

PROPERTIES OF FIXED POINT SET OF A MULTIVALUED MAP

ABDUL RAHIM KHAN

Received 15 September 2004 and in revised form 21 November 2004

Properties of the set of fixed points of some discontinuous multivalued maps in a strictly convex Banach space are studied; in particular, affirmative answers are provided to the questions related to set of fixed points and posed by Ko in 1972 and Xu and Beg in 1998. A result regarding the existence of best approximation is derived.

1. Introduction

The study of fixed points for multivalued contractions and nonexpansive maps using the Hausdorff metric was initiated by Markin [17]. Later, an interesting and rich fixed point theory for such maps has been developed. The theory of multivalued maps has applications in control theory, convex optimization, differential inclusions, and economics (see, e.g., [3, 8, 16, 22]).

The theory of multivalued nonexpansive mappings is harder than the corresponding theory of single-valued nonexpansive mappings. It is natural to expect that the theory of nonself-multivalued noncontinuous functions would be much more complicated.

The concept of a $*$ -nonexpansive multivalued map has been introduced and studied by Husain and Latif [9] which is a generalization of the usual notion of nonexpansiveness for single-valued maps. In general, $*$ -nonexpansive multivalued maps are neither nonexpansive nor continuous (see Example 3.7).

Xu [22] has established some fixed point theorems while Beg et al. [2] have recently studied the interplay between best approximation and fixed point results for $*$ -nonexpansive maps defined on certain subsets of a Hilbert space and Banach space. For this class of functions, approximating sequences to a fixed point in Hilbert spaces are constructed by Hussain and Khan [10] and its applications to random fixed points and best approximations in Fréchet spaces are given by Khan and Hussain [12].

In this paper, using the best approximation operator, we (i) establish certain properties of the set of fixed points of a $*$ -nonexpansive multivalued nonself-map in the setup of a strictly convex Banach space, (ii) prove fixed point results for $*$ -nonexpansive random maps in a Banach space under several boundary conditions, and (iii) provide affirmative answers to the questions posed by Ko [14] and Xu and Beg [24] related to the set of fixed points.

2. Notations and preliminaries

Let C be a subset of a normed space X . We denote by 2^X , $C(X)$, $K(X)$, $CC(X)$, $CK(X)$, and $CB(X)$ the families of all nonempty, nonempty closed, nonempty compact, nonempty closed convex, nonempty convex compact, and nonempty closed bounded subsets of X , respectively.

Define $d(x, C) = \inf_{y \in C} d(x, y)$. The Hausdorff metric on $CB(X)$ induced by the metric d on X is denoted by H and is defined as

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}. \tag{2.1}$$

A mapping $T : C \rightarrow CB(X)$ is a contraction if for any $x, y \in C$, $H(Tx, Ty) \leq kd(x, y)$, where $0 \leq k < 1$. If $k = 1$, then T is called a nonexpansive map. If $H(Tx, Ty) < d(x, y)$ whenever $x \neq y$ in C , then T is called a strictly nonexpansive mapping [14].

A multivalued map $T : C \rightarrow 2^X$ is said to be

- (i) $*$ -nonexpansive if for all $x, y \in C$ and $u_x \in Tx$ with $d(x, u_x) = d(x, Tx)$, there exists $u_y \in Ty$ with $d(y, u_y) = d(y, Ty)$ such that $d(u_x, u_y) \leq d(x, y)$ (see [9, 10]),
- (ii) strictly $*$ -nonexpansive if for all $x \neq y$ in C and $u_x \in Tx$ with $d(x, u_x) = d(x, Tx)$, there exists $u_y \in Ty$ with $d(y, u_y) = d(y, Ty)$ such that $d(u_x, u_y) < d(x, y)$,
- (iii) upper semicontinuous (usc) (lower semicontinuous (lsc)) if $T^{-1}(B) = \{x \in C : Tx \cap B \neq \emptyset\}$ is closed (open) for each closed (open) subset B of X . If T is both usc and lsc, then T is continuous,
- (iv) asymptotically contractive [19] if there exist some $c \in (0, 1)$ and $r > 0$ such that

$$\|y\| \leq c\|x\|, \quad \forall y \in Tx, \forall x \in C \setminus rB_X, \tag{2.2}$$

where B_X is the closed unit ball of X .

