APPROXIMATING FIXED POINTS OF NON-SELF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Received 23 March 2006; Revised 9 June 2006; Accepted 10 July 2006

Suppose K is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retraction. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T)$ is nonempty, where $F(T)$ denotes the fixed points set of T. Let $\{\alpha_n\}$, $\{\alpha_n'\}$, and $\{\alpha_n''\}$ be real sequences in (0,1) and $\epsilon \le \alpha_n, \alpha_n', \alpha_n'' \le 1-\epsilon$ for all $n \in \mathbb{N}$ and some $\epsilon > 0$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by $x_1 \in K$, $z_n = P(\alpha_n'' T(PT)^{n-1} x_n + (1-\alpha_n'')x_n)$, $y_n = P(\alpha_n' T(PT)^{n-1} z_n + (1-\alpha_n')x_n)$, $x_{n+1} = P(\alpha_n T(PT)^{n-1} y_n + (1-\alpha_n)x_n)$. (i) If the dual E^* of E has the *Kadec-Klee* property, then $\{x_n\}$ converges weakly to a fixed point $p \in F(T)$; (ii) if T satisfies condition (A), then $\{x_n\}$ converges strongly to a fixed point $p \in F(T)$.

1. Introduction and preliminaries

Let E be a real Banach space, let K be a nonempty subset of X and $F(T)$ denote the set of fixed points of T. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \quad \forall x, y \in K. \tag{1.1}$$

This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4]. They proved that if K is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of K has a fixed point. Moreover, the fixed points set $F(T)$ of T is closed and convex.

Many authors have contributed their efforts to investigate the problem of finding a fixed point of asymptotically nonexpansive mapping. In [8, 9], Schu introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive

Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2006, Article ID 21961, Pages 1–13
DOI 10.1155/JAMSA/2006/21961

self-maps defined on nonempty closed convex and bounded subsets of a Hilbert space H. More precisely, he proved the following theorems.

Theorem 1.1 (see [8]). *Let H be a Hilbert space, K a nonempty closed convex and bounded subset of H, and let $T : K \to K$ be a completely continuous asymptotically nonexpansive with sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $[0,1]$ satisfying the condition $\epsilon \le \alpha_n \le 1-\epsilon$ for all $n \ge 1$ and for some $\epsilon > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \ge 1, \tag{1.2}$$

converges strongly to a fixed point of T.

Theorem 1.2 (see [9]). *Let E be a uniformly convex Banach space satisfying Opial's condition, K a nonempty closed convex and bounded subset of E, and $T : K \to K$ an asymptotically nonexpansive with sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $[0,1]$ satisfying the condition $0 < a \le \alpha_n \le b < 1$ for all $n \ge 1$ and some $a, b \in (0,1)$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \ge 1, \tag{1.3}$$

converges weakly to a fixed point of T.

Subsequently, Tan and Xu [13] first proved that Schu's theorem remains true if the assumption that E satisfies *Opial's* condition is replaced by the one that E has a *Fréchet* differential norm. Meantime, Tan and Xu [13] proved the weak convergence of the modified Ishikawa iterative scheme in a uniformly convex Banach space which either satisfies *Opial's* condition or has a *Fréchet* differential norm. In [7], Rhoades extended [8, Theorem 1.1] to uniformly convex Banach space using a modified Ishikawa iteration method. In [6], Osilike and Aniagbosor proved that the theorems of Schu and Rhoades remain true without the boundedness condition imposed on K, provided that $F(T) = \{x \in K : Tx = x\} \neq \emptyset$.

In [12], Tan and Xu introduced a modified Ishikawa process to approximate fixed points of nonexpansive mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space E. More precisely, they proved the following theorem.

Theorem 1.3 (see [12]). *Let E be a uniformly convex Banach space which satisfies Opial's condition or has a* Fréchet *differentiable norm and C a nonempty closed convex bounded subset of E, $T : C \to C$ a nonexpansive mapping and let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty}\beta_n(1-\alpha_n) = \infty$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n T x_n], \quad n \ge 1, \tag{1.4}$$

converges weakly to a fixed point of T.

In the above results, T remains a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space if, however, the domain K of T is a proper subset

of E (and this is the case in several applications), and T maps K into E, then iteration processes of Mann and Ishikawa studied by these authors may fail to be well defined.

