PERIODIC SOLUTIONS FOR SOME PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

RACHID BENKHALTI AND KHALIL EZZINBI

Received 11 December 2002 and in revised form 25 August 2003

We study the existence of a periodic solution for some partial functional differential equations. We assume that the linear part is nondensely defined and satisfies the Hille-Yosida condition. In the nonhomogeneous linear case, we prove the existence of a periodic solution under the existence of a bounded solution. In the nonlinear case, using a fixed-point theorem concerning set-valued maps, we establish the existence of a periodic solution.

1. Introduction

Consider the partial functional differential equation

$$\begin{gathered}\frac{d}{dt}x(t)=Ax(t)+L(t,x_t)+G(t,x_t), \quad \text{for } t\geq 0,\\ x_0=\varphi\in C=C([-r,0];E),\end{gathered} \tag{1.1}$$

where $A:D(A)\subset E\to E$ is a nondensely defined linear operator on a Banach space E. Throughout this paper, we suppose that

(H_1) A is a Hille-Yosida operator: there exist $M_0\geq 1$ and $\omega_0\in\mathbb{R}$ such that

$$(\omega_0,\infty)\subset\rho(A), \qquad ||R(\lambda,A)^n||\leq\frac{M_0}{(\lambda-\omega_0)^n}, \quad \text{for } n\in\mathbb{N},\ \lambda>\omega_0, \tag{1.2}$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda,A)=(\lambda-A)^{-1}$.

C is the space of continuous functions from $[-r,0]$ into E endowed with the uniform norm topology, and for every $t\geq 0$, the history function $x_t\in C$ is defined by

$$x_t(\theta)=x(t+\theta), \quad \text{for } \theta\in[-r,0]. \tag{1.3}$$

$L:\mathbb{R}\times C\to E$ is continuous, linear with respect to the second argument and ω-periodic in t; $G:\mathbb{R}\times C\to E$ is continuous and ω-periodic in t.

When the operator A generates a strongly continuous semigroup on E, (1.1) has been treated extensively by several authors; for more details, we refer to [14]. Recently in [1, 8],

Journal of Applied Mathematics and Stochastic Analysis 2004:1 (2004) 9–18
2000 Mathematics Subject Classification: 34K20, 34K30, 34K40
URL: http://dx.doi.org/10.1155/S1048953304212011

the existence, the regularity of solutions, and the local stability have been treated when A is nondensely defined and satisfies the Hille-Yosida condition. In this work, we will deal with the existence of periodic solutions of (1.1) when A satisfies the Hille-Yosida condition. The problem of finding periodic solutions is an important subject in the qualitative study of functional differential equations. The famous Massera's theorem on two-dimensional periodic ordinary differential equations [11] explains the relationship between the boundedness of solutions and periodic solutions. In [15], using Browder's fixed-point theorem, it has been proved that if the solutions of an n-dimensional periodic ordinary differential equation are either uniformly bounded or uniformly ultimately bounded, then the system has a periodic solution. In [5], the existence of a periodic solution has been established under the existence of a bounded solution for some inhomogeneous, linear functional differential equation in infinite dimensional space. In [10], using Horn's fixed-point theorem, the existence of periodic solutions for functional differential equation with finite delay was established. Recently in [12], several criteria were obtained to ensure the existence and uniqueness of a periodic solution for some inhomogeneous linear partial functional differential equations with infinite delay. In [4], we developed some results dealing with the existence of a periodic solution for (1.1) when A generates a strongly continuous semigroup on E. In [7], it was established that the existence of bounded and ultimate bounded solutions of (1.1) implies the existence of periodic solutions. The approach that was used was based on Horn's fixed-point theorem. In this paper, we generalize the results obtained in [4, 5, 11] for (1.1), where the operator A is not necessarily densely defined but satisfies the Hille-Yosida condition. In Section 2, we prove the existence of periodic solutions in the nonhomogeneous linear case under the assumption that a bounded solution on $\mathbb{R}^+$ exists. In Section 3, we study the nonlinear case; our approach makes use of a fixed-point theorem for set-valued maps to obtain sufficient conditions, ensuring the existence of a periodic solution for (1.1). Section 4 is devoted to an example.

