A GENERALIZATION OF STRAUBE'S THEOREM: EXISTENCE OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR RANDOM MAPS

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A random map is a discrete-time dynamical system in which one of a number of transformations is randomly selected and applied at each iteration of the process. In this paper, we study random maps. The main result provides a necessary and sufficient condition for the existence of absolutely continuous invariant measure for a random map with constant probabilities and position-dependent probabilities.

1. Introduction

Random dynamical systems provide a useful framework for modeling and analyzing various physical, social, and economic phenomena. A random dynamical system of special interest is a random map where the process switches from one map to another according to fixed probabilities [5] or, more generally, position-dependent probabilities [1, 3, 4]. The existence and properties of invariant measures for random maps reflect their long-term behavior and play an important role in understanding their chaotic nature.

It is well known that if a map $\tau : I \to I$, $I = [0,1]$, is piecewise expanding, then it possesses an absolutely continuous invariant measure (ACIM) [2]. This result can be generalized to random maps where the condition of piecewise expanding is replaced by an average expanding condition where the weighting coefficients are the probabilities of switching [3, 4, 5]. Such results have been generalized in [1]. There are a number of interesting examples which do not fall into the average expanding condition for which the conditions of this paper may present a possible approach.

Consider the following simple random maps on I:

$$\tau_1(x) = \frac{x}{2}, \qquad \tau_2(x) = \frac{(x+1)}{2}, \tag{1.1}$$

with constant probabilities p_1 and p_2. τ_1 has an attracting fixed point at 0, while τ_2 has an attracting fixed point at 1. Thus, neither τ_1 nor τ_2 has an ACIM, yet any random map based on these two maps has Lebesgue measure as its unique ACIM. This shows that a random map does not necessarily inherit the properties of the underlying maps. Consider now an expanding map τ_1 on I and the logistic map τ_2 on I. Both maps have

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an ACIM, but the "average expanding" sufficiency condition for existence of an ACIM for the random map based on τ_1 and τ_2 fails since τ_2 has regions of arbitrarily small slope. Hence, in general, we cannot conclude that even such a simple random map admits an ACIM.

The foregoing suggests the need for results that can establish existence of an ACIM directly for random maps. To this end we generalize a theorem of Straube [7], which provides a necessary and sufficient condition for existence of an ACIM of a nonsingular map, to random maps. We consider both random maps with constant probabilities and random maps with position-dependent probabilities.

In Section 2, we present the notation and summarize the results that we will need in the sequel. In Section 3, we prove the main result.

2. Preliminaries

Let $(X,\mathcal{B},\lambda)$ be a measure space, where λ is an underlying measure and let $\tau_k : X \to X$, $k = 1,2,\dots,K$ be nonsingular transformations. A random map T with constant probabilities is defined as

$$T = \{\tau_1,\tau_2,\dots,\tau_K; p_1,p_2,\dots,p_K\}, \tag{2.1}$$

where $\{p_1,p_2,\dots,p_K\}$ is a set of constant probabilities. For any $x \in X$, $T(x) = \tau_k(x)$ with probability p_k and for any nonnegative integer N, $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)$ with probability $\Pi_{j=1}^{N} p_{k_j}$. A T-invariant measure satisfies the following condition [6]:

$$\mu(E) = \sum_{k=1}^{K} p_k \mu\left(\tau_k^{-1}(E)\right), \tag{2.2}$$

for any $E \in \mathcal{B}$.

A position-dependent random map T is defined as

$$T = \{\tau_1,\tau_2,\dots,\tau_K; p_1(x),p_2(x),\dots,p_K(x)\}, \tag{2.3}$$

where $\{p_1(x),p_2(x),\dots,p_K(x)\}$ is a set of position-dependent probabilities, that is, $\sum_{k=1}^{K} p_k(x) = 1$, for any $x \in X$, $T(x) = \tau_k(x)$ with probability $p_k(x)$ and for any nonnegative integer N, $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)$ with probability

$$p_{k_N}\left(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)\right) p_{k_{N-1}}\left(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)\right) \cdots p_{k_1}(x). \tag{2.4}$$

In [2], it was proved that a T-invariant measure μ is given by

$$\mu(E) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(E)} p_k(x)\, d\mu(x), \tag{2.5}$$

for any measurable set $E \in \mathcal{B}$.