Chidume [1] studied the iteration scheme defined by

$$x_1 \in K, \quad x_{n+1} = P\big((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n\big), \quad n \geq 1, \tag{1.5}$$

in the framework of uniformly convex Banach space, where K is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retraction. $T : K \to E$ is an asymptotically nonexpansive nonselfmap with sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$. $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0,1]$ satisfying the condition $\epsilon \leq \alpha_n \leq 1-\epsilon$ for all $n \geq 1$ and for some $\epsilon > 0$. They proved strong and weak convergence theorems for asymptotically nonexpansive nonselfmaps.

Recently, Shahzad [11] studied the sequence $\{x_n\}$ defined by

$$x_1 \in K, \quad x_{n+1} = P\big((1-\alpha_n)x_n + \alpha_n TP\big[(1-\beta_n)x_n + \beta_n Tx_n\big]\big), \tag{1.6}$$

where K is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retraction. He proved weak and strong convergence theorems for nonself nonexpansive mappings in Banach spaces.

Motivated by Chidume [1], Shahzad [11], and some others, the purpose of this paper is to construct an iterative scheme for approximating a fixed point of asymptotically nonexpansive nonself maps (when such a fixed point exists) and to prove some strong and weak convergence theorems for such maps.

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. In this paper, the following iteration scheme is studied:

$$\begin{aligned} x_1 &\in K, \\ z_n &= P\big(\alpha_n'' T(PT)^{n-1}x_n + (1-\alpha_n'')x_n\big), \\ y_n &= P\big(\alpha_n' T(PT)^{n-1}z_n + (1-\alpha_n')x_n\big), \\ x_{n+1} &= P\big(\alpha_n T(PT)^{n-1}y_n + (1-\alpha_n)x_n\big), \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, and $\{\alpha_n''\}$ are real sequences in $(0,1)$.

Our theorems improve and generalize some previous results. Our weak convergence result applies not only to L^p-spaces with $1 < p < \infty$ but also to other spaces which do not satisfy *Opial's* condition or have a *Fréchet* differentiable norm. More precisely, we prove weak convergence of the modified Noor-type iteration process (Noor-type iteration process was introduced by Xu and Noor [14]) in a uniformly convex Banach space whose dual has the *Kadec-Klee* property. It is worth mentioning that there are uniformly convex Banach spaces, which have neither a *Fréchet* differentiable norm nor *Opial's* property; however their dual does have the *Kadec-Klee* property (see, e.g., [3, 5]).

Let E be a real Banach space. A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to E$ such that $Px = x$ for all $x \in K$. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P. A set K is optimal if each point outside K can be moved to be closer to all

points of K. It is well known (see, e.g., [2]) that

(i) if E is a separable, strictly convex, smooth, reflexive Banach space, and if $K \subset E$ is an optimal set with interior, then K is a nonexpansive retract of E;

(ii) a subset of l^p, with $1 < p < \infty$, is a nonexpansive retract if and only if it is optimal.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. However, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

A Banach space E is said to have the *Kadec-Klee* property if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ strongly together imply $\|x_n - x\| \to 0$.

Recall that the mapping $T : K \to E$ with $F(T) \neq \emptyset$, where K is a subset of E, is said to satisfy [10, condition (A)] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in K$,

$$\|x - Tx\| \geq f\big(d(x,F(T))\big), \tag{1.8}$$

where $d(x,F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

In order to prove our main results, we will make use of the following lemmas.

Lemma 1.4 (Schu [9]). *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\begin{gathered} \limsup_{n\to\infty} \|x_n\| \leq r, \qquad \limsup_{n\to\infty} \|y_n\| \leq r, \\ \lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r \end{gathered} \tag{1.9}$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.5 (demiclosed principle for nonselfmap [1]). *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1,\infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed with respect to zero.*

Lemma 1.6 (see [3]). *Let E be a real reflexive Banach space such that its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $x^*, y^* \in w_w(x_n)$; here $w_w(x_n)$ denotes the weak w-limit set of $\{x_n\}$. Suppose $\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$ exists for all $t \in [0,1]$. Then $x^* = y^*$.*