2. Inhomogeneous linear case

Definition 2.1 [1, 8]. A continuous function $x:[-r,b]\to E$ $(b>0)$ is called an integral solution of (1.1) if

(i) $\int_0^t x(s)ds \in D(A)$, for $t\in[0,b]$,
(ii) $x(t)=\varphi(0)+A\int_0^t x(s)ds+\int_0^t L(s,x_s)ds+\int_0^t G(s,x_s)ds$, for $t\in[0,b]$,
(iii) $x_0=\varphi$.

It follows from the closedness of A that if x is an integral solution of (1.1), then $x(t)\in\overline{D(A)}$, for $t\geq 0$. The following result dealing with the existence and the uniqueness of the integral solution was established.

Theorem 2.2 [1, 8]. *Assume that (H_1) holds and G is Lipschitz with respect to the second argument. Then for all $\varphi\in C$ such that $\varphi(0)\in\overline{D(A)}$, (1.1) has a unique integral solution on $\mathbb{R}^+$. Moreover, the integral solution depends continuously on the initial data.*

Let A_0 be the part of A in $\overline{D(A)}$ given by

$$A_0 = A \quad \text{on } D(A_0)=\{x\in D(A): Ax\in\overline{D(A)}\}. \tag{2.1}$$

Then, from [2], A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$. Moreover, from [13], if the integral solution of (1.1) exists, then it is given by this variation of constant formula

$$x(t) = \begin{cases} T_0(t)\varphi(0) + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)B_\lambda(L(s,x_s)+G(s,x_s))\,ds, & t \geq 0, \\ \varphi(t), \quad t \in [-r,0], \end{cases} \tag{2.2}$$

where $B_\lambda = \lambda(\lambda - A)^{-1}$.

Consider the equation

$$\begin{gathered} \frac{d}{dt}x(t) = Ax(t) + L(t,x_t) + f(t), \quad \text{for } t \geq 0, \\ x_0 = \varphi \in C = C([-r,0];E), \end{gathered} \tag{2.3}$$

where f is continuous and ω-periodic in t, and suppose the hypothesis stated below.

(H_2) The semigroup $(T_0(t))_{t\geq 0}$ is compact on $\overline{D(A)}$, meaning that for $t > 0$, the operator $T_0(t)$ is compact on $\overline{D(A)}$.

Theorem 2.3. *Assume that (H_1) and (H_2) hold. Then the following are equivalent:*

(i) *there exists a $\varphi \in C$ such that (2.3) has a bounded integral solution defined on $\mathbb{R}^+$,*

(ii) *equation (2.3) has an ω-periodic solution.*

Let u be the bounded integral solution of (2.3) on $\mathbb{R}^+$, then the following two lemmas are needed in the proof of Theorem 2.3.

Lemma 2.4. *$\{u(t) : t \geq 0\}$ is relatively compact in E and u is uniformly continuous. Consequently, $\{u_t : t \geq 0\}$ is relatively compact in C.*

Proof of Lemma 2.4. For simplicity, we equate $F(t,\varphi) = L(t,\varphi) + f(t)$, and let $\varepsilon > 0$ and $t > \varepsilon$. Then,

$$u(t) = T_0(t)u(0) + \lim_{\lambda\to\infty} \int_0^{t-\varepsilon} T_0(t-s)B_\lambda F(s,u_s)\,ds + \lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda F(s,u_s)\,ds. \tag{2.4}$$

It follows that

$$\begin{aligned} u(t) &= T_0(\varepsilon)\left[T_0(t-\varepsilon)u(0) + \lim_{\lambda\to\infty}\int_0^{t-\varepsilon} T_0(t-\varepsilon-s)B_\lambda F(s,u_s)\,ds\right] \\ &\quad + \lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda F(s,u_s)\,ds, \\ u(t) &= T_0(\varepsilon)u(t-\varepsilon) + \lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda F(s,u_s)\,ds. \end{aligned} \tag{2.5}$$

The compactness property of the semigroup $(T_0(t))_{t\geq 0}$ and the boundedness of the solution u show that $\{T_0(\varepsilon)u(t-\varepsilon) : t > \varepsilon\}$ is relatively compact in E. Using the boundedness of B_λ and F, there exists a positive constant a such that

$$\left\|\lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda F(s,u_s)\,ds\right\| \leq a\varepsilon. \tag{2.6}$$

Hence, $\{u(t) : t \geq 0\}$ is relatively compact in E.