We now recall some definitions and results from [6, 7] which will be used to prove our main results in Section 3.

Definition 2.1. A set function $\phi : \mathcal{B} \to R$ is finitely additive measure if

(i) $-\infty < \phi(E) < \infty$, for all $E \in \mathcal{B}$;

(ii) $\phi(\emptyset) = 0$;

(iii) $\sup_{E\in\mathcal{B}} |\phi(E)| < \infty$;

(iv) $\phi(E_1 \cup E_2) = \phi(E_1) + \phi(E_2)$, for all $E_1, E_2 \in \mathcal{B}$ such that $E_1 \cap E_2 = \emptyset$.

Definition 2.2. A finitely additive positive measure μ is purely additive measure if every countably additive measure $\nu \geq 0$, $\nu \leq \mu$ is identically zero.

Theorem 2.3 [7]. *Let ϕ be finitely additive (positive) measure. Then ϕ has a unique representation $\phi = \phi_c + \phi_p$, where ϕ_c is countably additive $(\phi_c \geq 0)$ and ϕ_p is purely additive $(\phi_p \geq 0)$.*

Lemma 2.4 [7]. *If μ is a finitely additive positive measure on $\mathcal{B}$, then μ_c is the greatest measure among countably additive measures ν with $0 \leq \nu \leq \mu$.*

Theorem 2.5 [7]. *Let ϕ be a finitely additive positive measure on a σ-algebra $\mathcal{B}$ and let ν be a countably additive positive measure on $\mathcal{B}$. Then, there exists a decreasing sequence $\{E_n\}_{n\geq 1}$ of elements of $\mathcal{B}$ such that $\lim_{n\to\infty} \nu(E_n) = 0$ and $\phi(E_n) = \phi(X)$.*

Theorem 2.6 [6]. *Let (X,B,λ) be a measure space with normalized measure λ, and let $f : X \to X$ be a nonsingular transformation. Then, the following conditions are equivalent:*

(i) *there exists an f-invariant normalized measure μ which is absolutely continuous with respect to λ;*

(ii) *there exists $\delta > 0$, and $\alpha, 0 < \alpha < 1$ such that*

$$\lambda(E) < \delta \Longrightarrow \sup_{k\in N} \lambda\left(f^{-k}(E)\right) < \alpha, \quad E \in \mathcal{B}. \tag{2.6}$$

3. Existence of absolutely continuous invariant measures

In this section, we prove necessary and sufficient conditions for existence of an absolutely continuous invariant measure for random maps. For notational convenience, we consider $K = 2$, that is, we consider only two transformations τ_1, τ_2. The proofs for larger number of maps are analogous. We first consider random maps with constant probabilities, then random maps with position-dependent probabilities.

Theorem 3.1. *Let $(X,\mathcal{B},\lambda)$ be a measure space with normalized measure λ and let $\tau_i : X \to X$, $i = 1,2$ be nonsingular transformations. Consider the random map $T = \{\tau_1, \tau_2; p_1, p_2\}$ with constant probabilities p_1, p_2. Then, there exists a normalized absolutely continuous (w.r.t. λ) T-invariant measure μ if and only if there exists $\delta > 0$ and $0 < \alpha < 1$ such that for any measurable set E and any positive integer k, $\lambda(E) < \delta$ implies*

$$\begin{gathered}
p_1\lambda\left(\tau_1^{-1}(E)\right) + p_2\lambda\left(\tau_2^{-1}(E)\right) < \alpha; \\
p_1^2\lambda\left(\tau_1^{-2}(E)\right) + p_1p_2\lambda\left(\tau_2^{-1}\tau_1^{-1}(E)\right) + p_1p_2\lambda\left(\tau_1^{-1}\tau_2^{-1}(E)\right) + p_2^2\lambda\left(\tau_2^{-2}(E)\right) < \alpha; \\
\vdots \\
\sum_{(i_1,i_2,i_3,\ldots,i_k)} p_{i_1}p_{i_2}\cdots p_{i_k}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\cdots\tau_{i_k}^{-1}(E)\right) < \alpha.
\end{gathered} \tag{3.1}$$

To prove this theorem, we first prove the following two lemmas.