Lemma 1.7 (Tan and Xu [12]). *Let $\{r_n\}$, $\{s_n\}$, and $\{t_n\}$ be three nonnegative sequences satisfying the following conditions:*

$$r_{n+1} \leq (1 + s_n) r_n + t_n \quad \forall n \geq 1. \tag{1.10}$$

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

2. Main results

Definition 2.1 (see [1]). Let E be a real-normed linear space, K a nonempty subset of E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A map $T : K \to E$ is said to be an asymptotically nonexpansive mapping if there exists a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$ such that the following inequality holds:

$$||T(PT)^{n-1}x - T(PT)^{n-1}y|| \le k_n\|x - y\| \quad \forall x, y \in K,\ n \ge 1. \tag{2.1}$$

T is called uniformly L-Lipschitzian if there exists $L > 0$ such that

$$||T(PT)^{n-1}x - T(PT)^{n-1}y|| \le L\|x - y\| \quad \forall x, y \in K,\ n \ge 1. \tag{2.2}$$

Lemma 2.2. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.7). Then $\lim_{n\to\infty} \|x_n - p\|$ exists, for any $p \in F(T)$, where $F(T)$ denotes the nonempty fixed points set of T.*

Proof. For any given $p \in F(T)$, it follows from (1.7) that

$$\begin{aligned}
||z_n - p|| &= ||P(\alpha_n'' T(PT)^{n-1}x_n + (1 - \alpha_n'')x_n) - p|| \\
&\le ||\alpha_n'' T(PT)^{n-1}x_n + (1 - \alpha_n'')x_n - p|| \\
&\le \alpha_n''||T(PT)^{n-1}x_n - p|| + (1 - \alpha_n'')||x_n - p|| \\
&\le \alpha_n'' k_n||x_n - p|| + (1 - \alpha_n'')||x_n - p|| \\
&\le k_n||x_n - p||.
\end{aligned} \tag{2.3}$$

That is,

$$||z_n - p|| \le k_n||x_n - p||. \tag{2.4}$$

From (1.7) and (2.4) we get

$$\begin{aligned}
||y_n - p|| &= ||P(\alpha_n' T(PT)^{n-1}z_n + (1 - \alpha_n')x_n) - p|| \\
&\le ||\alpha_n' T(PT)^{n-1}z_n + (1 - \alpha_n')x_n - p|| \\
&\le \alpha_n'||T(PT)^{n-1}z_n - p|| + (1 - \alpha_n')||x_n - p|| \\
&\le \alpha_n' k_n||z_n - p|| + (1 - \alpha_n')||x_n - p|| \\
&\le k_n^2||x_n - p||.
\end{aligned} \tag{2.5}$$

That is,

$$||y_n - p|| \le k_n^2 ||x_n - p||. \tag{2.6}$$

Again, from (1.7) and (2.6) we have

$$\begin{aligned}
||x_{n+1} - p|| &= ||P(\alpha_n T(PT)^{n-1} y_n + (1-\alpha_n)x_n) - p|| \\
&= ||\alpha_n T(PT)^{n-1} y_n + (1-\alpha_n)x_n - p|| \\
&\le \alpha_n ||T(PT)^{n-1} y_n - p|| + (1-\alpha_n)||x_n - p|| \\
&\le \alpha_n k_n ||y_n - p|| + (1-\alpha_n)||x_n - p|| \\
&\le \alpha_n k_n^3 ||x_n - p|| + (1-\alpha_n)||x_n - p|| \\
&\le k_n^3 ||x_n - p||.
\end{aligned} \tag{2.7}$$

That is,

$$||x_{n+1} - p|| \le (1 + (k_n^3 - 1))||x_n - p||. \tag{2.8}$$

Note that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ is equivalent to $\sum_{n=1}^{\infty}(k_n^3 - 1) < \infty$, therefore by Lemma 1.7, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. This completes the proof. □