To show the uniform continuity of u, let $t > \tau > 0$. Then,

$$\begin{aligned} u(t) - u(\tau) = \big(T_0(t) - T_0(\tau)\big)u(0) + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B_\lambda F(s,u_s)\,ds \\ - \lim_{\lambda\to\infty}\int_0^\tau T_0(\tau-s)B_\lambda F(s,u_s)\,ds. \end{aligned} \tag{2.7}$$

Since

$$\begin{aligned} u(t) - u(\tau) = \big(T_0(t-\tau) - I\big)T_0(\tau)u(0) + \big(T_0(t-\tau) - I\big)\lim_{\lambda\to\infty}\int_0^\tau T_0(\tau-s)B_\lambda F(s,u_s)\,ds \\ + \lim_{\lambda\to\infty}\int_\tau^t T_0(t-s)B_\lambda F(s,u_s)\,ds, \end{aligned} \tag{2.8}$$

we have

$$u(t) - u(\tau) = \big(T_0(t-\tau) - I\big)u(\tau) + \lim_{\lambda\to\infty}\int_\tau^t T_0(t-s)B_\lambda F(s,u_s)\,ds. \tag{2.9}$$

Now the range of u is relatively compact, so

$$\lim_{h\to 0}\big(T_0(h) - I\big)\xi = 0, \quad \text{uniformly in } \xi \in \overline{\{u(t) : t \geq 0\}}. \tag{2.10}$$

Consequently,

$$\lim_{\substack{t-\tau\to 0\\ t>\tau}} \big\|\big(T_0(t-\tau) - I\big)u(\tau)\big\| = 0. \tag{2.11}$$

On the other hand, we have

$$\lim_{\substack{t-\tau\to 0\\ t>\tau}} \left\|\lim_{\lambda\to\infty}\int_\tau^t T_0(t-s)B_\lambda F(s,u_s)\,ds\right\| = 0. \tag{2.12}$$

Therefore,

$$\lim_{\substack{t-\tau\to 0\\ t>\tau}} \|u(t) - u(\tau)\| = 0. \tag{2.13}$$

Using a similar argument, one can also show that

$$\lim_{\substack{t-\tau\to 0\\ t<\tau}} \|u(t) - u(\tau)\| = 0. \tag{2.14}$$

From the uniform continuity of u, we determine that $\{u_t : t \geq 0\}$ is an equicontinuous family of functions on $[-r, 0]$; moreover, the range of u is relatively compact. Hence, by Arzèla-Ascoli theorem, we determine that $\{u_t : t \geq 0\}$ is relatively compact in C. □

Lemma 2.5 [9]. *Let X be a Banach space, let $\Phi : X \to X$ be a continuous linear operator, let $y \in X$ be given, and let $\Theta : X \to X$ be given by $\Theta x = \Phi x + y$. Suppose that there exists $x_0 \in X$ such that $\{\Theta^n x_0 : n \in \mathbb{N}\}$ is relatively compact. Then Θ has a fixed point.*