Lemma 3.2. *Let $(X,\mathcal{B},\lambda)$ be a probability measure space and let μ be absolutely continuous with respect to λ, $\mu = f \cdot \lambda$, for f an $L^1(X,\mathcal{B},\lambda)$ function. Then, there exists a constant $M \geq 0$ and a measurable set A_0 such that $\mu(A_0) \leq 1/10$ and $f \leq M$ on $X \setminus A_0$.*

Proof. Consider the following sets:

$$B_n = \{x \in X : n \leq f(x) < n+1\}, \quad n = 0,1,\dots. \tag{3.2}$$

Clearly, $\{B_n\}$ are disjoint measurable sets and $X = \cup_{n=0}^{\infty} B_n$ and $1 = \mu(X) = \sum_{n=0}^{\infty} \mu(B_n)$. Thus, there exists an $M \geq 0$ such that $\sum_{n=M}^{\infty} \mu(B_n) < 1/10$. Let $A_0 = \bigcup_{n=M}^{\infty} B_n$. Then on $X \setminus \bigcup_{n=M}^{\infty} B_n$, $f(x) \leq M$. □

For any measure ϕ, any integer k, and any measurable set E, define

$$\Lambda_k^{\phi}(E) := \sum_{(i_1,i_2,i_3,\dots,i_k)} p_{i_1} p_{i_2} \cdots p_{i_k} \phi\left(\tau_{i_1}^{-1} \tau_{i_2}^{-1} \cdots \tau_{i_k}^{-1}(E)\right). \tag{3.3}$$

It can be easily shown that Λ_k^{λ} and Λ_k^{μ} are normalized measures and Λ_k^{μ} are measures absolutely continuous with respect to Λ_k^{λ}.

Lemma 3.3. *Let M be the constant from the previous lemma and let δ be such that $M\delta + 1/10 < 1/4$. Then, for any $n \geq 1$, and any measurable set A, $\Lambda_n^{\lambda}(A) < \delta \Rightarrow \Lambda_n^{\mu}(A) < 1/4$.*

Proof. Let M and A_0 be as in the previous lemma. We have

$$\begin{aligned}
\Lambda_n^{\mu}(A) &= \sum_{(i_1,i_2,i_3,\dots,i_n)} p_{i_1} p_{i_2} \cdots p_{i_n} \mu\left(\tau_{i_1}^{-1} \tau_{i_2}^{-1} \cdots \tau_{i_n}^{-1}(A)\right) \\
&= \sum_{(i_1,i_2,i_3,\dots,i_n)} p_{i_1} p_{i_2} \cdots p_{i_n} \mu\left(\tau_{i_1}^{-1} \tau_{i_2}^{-1} \cdots \tau_{i_n}^{-1}(A) \cap A_0\right) \\
&\quad + \sum_{(i_1,i_2,i_3,\dots,i_n)} p_{i_1} p_{i_2} \cdots p_{i_n} \mu\left(\tau_{i_1}^{-1} \tau_{i_2}^{-1} \cdots \tau_{i_n}^{-1}(A) \cap (X \setminus A_0)\right) \\
&\leq \sum_{(i_1,i_2,i_3,\dots,i_n)} p_{i_1} p_{i_2} \cdots p_{i_n} \frac{1}{10} + \sum_{(i_1,i_2,i_3,\dots,i_n)} p_{i_1} p_{i_2} \cdots p_{i_n} M \lambda\left(\tau_{i_1}^{-1} \tau_{i_2}^{-1} \cdots \tau_{i_n}^{-1}(A)\right) \\
&\leq \frac{1}{10} + M \Lambda_n^{\lambda}(A) < \frac{1}{10} + M\delta < \frac{1}{4}.
\end{aligned} \tag{3.4}$$

□

Proof of Theorem 3.1. Suppose

$$\mu(E) = \sum_{i=1}^{2} p_i \mu\left(\tau_i^{-1}(E)\right), \quad E \in B, \qquad \mu(X) = 1, \quad \mu \ll \lambda. \tag{3.5}$$

