A DESCRIPTION OF STOCHASTIC SYSTEMS USING CHAOTIC MAPS

ABRAHAM BOYARSKY AND PAWEŁ GÓRA

Received 28 August 2003 and in revised form 4 February 2004

Let $\rho(x, t)$ denote a family of probability density functions parameterized by time t . We show the existence of a family $\{\tau_t : t > 0\}$ of deterministic nonlinear (chaotic) point transformations whose invariant probability density functions are precisely $\rho(x, t)$. In particular, we are interested in the densities that arise from the diffusions. We derive a partial differential equation whose solution yields the family of chaotic maps whose density functions are precisely those of the diffusion.

1. Introduction

In this paper, we establish a method for describing flows of probability density functions by means of discrete-time chaotic maps. We start with a standard map whose invariant probability density function is known and then use it to derive other invariant probability density functions by a simple conjugation process which solves the inverse Perron-Frobenius problem [2, 3] in a time-varying setting.

2. Notation and preliminary results

In this paper, we consider space to consist of 1 dimension although the extension to 2 and 3 dimensions is straightforward. In the sequel, we will need some notions from ergodic theory and nonlinear dynamics, which can be found in [1].

Let $\mathbb{R} = (-\infty, \infty)$ and let $T : \mathbb{R} \rightarrow \mathbb{R}$ possess a unique absolutely continuous invariant measure μ which has the probability density function f , that is,

$$\int_A f \, dx = \int_{T^{-1}A} f \, dx \quad (2.1)$$

for any measurable set $A \subset \mathbb{R}$. The Perron-Frobenius operator P_T acting on the space of integrable functions is defined by

$$\int_A f \, dx = \int_{T^{-1}A} P_T f \, dx. \quad (2.2)$$

The operator P_T transforms probability density functions into probability density functions under the transformation T , where T is assumed to be nonsingular. One of the most important properties of P_T is that its fixed points are the densities of measures invariant under T [1].

Let $h: \mathbb{R} \rightarrow \mathbb{R}$ be a diffeomorphism. Then $\tau = h^{-1} \circ T \circ h$ is a transformation from $\mathbb{R}$ into $\mathbb{R}$, which is differentiably conjugate to T and whose probability density function is given by

$$k = (f \circ h) \cdot |h'|. \quad (2.3)$$

We assume that T is a piecewise monotonic expanding C^1 map on $\mathbb{R}$ that admits a unique absolutely continuous invariant measure. Then the invariant density function $f(x)$ is a fixed point of the Perron-Frobenius operator P_T [1]. We now consider the inverse Perron-Frobenius problem: suppose we are given a probability density function $g(x)$ on $\mathbb{R}$, can we find a transformation τ such that $g(x)$ is the unique probability density function invariant under τ ? This problem has been dealt with by Ershov and Malinetskiĭ [2] and in [3] from a computational perspective.

We solve the inverse Perron-Frobenius problem by applying (2.3), that is, we find h such that

$$(f \circ h) \cdot h' = g, \quad (2.4)$$

where we have assumed, without loss of generality, that h is an increasing function on $\mathbb{R}$. Now, let

$$F(x) = \int_{-\infty}^x f(y) dy \quad (2.5)$$

be the distribution function associated with f . Then, from (2.4) and the change-of-variable formula, we have

$$F(h(x)) = \int_{-\infty}^x g(y) dy. \quad (2.6)$$

Since F is a monotonically increasing function, it has a unique inverse and

$$h(x) = F^{-1} \left(\int_{-\infty}^x g(y) dy \right). \quad (2.7)$$

Thus, we have found $h(x)$ such that $\tau = h^{-1} \circ T \circ h$ has the probability density function $g(x)$. Summarizing, given any probability density function $g(x)$, we have proven the existence of a point transformation τ whose probability density function is $g(x)$.

Example 2.1. Let

$$T(x) = a \tan x, \quad x \neq \frac{k\pi}{2}, \quad k = \pm 1, \pm 3, \dots, \quad (2.8)$$

and $a > 1$. Then the probability density function invariant under T is [1]

$$f(x) = \frac{p}{\pi(p^2 + x^2)}, \quad (2.9)$$

where $p > 0$ satisfies the equation $a \tanh(p) = p$. For $p > 2$, $a \approx p$ and we can assume that $T(x) = p \tanh x$. Hence, for $p = 4$, say,

$$\begin{aligned} f(x) &= \frac{4}{\pi(16 + x^2)}, \\ F(x) &= \int_{-\infty}^x \frac{4}{\pi(16 + y^2)} dy = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x}{4}\right), \\ F^{-1}(x) &= -4 \cot(\pi x). \end{aligned} \quad (2.10)$$

Now, suppose we want to find a map τ whose unique invariant probability density function is given by $g(x) = (\cos 8x) \exp(-3x^2)$. We obtain

$$\int_{-\infty}^x (\cos 8y) \exp(-3y^2) dy = \frac{2 \exp(-2x^2) (2/\pi) \cos^2(8x)}{1 + \exp(-64/3)} \quad (2.11)$$

from which we can determine $h(x)$ using (2.7). Once h is known, so is $\tau = h^{-1} \circ T \circ h$ whose probability density function is $g(x)$.

