates for the same.

Here, in Part II, we consider four types of boundary value problems associated with (L_1, L_2) and study the corresponding spectra.

Key words: Sturm-Liouville Equations, Interface Boundary Conditions, Boundary Value Problems, Eigenfunctions, Spectrum.

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1. Introduction

With the same notation as in Part I [4], consider the pair of Sturm-Liouville equations

$$L_1 y_1 \equiv -y_1'' + q_1(x)y_1 = \lambda y_1, \quad 0 \leq x \leq h, \quad (1)$$

$$L_2 y_2 \equiv -y_2'' + q_2(x)y_2 = \lambda y_2, \quad h \leq x \leq 1, \quad (2)$$

together with the matching conditions at the interface $x = h$ given by

$$y_1(h) = y_2(h), \quad w_1 y_1'(h) = w_2 y_2'(h) \quad (3)$$

where λ is a complex constant, $0 < h < 1$ and $(q_1, q_2) \in L_C^2[0, h] \times L_C^2[h, 1]$.

Also consider the following set of boundary conditions at the end points $x = 0$ and $x = 1$.

$$\text{Dirichlet boundary conditions: } y_1(0) = y_2(1) = 0 \quad (4)$$

$$\text{Neumann boundary condition: } y_1'(0) = y_2'(1) = 0 \quad (5)$$

$$\text{Mixed boundary condition of Type A: } y_1(0) = y_2'(1) = 0 \quad (6)$$

$$\text{Mixed boundary condition of Type B: } y_1'(0) = y_2(1) = 0. \quad (7)$$

We call the boundary value problem (BVP) defined by (1)-(3) and (4) as BVP (I), the BVP defined by (1)-(3) and (5) as BVP (II), the BVP defined by (1)-(3) and (6) as BVP (III), the BVP defined by (1)-(3) and (7) as BVP (IV).

We say that the complex constant λ is an eigenvalue of BVP (I), if BVP (I) has a nontrivial solution for that λ . The corresponding nontrivial solution(s) is (are) called the eigenfunction(s) for λ . The collection of all eigenvalues is called the spectrum of BVP (I). Similarly, we define the spectra of BVP (II, III, IV).

Note 1: It can be easily shown that the eigenvalues of BVP (I, II, III, IV) are identical with the zeros of the functions $y_{22}(1, \lambda)$, $y_{12}'(1, \lambda)$, $y_{22}'(1, \lambda)$, $y_{12}(1, \lambda)$, respectively, where (y_{11}, y_{12}) , (y_{21}, y_{22}) are as in part I [4].

Note 2: Even if w_1 and w_2 are real with $\frac{w_1}{w_2} > 0$, BVPs I-IV cannot be represented as selfadjoint operators acting on the product Hilbert space $L_C^2[0, h] \times L_C^2[h, 1]$ unless $w_1 = w_2$.

Before presenting the main theorem, we state the following easily verified lemma.

Lemma 1: If $|z - (\frac{2n+1}{2})\pi| \geq \pi/4$ for all integers n , then $\exp(|Imz|) \leq |4 \cos z|$.

Assumption 1: We assume that w_1, w_2 are reals and that $|1 - \frac{w_1}{w_2}| < 1/4$.

Note 3: We note that Assumption 1 necessarily implies $\frac{w_1}{w_2} > 0$.

Theorem 1: (Location of Eigenvalues) *Let Assumption 1 be true.*

(A) *Let N be an integer such that*

$$N > 2 \left(\frac{1 + |1 - \frac{w_1}{w_2}|}{1 - 4|1 - \frac{w_1}{w_2}|} \right) \exp \left((\|q_1\|^2 + \|q_2\|^2)^{1/2} \right). \quad (8)$$

Then $y_{22}(1, \lambda)$ has exactly N roots, counted with multiplicities, in the open half plane

$$Re\lambda < (N + \frac{1}{2})^2 \pi^2$$

and for each $n > N$, exactly one simple root in the egged shaped region

$$|\sqrt{\lambda} - n\pi| < \pi/2.$$

There are no other roots.

(B) *Let N be an integer such that*

$$N > 2 \left(\frac{1 + |1 - \frac{w_1}{w_2}|}{1 - 4|1 - \frac{w_1}{w_2}|} \right) (\|q_1\|^2 + \|q_2\|^2)^{1/2} \exp \left((\|q_1\|^2 + \|q_2\|^2)^{1/2} \right) \quad (9)$$

Then $y_{12}'(1, \lambda)$ has exactly N roots, counted with multiplicities, in the open half plane

$$Re\lambda < (N + \frac{1}{2})^2 \pi^2$$

and for each $n > N$, exactly one simple root in the egged shaped region

$$|\sqrt{\lambda} - n\pi| < \pi/2.$$

There are no other roots.

