A REMARK ON A COMPLEMENTARITY PROBLEM IN BANACH SPACE

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We give a modified proof of the theorem given in an earlier paper of the author (1994).

1. Introduction

Let B be a reflexive real Banach space and let B^* be its dual. Let the value of $u \in B^*$ at $x \in B$ be denoted by (u,x). Let C be a closed convex cone in B with the vertex at 0. The polar of C is the cone C^* defined by

$$C^* = \{u \in B^* : (u,x) \geq 0 \text{ for each } x \in C\}. \tag{1.1}$$

A mapping $T : C \to B^*$ is said to be monotone if

$$(Tx - Ty, x - y) \geq 0 \tag{1.2}$$

for all $x, y \in C$, and strictly monotone if strict inequality holds for $x \neq y$. T is said to be hemicontinuous on C if for all $x, y \in C$, the map $t \to T(ty + (1-t)x)$ of $[0,1]$ to B^* is continuous when B is endowed with the weak* topology. For any $e \in C^*$ and $r > 0$, define

$$D_r(e) = \{x \in C : 0 \leq (e,x) \leq r\}. \tag{1.3}$$

The following result was proved by the author in [3].

Theorem 1.1. *Let $T : C \to B^*$ be hemicontinuous and monotone such that there is an $x \in C$ with $Tx \in \operatorname{int} C^*$. Then there is an x_0 such that*

$$x_0 \in C, \quad Tx_0 \in C^*, \quad (Tx_0, x_0) = 0. \tag{1.4}$$

In order to prove this theorem, we first established that $D_r(e)$ is closed, bounded, and convex for $e \in \operatorname{int} C^*$. Although the proof that $D_r(e)$ is bounded given in [3] is correct for finite-dimensional case, Prof. Dr. W. Oettli, University of Mannheim, Germany, observed that it is not correct for infinite-dimensional case. In the proof in [3], the possibility that

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$\|y_n\| = 1$ for all n and $y_n \rightharpoonup 0$ (weakly) was eventually overlooked. This is possible since the norm boundary of the unit ball is not weakly closed, it is rather weakly dense in the unit ball. For example, observe that in the infinite-dimensional Hilbert sequence space l_2, the unit vectors converge weakly to zero. Nevertheless, Theorem 1.1 is true and the correct proof is given in the next section.

2. Proof of the theorem

We need the following result in the sequel (see [1, 2]).

Proposition 2.1. (a) *Let T be a monotone, hemicontinuous map of a closed, convex, bounded subset K of B, with $0 \in K$, into B^*. Then there exists $x_0 \in K$ with $(Tx_0, y - x_0) \geq 0$ for all $y \in K$.*

(b) *Let T be a continuous map from a closed convex bounded subset K of $\mathbb{R}^n$ into $\mathbb{R}^n$. Then there is*

$$x_0 \in K \quad \text{with } (Tx_0, y - x_0) \geq 0 \ \forall y \in K. \tag{2.1}$$

Proof of Theorem 1.1. Let P be a convex cone and $e \in \operatorname{int} P$. Then $0 \in \operatorname{int}(e - P)$.

Furthermore

$$x^* \in P^*, \quad (e, x^*) \leq 1 \Longrightarrow (\xi, x^*) \leq 1 \ \forall \xi \in e - P. \tag{2.2}$$

Hence

$$\begin{aligned} A &= \{x^* \in P^* : (e, x^*) \leq 1\} \subseteq K \\ &= \{x^* \in X^* : (\xi, x^*) \leq 1\} \quad \forall \xi \in e - P. \end{aligned} \tag{2.3}$$

Now, $u = e - P$ is a neighborhood of zero and K, as a polar of this neighborhood, is weak* compact by Alaoglu's theorem (see, e.g., Rudin [4] for the statement of Alaoglu's theorem). Hence, A is weak* compact. Now, set $P = K^*$ and use $K^{**} = K$ to obtain the desired result that

$$D_1(e) = \{x \in K : (e, x) \leq 1\} \tag{2.4}$$

is weakly compact if $e \in \operatorname{int} K^*$.

Therefore, now it follows from Proposition 2.1(a) that there exists an $x_0 \in D_1(e)$ such that

$$(Tx_0, y - x_0) \geq 0 \quad \forall y \in D_r(e). \tag{2.5}$$

Since $0 \in D_1(e)$, $(Tx_0, x_0) \leq 0$. If there exists $e \in \operatorname{int} C^*$ such that $(e, x_0) < 1$, then there exists $\lambda > 1$ such that $(e, \lambda x_0) = 1 \Rightarrow \lambda x_0 \in D_1(e)$. Then we have from (2.5) that

$$(Tx_0, x_0) \leq (Tx_0, \lambda x_0) = \lambda (Tx_0, x_0). \tag{2.6}$$

Since $(Tx_0,x_0) \le 0$, it is impossible unless $(Tx_0,x_0) = 0$. We now show that $Tx_0 \in C^*$. For every $y \in C$, with $y \in D_1(e)$, there exists $\lambda > 0$ such that $x_0 + \lambda(y - x_0) \in D_1(e)$. Hence $(Tx_0, x_0 + \lambda(y - x_0)) \ge 0$. Hence, since $(Tx_0,x_0) = 0$, $\lambda(Tx_0, y) \ge 0$, and so $(Tx_0, y) \ge 0$ for all $y \in C$. Thus $Tx_0 \in C^*$ and x_0 is a solution of (1.4).

Now assume that $(e,x_0) = 1$ for all $e \in \operatorname{int} C^*$. By the hypothesis, there exists an $x \in C$ with $Tx \in \operatorname{int} C^*$. Set $e = Tx$. Now $(Tx,x) < 1$. Since T is monotone, we have

$$(Tz, z - x) \ge (Tx, z - x) > 0 \tag{2.7}$$

for all z with $(Tx,z) = 1$. But $(Tx,x_0) = 1$,

$$(Tx_0, x - x_0) > 0. \tag{2.8}$$

Since $(Tx,x) < 1$, $x \in D_1(Tx)$, and it follows from (2.5) that

$$(Tx_0, x - x_0) \ge 0. \tag{2.9}$$

Now (2.8) and (2.9) contradict each other. Hence $(e,x_0) < 1$ for some $e \in \operatorname{int} C^*$ and the problem now reduces to the previous case. This completes the proof. □

We conclude this paper by stating another theorem.

Theorem 2.2. *Let $T : C \to \mathbb{R}^n$ be continuous and pseudomonotone such that there exists an $x \in C$ with $Tx \in \operatorname{int} C^*$. Then there exists an x_0 such that $x_0 \in C$, $Tx_0 \in C^*$, and $(Tx_0,x_0) = 0$.*

This result is known for continuous monotone mappings. This can be proved by using Proposition 2.1(b) and proceeding in a manner similar to the proof of [3, Theorem 1.1].

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