We want to prove that there exist $\delta > 0$, $0 < \alpha < 1$ such that for any $E \in \mathcal{B}$ and for any positive integer k,

$$\lambda(E) < \delta \Longrightarrow \Lambda_k^{\lambda}(E) < \alpha. \tag{3.6}$$

Suppose not. Then, for any α, $0 < \alpha < 1$, there exists $E \in \mathcal{B}$ and there exists a positive integer n_0 such that

$$\lambda(E) < \delta \Longrightarrow \Lambda^{\lambda}_{n_0}(E) > \alpha, \tag{3.7}$$

where $E \in \mathcal{B}$.

Choose $\delta > 0$ such that $M\delta + 1/10 < 1/4$, where M is the constant of Lemma 3.2. Let n_0 be the index corresponding to δ in formula (3.6). Then by Lemma 3.2, we have, for $A \in \mathcal{B}$,

$$\begin{aligned} \lambda(A) < \delta &\Longrightarrow \mu(A) < \frac{1}{4}; \\ \Lambda^{\lambda}_{n_0}(A) < \delta &\Longrightarrow \Lambda^{\mu}_{n_0}(A) < \frac{1}{4}. \end{aligned} \tag{3.8}$$

Let $\alpha = 1 - \delta/2$. Then,

$$\Lambda^{\lambda}_{n_0}(X \setminus E) = 1 - \Lambda^{\lambda}_{n_0}(E) < 1 - 1 + \delta = \delta. \tag{3.9}$$

From our choice of δ, we get

$$\Lambda^{\mu}_{n_0}(X \setminus E) < \frac{1}{4}. \tag{3.10}$$

Since μ is invariant, we have

$$\mu(X \setminus E) = \Lambda^{\mu}_{n_0}(X \setminus E) < \frac{1}{4}. \tag{3.11}$$

Thus,

$$1 = \mu(X) = \mu(E) + \mu(X \setminus E) < \frac{1}{4} + \frac{1}{4}, \tag{3.12}$$

a contradiction.

Conversely, suppose that there exists $\delta > 0$ and $0 < \alpha < 1$ such that for any measurable set E and any positive integer k, $\lambda(E) < \delta$ implies

$$\begin{gathered} p_1\lambda\big(\tau_1^{-1}(E)\big) + p_2\lambda\big(\tau_2^{-1}(E)\big) < \alpha; \\ p_1^2\lambda\big(\tau_1^{-2}(E)\big) + p_1p_2\lambda\big(\tau_2^{-1}\tau_1^{-1}(E)\big) + p_1p_2\lambda\big(\tau_1^{-1}\tau_2^{-1}(E)\big) + p_2^2\lambda\big(\tau_2^{-2}(E)\big) < \alpha; \\ \vdots \\ \sum_{(i_1,i_2,i_3,\dots,i_k)} p_{i_1}p_{i_2}\cdots p_{i_k}\lambda\big(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\cdots\tau_{i_k}^{-1}(E)\big) < \alpha. \end{gathered} \tag{3.13}$$

We want to show that there exists a measure μ such that $\mu(E) = \sum_{i=1}^{2} p_i\mu(\tau_i^{-1}(E))$, $E \in \mathcal{B}$, $\mu(X) = 1$ and $\mu \ll \lambda$.

Consider the measures λ_n defined by

$$\lambda_n(E) := \frac{1}{n}\sum_{k=0}^{n-1}\Lambda^{\lambda}_{k}(E), \quad E \in \mathcal{B}. \tag{3.14}$$