(C) *Let N be an integer satisfying inequality (8). Then $y_{22}'(1, \lambda)$ has exactly N roots, counted with multiplicities, in the open half plane*

$$Re\lambda < N^2 \pi^2$$

and for each $n > N$, exactly one simple root in the egged shaped region

$$\sqrt{\lambda} - \left(\frac{2n+1}{2}\right)\pi < \pi/2.$$

There are no other roots.

(D) Let N be an integer satisfying inequality (9). Then $y_{12}(1, \lambda)$ has exactly N roots, counted with multiplicities, in the open half plane

$$\operatorname{Re} \lambda < N^2 \pi^2$$

and for each $n > N$, exactly one simple root in the egged shaped region

$$\left| \sqrt{\lambda} - \left(\frac{2n+1}{2}\right)\pi \right| < \pi/2.$$

There are no other roots.

Proof: We prove only part (A). Parts (B), (C) and (D) can be proven similarly.

(A) Consider a fixed integer N , and let $k > N$ be another integer. Consider the contours

$$\left| \sqrt{\lambda} \right| + \left(k + \frac{1}{2}\right)\pi, \operatorname{Re} \sqrt{\lambda} = \left(N + \frac{1}{2}\right)\pi, \text{ and } \left| \sqrt{\lambda} - N + 1\pi \right| = \pi/2, n > N.$$

For λ belonging to the contours, it can be easily verified that the estimates

$$\frac{2N}{\left| \sqrt{\lambda} \right|} < 1 \quad (10)$$

and by Lemma 1 (on p. 27 [3])

$$\exp(|\operatorname{Im} \sqrt{\lambda}|) < 4 |\sin \sqrt{\lambda}| \quad (11)$$

hold.

Let N satisfy inequality (8). Then, for λ belonging to each one of the above contours, by Theorem 3 (on p. 13 [3]), we get

$$\begin{aligned} & \left| y_{22}(1, \lambda) - \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \right| \\ & \leq \frac{\left| 1 - \frac{w_1}{w_2} \right|}{\left| \sqrt{\lambda} \right|} \exp(|\operatorname{Im} \sqrt{\lambda}|) + \frac{\left(1 + \left| 1 - \frac{w_1}{w_2} \right| \right)}{|\lambda|} \exp\left(|\operatorname{Im} \sqrt{\lambda}| + (\|q_1\|^2 + \|q_1\|)^{1/2} \right) \end{aligned}$$

$$\leq \left[\left| 1 - \frac{w_1}{w_2} \right| + \frac{N \left(1 + \left| 1 - \frac{w_1}{w_2} \right| \right)}{2|\sqrt{\lambda}|} \right] \frac{\exp|\operatorname{Im} \sqrt{\lambda}|}{|\sqrt{\lambda}|} \quad (\text{by (8)})$$

$$< \left[4 \left| 1 - \frac{w_1}{w_2} \right| + \frac{2N}{|\sqrt{\lambda}|} \left(1 - 4 \left| 1 - \frac{w_1}{w_2} \right| \right) \right] \left| \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \right| \quad (\text{by (11)})$$

$$< \left| \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \right| \quad (\text{by (10)}).$$

Consequently, we get that $y_{22}(1, \lambda)$ does not vanish on the above contours. Also, we note that $y_{22}(1, \lambda)$ is an entire function of λ . Hence by Rouché's theorem ([2]), $y_{22}(1, \lambda)$ has as many roots, counted with multiplicities, as $\frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}}$ in each of the above regions. Since $\sin \sqrt{\lambda} \sqrt{\lambda}$ has only the simple roots $n^2 \pi^2$, $n \geq 1$ and since $k > N$ can be chosen arbitrarily large, the proof of part (A) is complete.

The next theorem gives conditions under which the spectrum is real.

Theorem 2: Let $(q_1, q_2) \in L_R^2[0, h] \times L_R^2[h, 1]$ and assume w_1, w_2 are reals with $\frac{w_1}{w_2} > 0$. Then the spectrum BVP (I) is an infinite sequence of real numbers, which is bounded from below and tends to $+\infty$, for each $i = I, II, III, IV$.