Proof of Theorem 2.3. As usual, define the Poincaré map $P(\varphi) = x_\omega(\cdot,\varphi,f)$ on the phase space $C_0 = \{\varphi \in C : \varphi(0) \in \overline{D(A)}\}$, where $x(\cdot,\varphi,f)$ is the integral solution of (2.3). Because of the uniqueness property, it is enough to show that P has a fixed point to get an ω-periodic solution of (2.3). Also, the uniqueness property of the solution with respect to φ allows the Poincaré map P to be decomposed as

$$P(\varphi) = x_\omega(\cdot,\varphi,0) + x_\omega(\cdot,0,f), \tag{2.15}$$

where $x_\omega(\cdot,\varphi,0)$ is the integral solution of (2.3) with $f = 0$, and $x_\omega(\cdot,0,f)$ is the integral solution of (2.3) with $\varphi = 0$. Let u be the bounded solution of (2.3) on $[0,+\infty)$ and $u_0 = \varphi$. Then, by Lemma 2.4,

$$\{P^n\varphi : n \in \mathbb{N}\} = \{u_{n\omega} : n \in \mathbb{N}\} \tag{2.16}$$

is relatively compact in C_0, and the mapping P has a fixed point in C_0 using Lemma 2.5. Hence, (2.3) has an ω-periodic solution. □

3. Nonlinear case

Consider the nonlinear equation

$$\frac{d}{dt}x(t) = Ax(t) + L(t,x_t) + G(t,x_t), \quad \text{for } t \geq 0, \tag{3.1}$$

and assume the hypothesis stated below.

(H_3) G takes every bounded set into a bounded set.

Let B_ω be the space of all continuous ω-periodic functions from $\mathbb{R}^+$ into E, endowed with the uniform norm topology.

Theorem 3.1. *Assume that (H_1), (H_2), and (H_3) hold. Further, assume that there exists a positive ρ such that for any $y \in S_\rho = \{v \in B_\omega : \|v\| \leq \rho\}$, the equation*

$$\frac{d}{dt}x(t) = Ax(t) + L(t,x_t) + G(t,y_t), \quad \text{for } t \in \mathbb{R}^+, \tag{3.2}$$

has an ω-periodic integral solution in S_ρ. Then, (3.1) has an integral ω-periodic solution on $\mathbb{R}^+$.

For the proof, we need the following definition and theorem.

Definition 3.2 (see [16, Definition 9.3]). Let $\mathcal{G} : M \to 2^M$ be a multivalued map, where M is a subset of a Banach space and 2^M is the power set of M.

(i) For $D \subset M$, the inverse image $\mathcal{G}^{-1}(D)$ is the set of all $x \in M$ such that $\mathcal{G}(x) \cap D \neq \emptyset$.

(ii) The map $\mathcal{G}$ is called upper semicontinuous if $\mathcal{G}^{-1}(D)$ is closed for all closed set D in M.

Theorem 3.3 (see [16, Corollary 9.8]). *Let $\mathcal{G}: M \to 2^M$ be a multivalued map, where M is a nonempty convex set in the Banach space X such that*

(i) *the set $\mathcal{G}(x)$ is nonempty, closed, and convex for all $x \in M$,*
(ii) *the set $\mathcal{G}(M)$ is relatively compact,*
(iii) *the map $\mathcal{G}: M \to 2^M$ is upper semicontinuous.*

Then $\mathcal{G}$ has a fixed point in the sense that there exists $x \in M$ such that $x \in \mathcal{G}(x)$.

Proof of Theorem 3.1. Define the set-valued mapping $\mathcal{G}: S_\rho \to 2^{S_\rho}$, for $y \in S_\rho$, by

$$\mathcal{G}(y) = \left\{x \in S_\rho : x(t) = T_0(t)x(0) + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B_\lambda\big(L(s,x_s)+G(s,y_s)\big)ds,\ t \geq 0\right\}. \tag{3.3}$$

We will show that the mapping $\mathcal{G}$ satisfies the conditions of Theorem 3.3.

(i) Let $y \in S_\rho$, $x_1, x_2 \in \mathcal{G}(y)$, and $\lambda \in [0,1]$. Then, $\lambda x_1 + (1-\lambda)x_2 \in \mathcal{G}(y)$, which implies that $\mathcal{G}(y)$ is convex. From the continuity of L and G, we obtain that $\mathcal{G}(y)$ is a closed set.