The map $T : C \rightarrow CB(X)$ is called (i) H -continuous (continuous with respect to Hausdorff metric H) if and only if for any sequence $\{x_n\}$ in C with $x_n \rightarrow x$, we have $H(Tx_n, Tx) \rightarrow 0$ (the two concepts of set-valued continuity are equivalent when T is compact-valued (cf. [8, Theorem 20.3, page 94])); (ii) demiclosed at 0 if the conditions $x_n \in C$, x_n converges weakly to x , $y_n \in Tx_n$ and $y_n \rightarrow 0$ imply that $0 \in Tx$. An element x in C is called a fixed point of a multivalued map T if and only if $x \in Tx$. The set of all fixed points of T will be denoted by $F(T)$.

For each $x \in X$, let $P_C(x) = \{z \in C : d(x, z) = d(x, C)\}$. Any $z \in P_C(x)$ is called a point of best approximation to x from C . If $P_C(x) \neq \emptyset$ (singleton) for each $x \in X$, then C is called a proximal (Chebyshev) set, respectively. If C is proximal, then the mapping $P_C : X \rightarrow 2^C$ is well defined and is called the metric projection.

The space X is said to have the Oshman property (see [18]) if it is reflexive and the metric projection on every closed convex subset is usc.

For the multivalued map T and each $x \in C$, we follow Xu [22] to define the best approximation operator, $P_T(x) = \{u_x \in Tx : d(x, u_x) = d(x, Tx)\}$, (possibly empty set).

A single-valued (multivalued) map $f : C \rightarrow X$ ($F : C \rightarrow 2^X$) is said to be a selector of T if $f(x) \in Tx$ ($Fx \subseteq Tx$), respectively, for each $x \in C$.

The space X is said to have the Opial condition if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\| \tag{2.3}$$

holds for all $y \neq x$.

Every Hilbert space and the spaces ℓ_p ($1 \leq p < \infty$) satisfy the Opial condition.

The inward set $I_C(x)$ of C at $x \in X$ is defined by $I_C(x) = \{x + \gamma(y - x) : y \in C \text{ and } \gamma > 0\}$. We will denote the closure of C by $cl(C)$.

Let (Ω, \mathcal{A}) denote a measurable space with \mathcal{A} sigma algebra of subsets of Ω . A mapping $T : \Omega \rightarrow 2^X$ is called measurable if for any open subset B of X , $T^{-1}(B) = \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$. A mapping $\zeta : \Omega \rightarrow X$ is said to be a measurable selector of a measurable mapping $T : \Omega \rightarrow 2^X$ if ζ is measurable and for any $\omega \in \Omega$, $\zeta(\omega) \in T(\omega)$. A mapping $T : \Omega \times C \rightarrow 2^X$ is a random operator if for any $x \in C$, $T(\cdot, x)$ is measurable. A mapping $\zeta : \Omega \rightarrow C$ is said to be a random fixed point of T if ζ is a measurable map such that for every $\omega \in \Omega$, $\zeta(\omega) \in T(\omega, \zeta(\omega))$.

A random operator $T : \Omega \times C \rightarrow 2^X$ is said to be continuous (nonexpansive, $*$ -nonexpansive, convex, etc.) if for each $\omega \in \Omega$, $T(\omega, \cdot)$ is continuous (nonexpansive, $*$ -nonexpansive, convex, etc.).

The following results are needed.

PROPOSITION 2.1 (see [3, Proposition 2.2]). *Let E be a metric space. If $T : \Omega \rightarrow C(E)$ is a multivalued mapping, then the following conditions are equivalent:*

- (i) T is measurable;
- (ii) $\omega \rightarrow d(x, T(\omega))$ is a measurable function of ω for each $x \in E$;
- (iii) there exists a sequence $\{f_n(\omega)\}$ of measurable selectors of T such that $cl\{f_n(\omega)\} = T(\omega)$ for all ω in Ω .