Proof: We prove the theorem for BVP (I). The proofs for the other BVPs can be given along similar lines.

Let λ be an eigenvalue of BVP (I) and let $(y_1(\cdot, \lambda), y_2(\cdot, \lambda))$ be a corresponding eigenfunction. Let $W[y_i(x, \lambda), \bar{y}_i(x, \lambda)]$ denote the Wronskian of y_i and $\bar{y}_i$ for $i = 1, 2$. Then as in Theorem 1 (on p. 29 [3]), it can be shown that

$$[W[y_1(x, \lambda), \bar{y}_1(x, \lambda)]]' = (\lambda - \bar{\lambda}) |y_1(x, \lambda)|^2, \quad 0 \leq x \leq h$$

$$[W[y_2(x, \lambda), \bar{y}_2(x, \lambda)]]' = (\lambda - \bar{\lambda}) |y_2(x, \lambda)|^2, \quad h \leq x \leq 1.$$

Integrating the above relations with respect to x on the respective intervals and using the Dirichlet boundary conditions, we get

$$[W[y_1(x, \lambda), \bar{y}_1(x, \lambda)]]_{x=h} = (\lambda - \bar{\lambda}) \int_0^h |y_1(x, \lambda)|^2 dx \quad (12)$$

$$[W[y_2(x, \lambda), \bar{y}_2(x, \lambda)]]_{x=h} = (\lambda - \bar{\lambda}) \int_h^1 |y_2(x, \lambda)|^2 dx. \quad (13)$$

By using the matching conditions (3) satisfied by $(y_1(\cdot, \lambda), y_2(\cdot, \lambda))$, Equations (12) and (13) yield

$$(\lambda - \bar{\lambda}) \left(\frac{w_1}{w_2} \int_0^h |y_1(x, \lambda)|^2 dx + \int_h^1 |y_2(x, \lambda)|^2 dx \right) = 0.$$

Since (y_1, y_2) is a nontrivial pair and $\frac{w_1}{w_2} > 0$, the above relation implies $\lambda = \bar{\lambda}$. Hence the spectrum of BVP (I) is real, and the rest follows from Theorem 1 (p. 29 [3]).

Finally, we present a theorem on the discreteness of the spectrum of BVPs (I)-(IV). Let $\dot{y}_{ii}(\cdot, \lambda)$ and $\dot{y}'_{ij}(\cdot, \lambda)$ denote the derivatives of $y_{ij}(\cdot, \lambda)$ and $y'_{ij}(\cdot, \lambda)$ with respect to λ , here $i, j = 1, 2$.

Theorem 3: Assume the hypotheses of Theorem 1 and let w_1, w_2 be reals with $\frac{w_1}{w_2} > 0$.

(A) If λ is an eigenvalue of BVP (I), then

$$\dot{y}_{22}(1, \lambda) y'_{22}(1, \lambda) = \frac{w_1}{w_2} \int_0^h |y_{21}^2(x, \lambda)|^2 dx + \int_h^1 |y_{22}^2(x, \lambda)|^2 dx.$$

In particular, $\dot{y}_{22}(1, \lambda) \neq 0$ and (hence) all roots of $y_{22}(1, \lambda)$ are simple.

(B) If λ is an eigenvalue of BVP (II), then

$$-y_{12}(1, \lambda) \dot{y}'_{12}(1, \lambda) = \frac{w_1}{w_2} \int_0^h |y_{12}^2(x, \lambda)|^2 dx + \int_h^l |y_{12}^2(x, \lambda)|^2 dx.$$

In particular, $\dot{y}'_{12}(1, \lambda) \neq 0$ and (hence) all roots of $y'_{12}(1, \lambda)$ are simple.

(C) If λ is an eigenvalue of BVP (III), then

$$-y_{22}(1, \lambda) \dot{y}'_{22}(1, \lambda) = \frac{w_1}{w_2} \int_0^h |y_{21}^2(x, \lambda)|^2 dx + \int_h^l |y_{22}^2(x, \lambda)|^2 dx.$$

In particular, $\dot{y}'_{12}(1, \lambda) \neq 0$ and (hence) all roots of $y'_{12}(1, \lambda)$ are simple.

(D) If λ is an eigenvalue of BVP (IV), then

$$-\dot{y}_{12}(1, \lambda) y'_{12}(1, \lambda) = \frac{w_1}{w_2} \int_0^h |y_{11}^2(x, \lambda)|^2 dx + \int_h^l |y_{12}^2(x, \lambda)|^2 dx.$$

In particular, $y_{12}(1, \lambda) \neq 0$ and (hence) all roots of $y_{12}(a, \lambda)$ are simple.