It can be shown that for all n, λ_n are normalized measures. Moreover, if $\lambda(E) = 0$, then

$$\begin{aligned}\lambda_n(E) &= \lambda(E) + p_1\lambda(\tau_1^{-1}(E)) + p_2\lambda(\tau_2^{-1}(E)) \\ &\quad + p_1^2\lambda(\tau_1^{-2}(E)) + p_1p_2\lambda(\tau_2^{-1}\tau_1^{-1}(E)) + p_1p_2\lambda(\tau_1^{-1}\tau_2^{-1}(E)) + p_2^2\lambda(\tau_2^{-2}(E)) \\ &\quad + \cdots + \sum_{(i_1,i_2,i_3,\dots,i_n)} p_{i_1}p_{i_2}\cdots p_{i_n}\lambda(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\cdots\tau_{i_n}^{-1}(E)) = 0,\end{aligned} \tag{3.15}$$

by nonsingularity of τ_1 and τ_2. Hence $\lambda_n \ll \lambda$. We imbed λ_n in the dual space $L_\infty(\lambda)^*$ of $L_\infty(\lambda)$ in the following way:

$$g_n(f) = \int_X f\,d\lambda_n, \quad f \in L_\infty(\lambda). \tag{3.16}$$

For every n,

$$|g_n(f)| = \left|\int_X f\,d\lambda_n\right| \le \|f\|_\infty \int_X d\lambda_n = \|f\|_\infty. \tag{3.17}$$

Hence, for each n, $\|g_n\| \le 1$. Thus, the λ_n can be thought of as elements of the unit ball of $L_\infty(\lambda)^*$. This unit ball is weak*-compact by Alaoglu's theorem [7]. Let ν be a cluster point in the weak*-topology of $L_\infty(\lambda)^*$ of the sequence $(\lambda_n)_{n\ge1}$.

Define a set function μ on $\mathcal{B}$ by

$$\mu(E) = \nu(\chi_E). \tag{3.18}$$

We claim that μ is finitely additive, bounded and it vanishes on sets of λ-measure zero: $\mu(\varnothing) = \nu(\chi_\varnothing) = \nu(0) = 0$, since ν is a linear functional. For any $E \in \mathcal{B}$,

$$\begin{aligned}\mu(E) = \nu(\chi_E) &= \lim_{s\to\infty} g_{n_s}(\chi_E) = \lim_{s\to\infty}\int_E d\lambda_{n_s} = \lim_{s\to\infty}\lambda_{n_s}(E) \\ &= \lim_{s\to\infty}\frac{1}{n_s}\sum_{k=0}^{n_s-1}\Lambda_k^\lambda(E) \ge 0,\end{aligned} \tag{3.19}$$

since Λ_k^λ is a measure. Thus,

$$0 \le \mu(E) \le \mu(X) = \lim_{s\to\infty}\lambda_{n_s}(X) = 1. \tag{3.20}$$

Now,

$$\begin{aligned}\mu\left(\bigcup_{i=1}^m E_i\right) &= \lim_{s\to\infty}\lambda_{n_s}\left(\bigcup_{i=1}^m E_i\right) = \lim_{s\to\infty}\sum_{i=1}^m \lambda_{n_s}(E_i) \\ &= \sum_{i=1}^m \lim_{s\to\infty}\lambda_{n_s}(E_i) = \sum_{i=1}^m \mu(E_i).\end{aligned} \tag{3.21}$$

Let $\lambda(E) = 0$. Then $\mu(E) = \lim_{s\to\infty}\lambda_{n_s}(E) = 0$, because $\lambda_{n_s} \ll \lambda$. Hence, μ is finitely additive, bounded, and it vanishes on sets of λ-measure zero.

μ is T-invariant:

$$\begin{aligned}\mu(E) &= \lim_{s\to\infty}\lambda_{n_s}(E) = \lim_{s\to\infty}\frac{1}{n_s}\sum_{k=0}^{n_s-1}\Lambda_k^{\lambda}(E)\\ &= \Lambda_0^{\lambda}(E)+\Lambda_1^{\lambda}(E)+\cdots+\Lambda_{n_s-1}^{\lambda}(E)\\ &= \lim_{s\to\infty}\frac{1}{n_s}\Bigg[\lambda(E)+p_1\lambda\left(\tau_1^{-1}(E)\right)+p_2\lambda\left(\tau_2^{-1}(E)\right)\\ &\qquad\qquad +\cdots+\sum_{(i_1,i_2,i_3,\ldots,i_{n_s-1})}p_{i_1}p_{i_2}\cdots p_{i_{n_s-1}}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\cdots\tau_{i_{n_s-1}}^{-1}(E)\right)\Bigg].\end{aligned} \tag{3.22}$$