THEOREM 2.2 (see [24, Theorem 3.1]). *Let C be a nonempty separable weakly compact convex subset of a Banach space X . Suppose that the map $T : \Omega \times C \rightarrow K(C)$ is a nonexpansive random mapping. If for each $\omega \in \Omega$, $I - T(\omega, \cdot)$ is demiclosed at 0, then the fixed point set function F of T given by $F(\omega) = \{x \in C : x \in T(\omega, x)\}$ is measurable (and hence T has a random fixed point).*

3. $*$ -nonexpansive maps

The properties of the set of fixed points of single-valued and multivalued maps have been considered by a number of authors (see, e.g., Agarwal and O'Regan [1], Browder [4], Bruck [5], Espínola et al. [6], Ko [14], Schöneberg [20], and Xu and Beg [24]). For a wide class of unbounded closed convex sets C in a Banach space, there exist nonexpansive maps $T : C \rightarrow K(C)$ which fail to have a fixed point (see [13]).

We obtain some properties of the set of fixed points of a $*$ -nonexpansive map on a Banach space with values which are not necessarily subsets of the domain.

Markin [17], Xu [22], and Jachymski [11] have utilized “selections;” we employ “non-expansive selector,” P_T , of a $*$ -nonexpansive map T to study the structure of the set of

fixed points of T . Consequently, we obtain generalized and improved versions of many results in the current literature.

In Theorem 8.2, Browder [4] has established the following result.

THEOREM 3.1. *Let C be a nonempty closed, convex, subset of a strictly convex Banach space X and let $T : C \rightarrow C$ be a nonexpansive map. Then the set $F(T)$ of fixed points of T is closed and convex.*

Ko [14] pointed out that Theorem 3.1 need not hold for multivalued nonexpansive mappings as follows.

Example 3.2 (see [14, Example 3]). Consider $C = [0, 1] \times [0, 1]$ with the usual norm. Define $T : C \rightarrow CK(C)$ by

$$T(x, y) = \text{the triangle with vertices } (0, 0), (x, 0), \text{ and } (0, y). \quad (3.1)$$

Note that T is nonexpansive and the norm in \mathbb{R}^2 is strictly convex. But the set $F(T) = \{(x, y) : (x, y) \in C \text{ and } xy = 0\}$ is not convex.

The following generalization of Theorem 3.1 for $*$ -nonexpansive continuous mappings is obtained in [15].

THEOREM 3.3. *Let X be a strictly convex Banach space and C a nonempty weakly compact convex subset of X . Let $T : C \rightarrow CC(C)$ be a $*$ -nonexpansive map such that $F(T)$ is nonempty. Then the set $F(T)$ is convex and is closed if T is continuous.*

We present a new proof, through the best approximation operator, of Theorem 3.3 without assuming any type of continuity of the map T and obtain the following structure theorem.

THEOREM 3.4. *Let X be a strictly convex Banach space and C a nonempty weakly compact convex subset of X . Let $T : C \rightarrow CC(C)$ be a $*$ -nonexpansive map such that $F(T)$ is nonempty. Then the set $F(T)$ is closed and convex.*

Proof. For each $x \in C$, its image Tx is weakly compact and convex and thus each Tx is Chebyshev. Hence, each u_x in $P_T(x)$ is unique. Thus by the definition of $*$ -nonexpansiveness of T , there is $u_y = P_T(y) \in Ty$ for all y in C such that

$$\|P_T(x) - P_T(y)\| = \|u_x - u_y\| \leq \|x - y\|. \quad (3.2)$$

Hence, $P_T : C \rightarrow C$ is a nonexpansive selector of T (see also [22]). By the definition of P_T , we have for each $y \in C$,

$$d(y, P_T(y)) = d(y, u_y) = d(y, Ty). \quad (3.3)$$

Equation (3.3) now implies that $F(T) = F(P_T)$. Thus $F(P_T)$ and hence $F(T)$ is closed and convex by Theorem 3.1. \square

The following example illustrates our results.