Lemma 2.3. *Let E be a normed linear space, K a nonempty closed convex subset which is also a nonexpansive retract of E, $T : K \to E$ a uniformly L-Lipschitzian, starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.7) and set $C_n = \|x_n - T(PT)^{n-1}x_n\|$ for all $n \ge 1$. If $\lim_{n\to\infty} C_n = 0$, then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. It follows from (1.7) that

$$\begin{aligned}
||x_{n+1} - x_n|| &\le ||\alpha_n T(PT)^{n-1} y_n + (1-\alpha_n)x_n - x_n|| \\
&\le ||T(PT)^{n-1} y_n - x_n|| \\
&\le ||T(PT)^{n-1} x_n - x_n|| + ||T(PT)^{n-1} y_n - T(PT)^{n-1} x_n|| \\
&\le C_n + L||y_n - x_n|| \\
&\le C_n + L||\alpha_n' T(PT)^{n-1} z_n + (1-\alpha_n')x_n - x_n|| \\
&\le C_n + L||T(PT)^{n-1} z_n - x_n|| \\
&\le C_n + L||T(PT)^{n-1} x_n - x_n|| + L||T(PT)^{n-1} z_n - T(PT)^{n-1} x_n|| \\
&\le C_n + LC_n + L^2||z_n - x_n|| \\
&\le C_n + LC_n + L^2||\alpha_n'' T(PT)^{n-1} x_n + (1-\alpha_n'')x_n - x_n|| \\
&\le C_n(1 + L + L^2),
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\|y_{n-1}-x_n\| &\le \|\alpha'_{n-1}T(PT)^{n-2}z_{n-1}+(1-\alpha'_{n-1})x_{n-1}-x_n\| \\
&\le \|T(PT)^{n-2}z_{n-1}-x_{n-1}\|+\|x_{n-1}-x_n\| \\
&\le \|T(PT)^{n-2}x_{n-1}-x_{n-1}\|+\|T(PT)^{n-2}z_{n-1}-T(PT)^{n-2}x_{n-1}\|+\|x_{n-1}-x_n\| \\
&\le C_{n-1}+LC_{n-1}+\|x_{n-1}-x_n\|.
\end{aligned} \tag{2.10}$$

Substituting (2.9) into (2.10) we obtain

$$\|y_{n-1}-x_n\| \le C_{n-1}(2+2L+L^2). \tag{2.11}$$

On the other hand, from (2.9) and (2.11) we have

$$\begin{aligned}
\|x_n-(PT)^{n-1}x_n\| &\le \|\alpha_{n-1}T(PT)^{n-2}y_{n-1}+(1-\alpha_{n-1})x_{n-1}-T(PT)^{n-2}x_n\| \\
&\le \|T(PT)^{n-2}y_{n-1}-T(PT)^{n-2}x_n\|+\|x_{n-1}-T(PT)^{n-2}x_n\| \\
&\le L\|y_{n-1}-x_n\|+\|x_{n-1}-T(PT)^{n-2}x_{n-1}\| \\
&\quad +\|T(PT)^{n-2}x_{n-1}-T(PT)^{n-2}x_n\| \\
&\le L\|y_{n-1}-x_n\|+C_{n-1}+L\|x_{n-1}-x_n\| \\
&\le LC_{n-1}(3+3L+2L^2)+C_{n-1}.
\end{aligned} \tag{2.12}$$

It follows from (2.12) that

$$\begin{aligned}
\|x_n-Tx_n\| &\le \|x_n-T(PT)^{n-1}x_n\|+\|T(PT)^{n-1}x_n-Tx_n\| \\
&\le C_n+L\|(PT)^{n-1}x_n-x_n\| \\
&\le C_n+LC_{n-1}(1+3L+3L^2+2L^3).
\end{aligned} \tag{2.13}$$

It follows from $\lim_{n\to\infty} C_n = 0$ that

$$\lim_{n\to\infty}\|x_n-Tx_n\| = 0. \tag{2.14}$$

This completes the proof. □

Theorem 2.4. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\alpha'_n\}$, and $\{\alpha''_n\}$ be real sequences in $(0,1)$ and $\epsilon \le \alpha_n, \alpha'_n, \alpha''_n \le 1-\epsilon$ for all $n \in \mathbb{N}$ and some $\epsilon > 0$, starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.7). Then* $\lim_{n\to\infty}\|x_n-Tx_n\| = 0$.