(ii) Let $x \in \mathcal{G}(S_\rho)$, then there exists $y \in S_\rho$ such that

$$x(t) = T_0(t)x(0) + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B_\lambda\big(L(s,x_s)+G(s,y_s)\big)ds, \quad t \geq 0. \tag{3.4}$$

We first show that $\{x(t) : x \in \mathcal{G}(S_\rho)\}$ is relatively compact in E. Let $t > 0$ and $\varepsilon > 0$ such that $t > \varepsilon$. Then,

$$\begin{aligned} x(t) = T_0(t)x(0) &+ T_0(\varepsilon)\lim_{\lambda\to\infty}\int_0^{t-\varepsilon} T_0(t-\varepsilon-s)B_\lambda\big(L(s,x_s)+G(s,y_s)\big)ds \\ &+ \lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda\big(L(s,x_s)+G(s,y_s)\big)ds. \end{aligned} \tag{3.5}$$

From the boundedness of L, G and (H_2), we deduce that

$$\left\{T(\varepsilon)\lim_{\lambda\to\infty}\int_0^{t-\varepsilon} T_0(t-\varepsilon-s)B_\lambda\big(L(s,x_s)+G(s,y_s)\big)ds : x \in \mathcal{G}(S_\rho)\right\} \tag{3.6}$$

is relatively compact in E. On the other hand, for some positive constant b, we have

$$\left\|\lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda\big(L(s,x_s)+G(s,y_s)\big)\right\|ds \leq b\varepsilon, \quad \forall x \in \mathcal{G}(S_\rho). \tag{3.7}$$

Hence, $\{x(t) : x \in \mathcal{G}(S_\rho)\}$ is relatively compact in E, for every $t > 0$, and by periodicity, we also have that $\{x(0) : x \in \mathcal{G}(S_\rho)\}$ is relatively compact in E. For the equicontinuity, one has, for $t > \tau > 0$,

$$\begin{aligned} \|x(t)-x(\tau)\| \leq \|T_0(t)-T_0(\tau)\|\rho &+ \left\|\lim_{\lambda\to\infty}\int_\tau^t T_0(t-s)B_\lambda\big(L(s,x_s)+G(s,y_s)\big)ds\right\| \\ &+ \left\|\big(T_0(t-\tau)-I\big)\lim_{\lambda\to\infty}\int_0^\tau T_0(\tau-s)B_\lambda\big(L(s,x_s)+G(s,y_s)\big)ds\right\|. \end{aligned} \tag{3.8}$$

The semigroup $(T_0(t))_{t\geq 0}$ is compact, so $(T_0(t))_{t\geq 0}$ is continuous in the uniform topology whenever $t > 0$. Hence,

$$\lim_{t\to\tau} \|T_0(t) - T_0(\tau)\| = 0. \tag{3.9}$$

By (H_3), we deduce that for some positive constant c,

$$\int_\tau^t \|T_0(t-s)B_\lambda(L(s,x_s)+G(s,y_s))\|ds \leq c(t-\tau), \quad \text{uniformly for } x,y \in S_\rho. \tag{3.10}$$

Since $\{x(t) : x \in \mathcal{G}(S_\rho)\}$ is relatively compact in E for every $t \geq 0$, $\{x(t) - T(t)x(0) : x \in \mathcal{G}(S_\rho)\}$ is also relatively compact in E. Moreover, there exists a compact set K in E such that

$$\lim_{\lambda\to\infty} \int_0^\tau T_0(\tau-s)B_\lambda(L(s,x_s)+G(s,y_s))ds \in K, \quad \forall x \in \mathcal{G}(S_\rho). \tag{3.11}$$

Consequently,

$$\begin{gathered} \lim_{h\to 0}(T_0(h)-I)\xi = 0, \quad \text{uniformly in } \xi \in K, \\ \lim_{\substack{t\to\tau \\ t>\tau}} \sup_{x\in\mathcal{G}(S_\rho)} \|x(t)-x(\tau)\| = 0. \end{gathered} \tag{3.12}$$

Similarly, one can also prove that

$$\lim_{\substack{t\to\tau \\ t<\tau}} \sup_{x\in\mathcal{G}(S_\rho)} \|x(t)-x(\tau)\| = 0. \tag{3.13}$$

Therefore, $\mathcal{G}(S_\rho)$ is a family of uniformly bounded and equicontinuous ω-periodic functions. By the Arzèla-Ascoli theorem, we deduce that $\mathcal{G}(S_\rho)$ is relatively compact in B_ω.