Example 3.5. Let $T : [0, 1] \rightarrow 2^{[0,1]}$ be a multivalued map defined by

$$Tx = \begin{cases} \left[0, \frac{1}{2}\right], & x \neq \frac{1}{2}, \\ [0, 1], & x = \frac{1}{2}. \end{cases} \tag{3.4}$$

Then

$$P_T(x) = \begin{cases} x, & x \in \left[0, \frac{1}{2}\right], \\ \frac{1}{2}, & x \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{3.5}$$

This implies that T is a $*$ -nonexpansive map. Further, T is usc but not lsc (see [8, Remark 15.2, page 71]) and hence T is not continuous according to both definitions as T is compact-valued. Note that $F(T) = [0, 1/2]$ is closed and convex.

If T is a single-valued strictly nonexpansive map, then $F(T)$ is a singleton. In general, this is not true for a multivalued nonexpansive map [17]. The set $F(T)$ is said to be singleton in a generalized sense if there exists $x \in F(T)$ such that $F(T) \subseteq Tx$. Ko has given an example of a strictly nonexpansive mapping $T : C \rightarrow CC(C)$, in a strictly convex Banach space, for which the set $F(T)$ is not singleton in a generalized sense (cf. [17, Example 4]). Ko raised the following question: is $F(T)$ singleton in a generalized sense if T is nonexpansive, I is the identity operator, and $I - T$ is convex?

The following proposition provides an affirmative answer to this question for strictly $*$ -nonexpansive multivalued mappings.

PROPOSITION 3.6. *Let C be a nonempty closed convex subset of a reflexive strictly convex Banach space X and let $T : C \rightarrow CC(C)$ be a strictly $*$ -nonexpansive map such that $F(T)$ is nonempty. Then the set $F(T)$ is singleton in a generalized sense.*

Proof. Any closed convex subset of a reflexive strictly convex Banach space is Chebyshev, so each Tx is Chebyshev. Thus as in the proof of Theorem 3.4, $P_T : C \rightarrow C$ is a strictly nonexpansive selector of T satisfying (3.3). Hence, $F(T) = F(P_T)$ is singleton in a generalized sense as required. □

The following example supports the above proposition.

Example 3.7. Let $T : [0, 1] \rightarrow 2^{[0,1]}$ be a multivalued map defined by

$$Tx = \begin{cases} \left\{\frac{1}{2}\right\}, & x \in \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right], \\ \left[\frac{1}{4}, \frac{3}{4}\right], & x = \frac{1}{2}. \end{cases} \tag{3.6}$$

Then $P_T(x) = \{1/2\}$ for every $x \in [0, 1]$. This implies that T is a strictly $*$ -nonexpansive map:

$$\begin{aligned} H\left(T\left(\frac{1}{3}\right), T\left(\frac{1}{2}\right)\right) &= H\left(\left\{\frac{1}{2}\right\}, \left[\frac{1}{4}, \frac{3}{4}\right]\right) \\ &= \max \left\{ \sup_{a \in \{1/2\}} d\left(a, \left[\frac{1}{4}, \frac{3}{4}\right]\right), \sup_{b \in [1/4, 3/4]} d\left(b, \frac{1}{2}\right) \right\} \quad (3.7) \\ &= \max \left\{ 0, \frac{1}{4} \right\} = \frac{1}{4} > \frac{1}{6} = \left| \frac{1}{3} - \frac{1}{2} \right|. \end{aligned}$$

This implies that T is not nonexpansive. Obviously, T is compact-valued. Next we show that T is not lsc.

Let $V_{1/4}$ be any small open neighbourhood of $1/4$. Then the set

$$T^{-1}(V_{1/4}) = \{x \in [0, 1] : Tx \cap V_{1/4} \neq \emptyset\} = \left\{ \frac{1}{2} \right\} \quad (3.8)$$

is not open. Thus T is not continuous in the sense of both definitions.