Proof. Take $p \in F(T)$, by Lemma 2.2 we know $\lim_{n\to\infty}\|x_n-p\|$ exists. Let $\lim_{n\to\infty}\|x_n-p\| = c$. If $c = 0$, then by the continuity of T the conclusion follows. Now suppose $c > 0$. We claim $\lim_{n\to\infty}\|Tx_n-x_n\| = 0$. Taking limsup on both sides in the inequality (2.4), we

have

$$\limsup_{n\to\infty} \|z_n - p\| \le c. \tag{2.15}$$

Similarly, taking limsup on both sides in the inequality (2.6), we have

$$\limsup_{n\to\infty} \|y_n - p\| \le c. \tag{2.16}$$

Next, we consider

$$\|T(PT)^{n-1} y_n - p\| \le k_n \|y_n - p\|. \tag{2.17}$$

Taking lim sup on both sides in the above inequality and using (2.16), we get

$$\limsup_{n\to\infty} \|T(PT)^{n-1} y_n - p\| \le c. \tag{2.18}$$

Again, $\lim_{n\to\infty} \|x_{n+1} - p\| = c$ means that

$$\liminf_{n\to\infty} \|\alpha_n (T(PT)^{n-1} y_n - p) + (1-\alpha_n)(x_n - p)\| \ge c. \tag{2.19}$$

On the other hand, we have

$$\begin{aligned}
&\|\alpha_n (T(PT)^{n-1} y_n - p) + (1-\alpha_n)(x_n - p)\| \\
&\quad \le \alpha_n \|T(PT)^{n-1} y_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\quad \le \alpha_n k_n \|y_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\quad \le \alpha_n k_n^3 \|x_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\quad \le k_n^3 \|x_n - p\|.
\end{aligned} \tag{2.20}$$

Therefore, we obtain

$$\limsup_{n\to\infty} \|\alpha_n (T(PT)^{n-1} y_n - p) + (1-\alpha_n)(x_n - p)\| \le c. \tag{2.21}$$

Combining (2.19) and (2.21), we obtain

$$\lim_{n\to\infty} \|\alpha_n (T(PT)^{n-1} y_n - p) + (1-\alpha_n)(x_n - p)\| = c. \tag{2.22}$$

Hence applying Lemma 1.4, we have

$$\lim_{n\to\infty} \|T(PT)^{n-1} y_n - x_n\| = 0. \tag{2.23}$$

Next, it follows from

$$\begin{aligned}
\|x_n - p\| &\le \|T(PT)^{n-1} y_n - x_n\| + \|T(PT)^{n-1} y_n - p\| \\
&\le \|T(PT)^{n-1} y_n - x_n\| + k_n \|y_n - p\|
\end{aligned} \tag{2.24}$$

that we have

$$c \le \liminf_{n\to\infty} \|y_n - p\| \le \limsup_{n\to\infty} \|y_n - p\| \le c. \tag{2.25}$$

That is,

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{2.26}$$

Again, $\lim_{n\to\infty} \|y_n - p\| = c$ gives that

$$\liminf_{n\to\infty} \|\alpha_n'(T(PT)^{n-1}z_n - p) + (1-\alpha_n')(x_n - p)\| \ge c. \tag{2.27}$$

Similarly,

$$\begin{aligned}
&\|\alpha_n'(T(PT)^{n-1}z_n - p) + (1-\alpha_n')(x_n - p)\| \\
&\quad \le \alpha_n'\|T(PT)^{n-1}z_n - p\| + (1-\alpha_n')\|x_n - p\| \\
&\quad \le \alpha_n' k_n\|z_n - p\| + (1-\alpha_n')\|x_n - p\| \\
&\quad \le \alpha_n' k_n^2\|x_n - p\| + (1-\alpha_n')\|x_n - p\| \\
&\quad \le k_n^2\|x_n - p\|.
\end{aligned} \tag{2.28}$$

Therefore, we have

$$\limsup_{n\to\infty} \|\alpha_n'(T(PT)^{n-1}z_n - p) + (1-\alpha_n')(x_n - p)\| \le c. \tag{2.29}$$

Combining (2.27) and (2.29) yields

$$\lim_{n\to\infty} \|\alpha_n'(T(PT)^{n-1}z_n - p) + (1-\alpha_n')(x_n - p)\| = c. \tag{2.30}$$