(iii) To prove that $\mathcal{G}$ is upper semicontinuous, it is enough to show that $\mathcal{G}$ is closed. Let $(y^n)_{n\geq 0}$ and $(z^n)_{n\geq 0}$ be sequences, respectively, in S_ρ and $\mathcal{G}(S_\rho)$ such that

$$y^n \longrightarrow y, \quad z^n \longrightarrow z \quad \text{as } n \longrightarrow \infty,\ z^n \in \mathcal{G}(y^n),\ \forall n \geq 0. \tag{3.14}$$

Then,

$$z^n(t) = T_0(t)z^n(0) + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B_\lambda(L(s,z_s^n)+G(s,y_s^n))ds, \quad t \geq 0. \tag{3.15}$$

Letting n go to infinity and by a continuity argument, we obtain

$$z(t) = T_0(t)z(0) + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B_\lambda(L(s,z_s)+G(s,y_s))ds, \quad t \geq 0. \tag{3.16}$$

Hence, $z \in \mathcal{G}(y)$, which implies that $\mathcal{G}$ is closed. Now let D be a closed set in S_ρ and take a sequence $(x_n)_n \subset \mathcal{G}^{-1}(D)$ such that $x_n \to x$ as $n \to \infty$. Since $x_n \in \mathcal{G}^{-1}(D)$, it follows that there exists $y_n \in D$ such that $y_n \in \mathcal{G}(x_n)$. Moreover, $\mathcal{G}(S_\rho)$ is compact; thus, there exists a subsequence $(y_n')_n$ of $(y_n)_n$ such that $y_n' \to y$ as $n \to \infty$. Therefore, $\mathcal{G}$ is closed and it follows that $y \in \mathcal{G}(x)$ and $y \in \mathcal{G}^{-1}(D)$. Consequently, $\mathcal{G}$ is upper semicontinuous. All the assumptions of Theorem 3.3 hold. Hence, there exists $x \in S_\rho$ such that $x \in \mathcal{G}(x)$. Finally, x is an ω-periodic solution of (3.1) on $\mathbb{R}^+$. □

To prove that (3.2) has an ω-periodic solution in S_ρ, it suffices, by Theorem 2.3, to show that it has a solution which is bounded by ρ.

Corollary 3.4. *Assume that (H_1), (H_2), and (H_3) hold. If there exists a positive ρ such that for any $y \in S_\rho = \{v \in B_\omega : \|v\| \le \rho\}$, the nonhomogeneous linear equation (3.2) has an integral solution that is bounded by ρ. Then, (3.1) has an integral ω-periodic solution on $\mathbb{R}^+$.*

Proof. Let u be a bounded solution of (3.2) such that $u_0 = \varphi$. Following the proof of [9, Theorem 2.5], the map P has a fixed point which belongs to $\overline{\text{co}}\{P^n\varphi : n \ge 0\}$, where $\overline{\text{co}}$ denotes the closure of the convex hull. Let ψ be the fixed point of P and $x(\cdot,\psi,f)$ the associated integral solution; by virtue of the continuous dependence on the initial data, the solution $x(\cdot,\psi,f)$ is also bounded by ρ. □

4. Application

To apply the previous results, we consider the partial differential equation with delay:

$$\begin{gathered}\frac{\partial}{\partial t}w(t,x)=\frac{\partial^2}{\partial x^2}w(t,x)+b_1(t)w(t-r,x)+b_2(t)h\big(w(t-r,x)\big)+g(t,x), \quad t\ge 0,\ x\in[0,\pi],\\ w(t,0)=w(t,\pi)=0, \quad t\ge 0,\\ w(\theta,x)=\phi(\theta,x), \quad \theta\in[-r,0],\ x\in[0,\pi],\end{gathered} \tag{4.1}$$