On the other hand,

$$\begin{aligned}&\sum_{i=1}^{2}p_i\mu\left(\tau_i^{-1}(E)\right) = p_1\mu\left(\tau_1^{-1}(E)\right)+p_2\mu\left(\tau_2^{-1}(E)\right)\\ &= p_1\lim_{s\to\infty}\frac{1}{n_s}\sum_{k=0}^{n_s-1}\Lambda_k^{\lambda}\left(\tau_1^{-1}(E)\right)+p_2\lim_{s\to\infty}\frac{1}{n_s}\sum_{k=0}^{n_s-1}\Lambda_k^{\lambda}\left(\tau_2^{-1}(E)\right)\\ &= \lim_{s\to\infty}\frac{1}{n_s}\Bigg[p_1\Big\{\lambda\left(\tau_1^{-1}(E)\right)+p_1\lambda\left(\tau_1^{-2}(E)\right)+p_2\lambda\left(\tau_2^{-1}\tau_1^{-1}(E)\right)+\cdots\\ &\qquad +\sum_{(i_1,i_2,i_3,\ldots,i_{n_s-1})}p_{i_1}p_{i_2}\cdots p_{i_{n_s-1}}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\cdots\tau_{i_{n_s-1}}^{-1}\left(\tau_1^{-1}(E)\right)\right)\Big\}\\ &\qquad +p_2\Big\{\lambda\left(\tau_2^{-1}(E)\right)+p_1\lambda\left(\tau_1^{-1}\tau_2^{-1}(E)\right)+p_2\lambda\left(\tau_2^{-2}(E)\right)+\cdots\\ &\qquad +\sum_{(i_1,i_2,i_3,\ldots,i_{n_s-1})}p_{i_1}p_{i_2}\cdots p_{i_{n_s-1}}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\cdots\tau_{i_{n_s-1}}^{-1}\left(\tau_2^{-1}(E)\right)\right)\Big\}\Bigg]\\ &= \lim_{s\to\infty}\frac{1}{n_s}\Bigg[p_1\lambda\left(\tau_1^{-1}(E)\right)+p_1^2\lambda\left(\tau_1^{-2}(E)\right)+p_1p_2\lambda\left(\tau_2^{-1}\tau_1^{-1}(E)\right)+\cdots\\ &\qquad +p_1\sum_{(i_1,i_2,i_3,\ldots,i_{n_s-1})}p_{i_1}p_{i_2}\cdots p_{i_{n_s-1}}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\cdots\tau_{i_{n_s-1}}^{-1}\left(\tau_1^{-1}(E)\right)\right)\\ &\qquad +p_2\lambda\left(\tau_2^{-1}(E)\right)+p_2p_1\lambda\left(\tau_1^{-1}\tau_2^{-1}(E)\right)+p_2^2\lambda\left(\tau_2^{-2}(E)\right)+\cdots\\ &\qquad +p_2\sum_{(i_1,i_2,i_3,\ldots,i_{n_s-1})}p_{i_1}p_{i_2}\cdots p_{i_{n_s-1}}\lambda\left(\tau_{i_1}^{-1}\tau_{i_2}^{-1}\cdots\tau_{i_{n_s-1}}^{-1}\left(\tau_2^{-1}(E)\right)\right)\Bigg].\end{aligned} \tag{3.23}$$

Clearly,

$$\mu(E) = \sum_{i=1}^{2}p_i\mu\left(\tau_i^{-1}(E)\right). \tag{3.24}$$

Thus, we have shown that μ is a finitely additive T-invariant measure. By Theorem 2.3, μ has a unique representation

$$\mu = \mu_c + \mu_p, \tag{3.25}$$

where μ_c is countably additive and $\mu_c \geq 0$ and μ_p is purely additive and $\mu_p \geq 0$. We claim that $\mu_c \neq 0$. Suppose $\mu_c = 0$. Then by Theorem 2.5 there exists a decreasing sequence $\{E_n\}_{n\geq 1}$ of elements of $\mathcal{B}$ such that $\lim_{n\to\infty} \lambda(E_n) = 0$ and $\mu(E_n) = \mu(X) = 1$. Thus, there exists an integer n_0 such that for all $n \geq n_0$, $\lambda(E_n) < \delta$ and, as a consequence of our hypothesis, we have, for all k,