On the other hand, we have

$$\|T(PT)^{n-1}z_n - p\| \le k_n\|z_n - p\|. \tag{2.31}$$

Taking lim sup on both sides in the above inequality and using (2.15), we have

$$\limsup_{n\to\infty} \|T(PT)^{n-1}z_n - p\| \le c. \tag{2.32}$$

Applying Lemma 1.4, it follows from (2.30) and (2.32) that

$$\lim_{n\to\infty} \|T(PT)^{n-1}z_n - x_n\| = 0. \tag{2.33}$$

Notice that

$$\begin{aligned} \|x_n - p\| &\le \|T(PT)^{n-1}z_n - x_n\| + \|T(PT)^{n-1}z_n - p\| \\ &\le \|T(PT)^{n-1}z_n - x_n\| + k_n\|z_n - p\| \end{aligned} \tag{2.34}$$

which yields

$$c \le \liminf_{n\to\infty} \|z_n - p\| \le \limsup_{n\to\infty} \|z_n - p\| \le c. \tag{2.35}$$

That is,

$$\lim_{n\to\infty} \|z_n - p\| = c. \tag{2.36}$$

Using the same method, we have

$$\lim_{n\to\infty} \|\alpha_n''(T(PT)^{n-1}x_n - p) + (1-\alpha_n'')(x_n - p)\| = c. \tag{2.37}$$

Moreover,

$$\begin{aligned} \|T(PT)^{n-1}x_n - p\| &\le \|T(PT)^{n-1}x_n - p\| \\ &\le k_n\|x_n - p\| \end{aligned} \tag{2.38}$$

which implies that

$$\limsup_{n\to\infty} \|T(PT)^{n-1}x_n - p\| \le c. \tag{2.39}$$

Lemma 1.4 combined with (2.37) and (2.39) yields

$$\lim_{n\to\infty} \|T(PT)^{n-1}x_n - x_n\| = 0. \tag{2.40}$$

Since T is uniformly *L-Lipschitzian* for some $L > 0$, it follows from Lemma 2.3 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.41}$$

This completes the proof. □

Lemma 2.5. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\alpha_n'\}$, and $\{\alpha_n''\}$ be real sequences in $[0,1]$ and $\epsilon \le \alpha_n, \alpha_n', \alpha_n'' \le 1-\epsilon$ for all $n \in \mathbb{N}$ and some $\epsilon > 0$, starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.7). Then for all $x^*, x^{**} \in F(T)$, the limit*

$$\lim_{n\to\infty} \|tx_n + (1-t)x^* - x^{**}\| \tag{2.42}$$

exists for all $t \in [0,1]$

Proof. It follows from Lemma 2.2 that $\{x_n\}$ is bounded, there exists $\mathbb{R} > 0$ such that $\{x_n\} \subset C := B_R(0) \cap K$, where $B_R(0) = \{x \in E; \|x\| \le \mathbb{R}\}$. Then C is a nonempty closed convex bounded subset of E. We follow basically the idea of [12]. Let $a_n(t) = \|tx_n + (1-t)x^* - x^{**}\|$. Then $\lim_{n\to\infty} a_n(0) = \|x^* - x^{**}\|$, and from Lemma 2.2, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - x^{**}\|$ exists. Without loss of generality, we may assume that $\lim_{n\to\infty} \|x_n - x^{**}\| = b > 0$ and $t \in (0,1)$. Define $T_n : C \to C$ by

$$\begin{aligned} T_n x = P\big(\alpha_n T(PT)^{n-1}\big(P\big(\alpha_n' T(PT)^{n-1}\big)\big(P\big(\alpha_n'' T(PT)^{n-1}x\big) + (1-\alpha_n'')x\big) \\ + (1-\alpha_n')x\big) + (1-\alpha_n)x\big). \end{aligned} \tag{2.43}$$

Then

$$\|T_n x - T_n y\| \le k_n^3 \|x - y\|. \tag{2.44}$$

Set $S_{n,m} := T_{n+m-1} T_{n+m-2} \dots T_n$, $m \ge 1$, and

$$b_{n,m} = \|S_{n,m}(tx_n + (1-t)x^*) - (tS_{n,m}x_n + (1-t)x^{**})\|. \tag{2.45}$$