where $b_1, b_2 : \mathbb{R}^+ \to \mathbb{R}$ are continuous and ω-periodic, $h : \mathbb{R} \to \mathbb{R}$ is continuous such that

$$\big|h(x)\big| \le k|x|, \quad x \in \mathbb{R}, \tag{4.2}$$

$g : \mathbb{R}^+ \times [0,\pi] \to \mathbb{R}$ is continuous and ω-periodic in t, and $\phi : [-r,0] \times [0,\pi] \to \mathbb{R}$ is continuous. Let $Y = C([0,\pi];\mathbb{R})$ and Δ the Laplacian operator on $[0,\pi]$ with domain

$$D(\Delta) = \{z \in C([0,\pi];\mathbb{R}) : \Delta z \in C([0,\pi];\mathbb{R}),\ z(0) = z(\pi) = 0\}. \tag{4.3}$$

Then, by [6], Δ satisfies the Hille-Yosida condition in Y; more precisely, one has

$$(0,+\infty) \subset \rho(\Delta), \qquad \|R(\lambda,\Delta)\| \le \frac{1}{\lambda}, \quad \text{for } \lambda > 0. \tag{4.4}$$

Moreover,

$$\overline{D(\Delta)} = \{z \in C([0,\pi];\mathbb{R}) : z(0) = z(\pi) = 0\} = C_0([0,\pi];\mathbb{R}). \tag{4.5}$$

Let Δ_0 be the part of Δ in $\overline{D(\Delta)}$ given by

$$D(\Delta_0) = \{z \in C_0([0,\pi];\mathbb{R}) : \Delta z \in C_0([0,\pi];\mathbb{R})\}, \qquad \Delta_0 z = \Delta z. \tag{4.6}$$

Then, by [3], Δ_0 generates a compact semigroup $(T_0(t))_{t\ge 0}$ on $C_0([0,\pi];\mathbb{R})$ such that

$$\|T_0(t)\| \le e^{-t}, \quad t \ge 0. \tag{4.7}$$

Let $L,G:\mathbb{R}\times C([-r,0];Y)\to Y$ be defined, for $t\in\mathbb{R}^+$, $\varphi\in C([-r,0];Y)$, and $x\in[0,\pi]$, by

$$\begin{aligned}(L(t,\varphi))(x)&=b_1(t)\varphi(-r)(x),\\ (G(t,\varphi))(x)&=b_2(t)h(\varphi(-r)(x))+g(t,x).\end{aligned}\tag{4.8}$$

Then, (4.1) takes the abstract form

$$\frac{d}{dt}x(t)=\Delta x(t)+L(t,x_t)+G(t,x_t),\quad \text{for } t\geq 0.\tag{4.9}$$

Hence, (H_1), (H_2), and (H_3) are satisfied, and we have the following proposition.

Proposition 4.1. *Assume that there exists $d\in(0,1)$ such that*

$$|b_1(t)|+|b_2(t)|k\leq 1-d,\quad \textit{for } t\in[0,\omega].\tag{4.10}$$

Then, (4.9) has an ω-periodic solution.

Proof. Let $m=\max_{t\in[0,\omega],x\in[0,\pi]}|g(t,x)|$ and $\rho=1+m/d$. We claim that if y is a continuous ω-periodic function such that $\|y\|\leq\rho$, then for all φ with $\|\varphi\|<\rho$, the solution x of

$$\begin{gathered}\frac{d}{dt}x(t)=\Delta x(t)+L(t,x_t)+G(t,y_t),\quad \text{for } t\geq 0,\\ x_0=\varphi\in C([-r,0];Y),\end{gathered}\tag{4.11}$$

satisfies $\|x(t)\|\leq\rho$, for all $t\geq 0$. Proceeding by contradiction, suppose that there exists t_1 such that $\|x(t_1)\|>\rho$ and let

$$t_0=\inf\{t>0:||x(t)||>\rho\}.\tag{4.12}$$

By continuity, we get $\|x(t_0)\|=\rho$ and there exists $\delta>0$ such that $\|x(t)\|>\rho$, for $t\in(t_0,t_0+\delta)$. By using the variation of constant formula (2.2),

$$x(t_0)=T_0(t_0)\varphi(0)+\lim_{\lambda\to\infty}\int_0^{t_0}T_0(t_0-s)B_\lambda(L(s,x_s)+G(s,y_s))ds,\quad t\geq 0.\tag{4.13}$$