Proof: We prove part (A). The proofs of parts (B)-(D) follow along similar lines.

(A) Let λ be an eigenvalue of BVP (I). Then $(y_{21}(\cdot, \lambda), y_{22}(\cdot, \lambda))$ satisfies Equations (1)-(3), the initial conditions

$$y_{21}(0, \lambda) = 0, \quad y'_{21}(0, \lambda) = 1$$

and the boundary conditions

$$y_{22}(1, \lambda) = 0.$$

Proceeding as in Theorem 2 (on p. 30 [3]), we get

$$W[y_{21}(y, \lambda), \dot{y}_{21}(h, \lambda)] = - \int_0^h y_{21}^2(x, \lambda) dx \quad (14)$$

and

$$W[y_{22}(h, \lambda), \dot{y}_{22}(h, \lambda)] - \dot{y}_{22}(1, \lambda) y'_{22}(1, \lambda) = - \int_h^l y_{22}^2(x, \lambda) dx. \quad (15)$$

Multiplying Equation (14) by $\frac{w_1}{w_2}$ and adding this to Equation (15) of matching conditions at $x = h$, we get

$$\begin{aligned} \dot{y}_{22}(1, \lambda) y'_{22}(1, \lambda) &= \frac{w_1}{w_2} \int_0^h y_{21}^2(x, \lambda) dx + \int_h^l y_{22}^2(x, \lambda) dx \\ &= \frac{w_1}{w_2} \int_0^h |y_{21}|^2 dx + \int_h^l |y_{22}|^2 dx, \end{aligned}$$

since y_{21} and y_{22} are real-valued. Consequently, $\dot{y}_{22}(1, \lambda) \neq 0$ and λ is a simple root of $y_{22}(1, \lambda)$.

2. Physical Application

Acoustic wave guides in oceans [1]: The following problem is encountered in the study of 'acoustic wave guides in oceans' when the ocean is considered to consist of two layers:

$$- [z_1''(t) + k_1^2 z_1(t)] = \lambda z_1(t), 0 \leq t \leq d_1, \quad (16)$$

$$= [z_2''(t) + k_2^2 z_2(t)] = \lambda z_2(t), d_1 \leq t \leq d_2, \quad (17)$$

together with mixed boundary conditions given by

$$z_1(d_1) = z_2(d_1), \frac{1}{\rho_1} z_1'(d_1) = \frac{1}{\rho_2} z_2'(d_1) \quad (18)$$

$$z_1(0) = z_2'(d_2) = 0 \quad (19)$$

where ρ_1, ρ_2 are (positive) constant densities of the two layers; k_1, k_2 are the constants which depend upon the frequency constant w and constant sound velocities c_1, c_2 of the two layers, λ is an unknown constant (eigenvalue), $[0, d_1]$ and $[d_1, d_2]$ denote the two layers and z_1, z_2 denote the depth eigenfunctions. From the physics of the problem, it follows that the absolute difference between the densities ρ_1, ρ_2 is relatively very small, and therefore we may assume that

$$\left| 1 - \frac{\rho_2}{\rho_1} \right| < 1/4.$$

Under the transformation $x = t/d_2$, BVP (16)-(19) can be shown to be of the type BVP III. Now, choose N such that

$$N > 2 \left(\frac{1 + \left| 1 - \frac{\rho_2}{\rho_1} \right|}{1 - 4 \left| 1 - \frac{\rho_2}{\rho_1} \right|} \right) d_2 (k_1^4 d_2 d_1 + k_2^4 d_2 (d_2 - d_1))^{1/2} \\ \times \exp(d_2 (k_1^4 d_2 d_1 + k_2^4 d_2 (d_2 - d_1))^{1/2}).$$

Then applying Theorem 1 (C), it can be shown that the problem (16)-(19) has exactly N eigenvalues in the open half plane $\operatorname{Re} \lambda < \frac{N^2 \pi^2}{d_2^2}$, and for each $n > N$, exactly one eigenvalue in the egg-shaped region

$$\left| \sqrt{\lambda} - \left(\frac{2n+1}{2} \right) \frac{\pi}{d_2} \right| < \frac{\pi}{2d_2}.$$

Also, by Theorems 3 (C) and 2, these eigenvalues are simple and form an infinite sequence of real numbers, which is bounded from below and tends to $+\infty$.

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