Then

$$\|S_{n,m}x - S_{n,m}y\| \le \prod_{j=n}^{n+m-1} k_j^3 \|x - y\|, \tag{2.46}$$

$S_{n,m}x_n = x_{n+m}$, and $S_{n,m}p = p$ for all $p \in F(T)$. It follows from [3, Lemma 2.1 and Theorem 2.3] that $\lim_{n\to\infty} b_{n,m} = 0$. Note that

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)x^* - x^{**}\| \\ &\le b_{n,m} + \|S_{n,m}(tx_n + (1-t)x^*) - x^{**}\| \\ &\le b_{n,m} + \prod_{j=n}^{n+m-1} k_j^3 a_n(t). \end{aligned} \tag{2.47}$$

Note that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ is equivalent to $\sum_{n=1}^{\infty}(k_n^3 - 1) < \infty$. Since $k_n \in [1,\infty)$, $\lim_{n\to\infty} \prod_{j=n}^{\infty} k_j^3 = 1$. Therefore, we have

$$\limsup_{n\to\infty} a_n(t) \le \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} a_n(t) = \liminf_{n\to\infty} a_n(t). \tag{2.48}$$

That is,

$$\lim_{n\to\infty} \|tx_n + (1-t)x^* - x^{**}\| \tag{2.49}$$

exists for all $t \in [0,1]$. This completes the proof. □

Theorem 2.6. *Let E be a uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T : K \to E$ be an asymptotically nonexpansive mapping with $\{k_n\} \subset [1,\infty)$,*

$k_n \to 1$ as $n \to \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\alpha'_n\}$, and $\{\alpha''_n\}$ be real sequences in $[0,1]$ and $\epsilon \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1-\epsilon$ for all $n \in \mathbb{N}$ and some $\epsilon > 0$, starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.7). Then $\{x_n\}$ converges weakly to some fixed point of T.

Proof. Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to some $u \in k$. It follows from Theorem 2.4 that $\lim_{i\to\infty} \|Tx_{n_i} - x_{n_i}\| = 0$. By Lemma 1.5, we have $u = Tu$. Next we claim $\{x_n\}$ converges weakly to u. Suppose $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ converging to some $v \in K$. Then $u, v \in w_w(x_n) \cap F(T)$. Using Lemma 2.5 yields that the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)u - v\| \tag{2.50}$$

exists for all $t \in (0,1)$. By Lemma 1.6 we have that $u = v$. Then $\{x_n\}$ converges weakly to some fixed point of T. □

Next, we will prove a strong convergence theorem.

Theorem 2.7. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T : K \to E$ be a nonexpansive mapping with $p \in F(T) := \{x \in K : Tx = x\}$. Let $\{\alpha_n\}$, $\{\alpha'_n\}$, and $\{\alpha''_n\}$ be real sequences in $[0,1]$ and $\epsilon \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1-\epsilon$ for all $n \in \mathbb{N}$ and some $\epsilon > 0$, starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.7). Suppose T satisfies condition (A). Then $\{x_n\}$ converges strongly to some fixed point of T.*

Proof. By Lemma 2.2, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F = F(T)$. Let $\lim_{n\to\infty} \|x_n - p\| = c$ for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. By Theorem 2.4, $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, and (2.8) gives

$$\inf_{p\in F} \|x_{n+1} - p\| \leq \left(1 + \left(k_n^3 - 1\right)\right) \inf_{p\in F} \|x_n - p\|. \tag{2.51}$$

That is,

$$d(x_{n+1}, F) \leq \left(1 + \left(k_n^3 - 1\right)\right) d(x_n, F) \tag{2.52}$$

gives that $\lim_{n\to\infty} d(x_n, F)$ exists by the virtue of Lemma 1.7. Now by condition (A), $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Now we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and sequence $\{y_j\} \subset F$ such that $\|x_{n_j} - y_j\| < 2^{-j}$. Then following the method of the proof of Tan and Xu [12], we get that $\{y_j\}$ is a Cauchy sequence in F and so it converges. Let $y_j \to y$. Since F is closed, therefore $y \in F$ and then $x_{n_j} \to y$. As $\lim_{n\to\infty} \|x_n - p\|$ exists, $x_n \to y \in F = F(T)$ thereby completing the proof. □

Acknowledgment

The authors are greatly indebted to the referees for their valuable comments and suggestions which helped to improve this manuscript.

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