The foregoing method can be extended to a family of probability density functions $\{g_t(y) : t \in I\}$. In this case, the homeomorphism h becomes a family of homeomorphisms $\{h_t : t \in I\}$ parameterized by t , where

$$h_t(x) = F^{-1} \left(\int_{-\infty}^x g_t(y) dy \right). \quad (2.12)$$

3. Chaotic map description of diffusions

Consider the diffusion equation

$$\frac{\partial \rho(x, t)}{\partial t} = -\nabla(v(x, t)\rho(x, t)) = -\frac{\partial}{\partial x} \left[b(x, t)\rho(x, t) - \frac{1}{2} \frac{\partial \rho(x, t)}{\partial x} \right], \quad (3.1)$$

where $b(x, t)$ is the forward drift coefficient. Our objective is to prove the existence of a family of point transformations $\{\tau_t \in \Gamma : t > 0\}$ whose invariant probability density functions are $\{\rho_t : t > 0\}$. To do this, we let T be the transformation defined by (2.8) and we derive a partial differential equation for $h_t(x)$ such that $\{\tau_t = h_t^{-1} \circ T \circ h_t\}$ possesses $\{\rho(x, t) : t > 0\}$ as the associated family of invariant probability density functions.

Since

$$h_t(x) = F^{-1} \left(\int_{-\infty}^x \rho(y, t) dy \right), \quad (3.2)$$

we have

$$\frac{\partial}{\partial t} h_t(x) = (F^{-1})' \left(\int_{-\infty}^x \rho(y, t) dy \right) \left(\int_{-\infty}^x \frac{\partial}{\partial t} \rho(y, t) dy \right). \quad (3.3)$$

Noting that

$$(F^{-1})' \left(\int_{-\infty}^x \rho(y, t) dy \right) = \frac{1}{F'(F^{-1}(\int_{-\infty}^x \rho(y, t) dy))} \quad (3.4)$$

and using (3.1), we obtain

$$\frac{\partial}{\partial t} h_t(x) = \frac{1}{f(h_t)} \int_{-\infty}^x \left[-\frac{\partial}{\partial y} b(y, t) \rho(y, t) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (\rho(y, t)) \right] dy. \quad (3.5)$$

Thus,

$$f(h_t(x)) \frac{\partial}{\partial t} h_t(x) = -b(x, t) \rho(x, t) + \frac{1}{2} \frac{\partial}{\partial x} (\rho(x, t)) + q(t), \quad (3.6)$$

where $q(t)$ is an unknown function of t only. To find $q(t)$, we return to (3.3) and write

$$f(h_t(x)) \frac{\partial}{\partial t} h_t(x) = \frac{\partial}{\partial t} \int_{-\infty}^x \rho(y, t) dy. \quad (3.7)$$

We assume that $\rho(y, t)$ and $(\partial/\partial x)(\rho(x, t))$ both go to 0 as $x \rightarrow \infty$; then the right-hand side of (3.7) also goes to 0 since

$$\int_{-\infty}^{\infty} \rho(y, t) dy = 1 \quad (3.8)$$

for all $t \geq 0$. Hence, $q(t) = 0$ for all $t \geq 0$. Thus, (3.6) reduces to

$$f(h_t(x)) \frac{\partial}{\partial t} h_t(x) = -b(x, t) f(h_t(x)) \frac{\partial}{\partial t} h_t(x) + \frac{1}{2} \frac{\partial}{\partial x} \left(f(h_t(x)) \frac{\partial}{\partial t} h_t(x) \right) \quad (3.9)$$

or

$$(1 + b(x, t)) f(h_t(x)) \frac{\partial}{\partial t} h_t(x) = \frac{1}{2} \frac{\partial}{\partial x} \left(f(h_t(x)) \frac{\partial}{\partial t} h_t(x) \right), \quad (3.10)$$

whose solution is the family of homeomorphisms $\{h_t\}$ which determine the family of deterministic chaotic maps $\{\tau_t = h_t^{-1} \circ T \circ h_t\}$, whose probability density functions are equal to $\rho(x, t)$.

Acknowledgment

This research has been supported by Natural Sciences and Engineering Research Council of Canada (NSERC) grants.

References

- [1] A. Boyarsky and P. Góra, *Laws of Chaos*, Probability and Its Applications, Birkhäuser Boston, Massachusetts, 1997.
- [2] S. V. Ershov and G. G. Malinetskii, *Solution of an inverse problem for the Perron-Frobenius equation*, U.S.S.R. Comput. Math. and Math. Phys. **28** (1988), no. 5, 136–141 (Russian).
- [3] P. Góra and A. Boyarsky, *A matrix solution to the inverse Perron-Frobenius problem*, Proc. Amer. Math. Soc. **118** (1993), no. 2, 409–414.

Abraham Boyarsky: Department of Mathematics and Statistics, Concordia University, 7141 Sherbrooke Street West, Montreal, Québec, Canada H4B 1R6

E-mail address: boyar@alcor.concordia.ca

Paweł Góra: Department of Mathematics and Statistics, Concordia University, 7141 Sherbrooke Street West, Montreal, Québec, Canada H4B 1R6

E-mail address: pgora@vax2.concordia.ca