By (4.8), we get that

$$||x(t_0)||\leq e^{-t_0}\rho+((|b_1|+|b_2|k)\rho+m)(1-e^{-t_0}),\tag{4.14}$$

and by condition (4.10), we obtain

$$||x(t_0)||\leq\rho+(m-\rho d)(1-e^{-t_0})\tag{4.15}$$

or $\|x(t_0)\|\leq\rho-d(1-e^{-t_0})$, which gives that $\|x(t_0)\|<\rho$. This contradicts the definition of t_0. Consequently, $\|x(t)\|\leq\rho$ for all $t\geq 0$, and by Corollary 3.4, (4.9) has an ω-periodic solution in S_ρ. □

Acknowledgments

The authors would like to thank the referee for his valuable suggestions and comments which allowed them to improve the original manuscript. This research is supported by Third World Academy of Sciences (TWAS) Grant under contract no. 00-412 RG/MATHS /AF/AC.

References

[1] M. Adimy and K. Ezzinbi, *Local existence and linearized stability for partial functional-differential equations*, Dynam. Systems Appl. **7** (1998), no. 3, 389–403.

[2] W. Arendt, *Vector-valued Laplace transforms and Cauchy problems*, Israel J. Math. **59** (1987), no. 3, 327–352.

[3] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck, *One-Parameter Semigroups of Positive Operators*, Lecture Notes in Mathematics, vol. 1184, Springer-Verlag, Berlin, 1986, edited by R. Nagel.

[4] R. Benkhalti and K. Ezzinbi, *A Massera type criterion for some partial functional differential equations*, Dynam. Systems Appl. **9** (2000), no. 2, 221–228.

[5] S. N. Chow, *Remarks on one-dimensional delay-differential equations*, J. Math. Anal. Appl. **41** (1973), 426–429.

[6] G. Da Prato and E. Sinestrari, *Differential operators with nondense domain*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), no. 2, 285–344.

[7] K. Ezzinbi and J. H. Liu, *Periodic solutions of non-densely defined delay evolution equations*, J. Appl. Math. Stochastic Anal. **15** (2002), no. 2, 113–123.

[8] K. Ezzinbi and H. Tamou, *Abstract semilinear functional differential equations of retarded type*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **8** (2001), no. 2, 291–303.

[9] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, vol. 25, American Mathematical Society, Rhode Island, 1988.

[10] J. K. Hale and O. Lopes, *Fixed point theorems and dissipative processes*, J. Differential Equations **13** (1973), 391–402.

[11] J. L. Massera, *The existence of periodic solutions of systems of differential equations*, Duke Math. J. **17** (1950), 457–475.

[12] J. S. Shin and T. Naito, *Semi-Fredholm operators and periodic solutions for linear functional-differential equations*, J. Differential Equations **153** (1999), no. 2, 407–441.

[13] H. R. Thieme, *Semiflows generated by Lipschitz perturbations of non-densely defined operators*, Differential Integral Equations **3** (1990), no. 6, 1035–1066.

[14] J. Wu, *Theory and Applications of Partial Functional-Differential Equations*, Applied Mathematical Sciences, vol. 119, Springer-Verlag, New York, 1996.

[15] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, Publications of the Mathematical Society of Japan, No. 9, Mathematical Society of Japan, Tokyo, 1966.

[16] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. I. Fixed-Point Theorems*, Springer-Verlag, New York, 1986.

Rachid Benkhalti: Department of Mathematics, Pacific Lutheran University, Tacoma, WA 98447, USA

E-mail address: benkhar@plu.edu

Khalil Ezzinbi: Département de Mathématiques, Faculté des Sciences Semlalia, Université Cadi Ayyad, B.P. 2390, Marrakech 40000, Morocco

E-mail address: ezzinbi@ucam.ac.ma