$$\Lambda_k^{\lambda}(E_n) < \alpha. \tag{3.26}$$

Hence

$$\lambda_k(E_n) < \alpha, \quad k = 1,2,3,\dots. \tag{3.27}$$

Thus, $\mu(E_n) = \lim_{s\to\infty} g_{n_s}(E_n) < \alpha < 1$, a contradiction. Now,

$$\begin{aligned}\mu(E) &= p_1\mu(\tau_1^{-1}(E)) + p_2\mu(\tau_2^{-1}(E)) \\ &= p_1\{\mu_c(\tau_1^{-1}(E)) + \mu_p(\tau_1^{-1}(E))\} + p_2\{\mu_c(\tau_2^{-1}(E)) + \mu_p(\tau_2^{-1}(E))\} \\ &= \{p_1\mu_c(\tau_1^{-1}(E)) + p_2\mu_c(\tau_2^{-1}(E))\} + \{p_1\mu_p(\tau_1^{-1}(E)) + p_2\mu_p(\tau_2^{-1}(E))\}.\end{aligned} \tag{3.28}$$

Clearly $m : \mathcal{B} \to R$, defined by

$$m(E) = p_1\mu_c(\tau_1^{-1}(E)) + p_2\mu_c(\tau_2^{-1}(E)), \tag{3.29}$$

is a countably additive measure, and $m \leq \mu$. Thus, by Lemma 2.4, we have $m \leq \mu_c$ and hence

$$E \mapsto \mu_c(E) - m(E) = \mu_c(E) - \{p_1\mu_c(\tau_1^{-1}(E)) + p_2\mu_c(\tau_2^{-1}(E))\} \tag{3.30}$$

is a positive measure. But this measure has total mass zero. Hence, it is a zero measure. Thus μ_c is T-invariant. Because μ vanishes on sets of λ-measure zero and $0 \leq \mu_c \leq \mu$, we have $\mu_c \ll \lambda$. Finally, $\gamma(E) = \mu_c(E)/\mu_c(X)$ is normalized, T-invariant, and absolutely continuous with respect to λ. □

We now state the analogous result for position-dependent random maps.

Theorem 3.4. *Let (X, B, λ) be a measure space with normalized measure λ and let $\tau_i : X \to X$, $i = 1,2$ be nonsingular transformations. Consider the random map $T = \{\tau_1, \tau_2; p_1, p_2\}$ with position-dependent probabilities p_1, p_2. Then there exists a normalized absolutely continuous (w.r.t. λ) T-invariant measure μ if and only if there exists $\delta > 0$ and $0 < \alpha < 1$ such* *that for any measurable set E and any positive integer k, $\lambda(E) < \delta$ implies*

$$\int_{\tau_1^{-1}(E)} p_1(x)d\lambda + \int_{\tau_2^{-1}(E)} p_2(x)d\lambda < \alpha;$$

$$\begin{aligned}&\int_{\tau_1^{-2}(E)} p_1(x)p_1(\tau_1(x))d\lambda + \int_{\tau_2^{-1}\tau_1^{-1}(E)} p_1(x)p_2(\tau_1(x))d\lambda \\ &\quad + \int_{\tau_1^{-1}\tau_2^{-1}(E)} p_2(x)p_1(\tau_2(x))d\lambda + \int_{\tau_2^{-2}(E)} p_2(x)p_2(\tau_2(x))d\lambda < \alpha;\end{aligned} \tag{3.31}$$

$$\vdots$$

$$\sum_{(i_1,i_2,i_3,\dots,i_k)} \int_{\tau_{i_1}^{-1}\tau_{i_2}^{-1}\cdots\tau_{i_k}^{-1}(E)} p_{i_1}(x)p_{i_2}(\tau_{i_1}(x))\cdots p_{i_k}(\tau_{i_1}\tau_{i_2}\cdots\tau_{i_{k-1}}(x))d\lambda < \alpha.$$

Proof. The proof is analogous to the proof of Theorem 3.1. □

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