Note that $F(T) = \{1/2\}$ is singleton in a generalized sense.

The conclusion of Proposition 3.6 does not hold for $*$ -nonexpansive maps as follows.

Example 3.8. Let $C = [0, \infty)$ and $T : C \rightarrow CK(C)$ be defined by

$$Tx = [x, 2x] \quad \text{for } x \in C. \quad (3.9)$$

Then $P_T(x) = \{x\}$ for every $x \in C$. This clearly implies that T is $*$ -nonexpansive but not nonexpansive (cf. [22]). Note that $F(T) = C$ and there does not exist any x in $F(T)$ such that $F(T) \subseteq Tx$. Thus $F(T)$ is not singleton in a generalized sense.

The above example also indicates that the fixed point set of a $*$ -nonexpansive map need not be bounded in general. However, if T is asymptotically contractive, then we have the following affirmative result.

THEOREM 3.9. *Let X be a uniformly convex Banach space and C a nonempty closed convex subset of X . Let $T : C \rightarrow CC(C)$ be a $*$ -nonexpansive map which is asymptotically contractive on C . Then $F(T)$ is nonempty closed, convex, and bounded.*

Proof. The map T has a nonexpansive selector f which is also asymptotically contractive by the asymptotic contractivity of T . Further, $F(T) = F(f)$ is nonempty closed bounded and convex (see [19, Corollary 3 and Remark (a)]). □

We are now ready to derive a version of the Ky-Fan best approximation theorem [7] (compare the result with [10, Theorem 3.1] and [21, Theorem 4.3]).

THEOREM 3.10. *Let C be a nonempty closed convex subset of a strictly convex Banach space X with the Oshman property. If $T : C \rightarrow CC(X)$ is an H -continuous (or a $*$ -nonexpansive) map and $T(C)$ is relatively compact, then there exists $y \in C$ such that*

$$d(y, Ty) = \|y - fy\| = d(fy, \text{cl}(I_C(y))), \quad \text{for some continuous selector } f \text{ of } T. \quad (3.10)$$

Proof. The Hausdorff continuity of T implies that $f = P_T : C \rightarrow X$ is a continuous selector of T . Since $T(C)$ is relatively compact and $f(C) \subseteq T(C)$, therefore $f(C)$ is relatively compact. By [18, Theorem 3(0)] and (3.3), we obtain

$$d(y, Ty) = d(y, fy) = d(fy, \text{cl}(I_C(y))), \quad \text{for some } y \in C. \tag{3.11}$$

The proof for $*$ -nonexpansive map is similar. □

As an application of Theorems 3.4 and 3.10, we obtain the following extension of [9, Theorem 3.2], [22, Corollary 1], and Theorem 3.3.

COROLLARY 3.11. *Let C be a nonempty closed convex subset of a strictly convex Banach space X with the Oshman property. If $T : C \rightarrow CC(C)$ is a $*$ -nonexpansive map and $T(C)$ is relatively compact, then $F(T)$ is nonempty closed and convex.*

Remark 3.12. The map T in Example 3.2 is neither $*$ -nonexpansive nor has a nonexpansive selection f with $F(f) = F(T)$; for if T is so, then using the same argument as in the proof of Theorem 3.4, T should have a nonexpansive selector $P_T : C \rightarrow C$ such that $F(T) = F(P_T)$, which should be convex by Theorem 3.1; a contradiction.

Xu [23] obtained the randomization of a remarkable fixed point theorem for multivalued nonexpansive maps due to Lim [16]. Further, Xu and Beg stated that it is unknown whether the fixed point set function F in this case is measurable (see [24, page 69]). We prove that the fixed point set function F is measurable if the underlying map is $*$ -nonexpansive.

THEOREM 3.13. *Let C be a nonempty separable closed bounded convex subset of a Banach space X and $T : \Omega \times C \rightarrow K(C)$ a $*$ -nonexpansive random operator. Then the fixed point set function F of T given by $F(\omega) = \{x \in C : x \in T(\omega, x)\}$ is measurable (and hence T has a random fixed point) provided one of the following conditions holds:*

- (i) $T(\omega, \cdot)$ is convex for each $\omega \in \Omega$ and X is a uniformly convex space,
- (ii) C is weakly compact and $I - T$ is demiclosed at 0,
- (iii) C is weakly compact and X satisfies the Opial condition.

Proof. (i) As before, for each $\omega \in \Omega$, $P_T(\omega, \cdot) : \Omega \times C \rightarrow C$ is a nonexpansive selector of $T(\omega, \cdot)$ and for each $y \in C$, $\omega \in \Omega$,

$$d(y, P_T(\omega, y)) = d(y, u_y) = d(y, T(\omega, y)). \tag{3.12}$$

By Proposition 2.1, $T(\cdot, x)$ is measurable if and only if for each x in C , the function $d(x, T(\cdot, x))$ is measurable. Thus by (3.12), for each x in C , $d(x, P_T(\cdot, x))$ is measurable and hence again by Proposition 2.1, $P_T(\cdot, x)$ is measurable (see also [12, Proposition 3.6]). Thus $P_T : \Omega \times C \rightarrow C$ is a nonexpansive random operator.

We observe that if X is a uniformly convex space, then $I - P_T(\omega, \cdot)$ is demiclosed. Also (3.12) implies that fixed point set function G of P_T given by $G(\omega) = \{x \in C : x = P_T(\omega, x)\}$ is equal to $F(\omega) = \{x \in C : x \in T(\omega, x)\}$ for each $\omega \in \Omega$. Consequently, G , and hence F , is measurable by Theorem 2.2.

(ii) Note that $P_T(\omega, \cdot) : C \rightarrow K(C)$ is a nonexpansive selector of $T(\omega, \cdot)$. Also for each $y \in C$, $\omega \in \Omega$,

$$d(y, P_T(\omega, y)) \leq d(y, u_y) = d(y, T(\omega, y)) \leq d(y, P_T(\omega, y)). \quad (3.13)$$

The measurability of P_T follows from the arguments adopted in part (i) using (3.13) instead of (3.12). The demiclosedness of $I - T(\omega, \cdot)$ at 0 implies that $I - P_T(\omega, \cdot)$ is also demiclosed at 0 as follows.

Suppose that $x_n \rightarrow x_0$ weakly and $y_n \in I - P_T(\omega, x_n)$ with $y_n \rightarrow 0$ strongly. Note that $y_n \in I - P_T(\omega, x_n) \subseteq I - T(\omega, x_n)$ and $I - T(\omega, \cdot)$ is demiclosed at 0 so $0 \in I - T(\omega, x_0)$ for each $\omega \in \Omega$. This implies that $x_0 \in T(\omega, x_0)$ and hence $0 = d(x_0, T(\omega, x_0)) = d(x_0, P_T(\omega, x_0))$ for each $\omega \in \Omega$. Thus $x_0 \in P_T(\omega, x_0)$ implies that $I - P_T(\omega, \cdot)$ is demiclosed at 0 for each $\omega \in \Omega$. Thus G , and hence, F is measurable by Theorem 2.2.

(iii) It is well known that if C is a weakly compact subset of a Banach space X satisfying the Opial condition and $f : C \rightarrow K(C)$ is nonexpansive, then $I - f$ is demiclosed on C . Hence, $I - P_T(\omega, \cdot)$ is demiclosed for each $\omega \in \Omega$ and the conclusion now follows from part (ii). \square

Remark 3.14. It is not common at all that a nonexpansive multivalued mapping admits a single-valued nonexpansive selection (cf. Example 3.2 and Remark 3.12). However, in the general setup of metric linear spaces, $*$ -nonexpansive maps have nonexpansive selector satisfying a very useful relation (3.3).

Acknowledgment

The author gratefully acknowledges the support provided by King Fahd University of Petroleum & Minerals during this research.

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Abdul Rahim Khan: Department of Mathematical Sciences, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia
E-mail address: arahim@kfupm.edu.sa



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