A POSITIVE SOLUTION FOR SINGULAR DISCRETE BOUNDARY VALUE PROBLEMS WITH SIGN-CHANGING NONLINEARITIES

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This paper presents new existence results for the singular discrete boundary value problem $-\Delta^2 u(k-1) = g(k, u(k)) + \lambda h(k, u(k))$, $k \in [1, T]$, $u(0) = 0 = u(T+1)$. In particular, our nonlinearity may be singular in its dependent variable and is allowed to change sign.

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1. Introduction

Let a, b ($b > a$) be nonnegative integers. We define the discrete interval $[a, b] = \{a, a+1, \dots, b\}$. All other intervals will carry its standard meaning, for example, $[0, \infty)$ denotes the set of nonnegative real numbers. The symbol Δ denotes the forward difference operator with step size 1, that is, $\Delta u(k) = u(k+1) - u(k)$. Furthermore for a positive m , Δ^m is defined as $\Delta^m u(k) = \Delta^{m-1}(\Delta u(k))$. In this paper, we will study positive solutions of the second-order discrete boundary value problem

$$\begin{aligned} -\Delta^2 u(k-1) &= g(k, u(k)) + \lambda h(k, u(k)), & k \in [1, T], \\ u(0) &= 0 = u(T+1), \end{aligned} \tag{1.1}$$

where $\lambda > 0$ is a constant and $T > 2$ is a positive integer. Here, $g : [1, T] \times (0, \infty) \rightarrow \mathbb{R}$ and $h : [1, T] \times [0, \infty) \rightarrow [0, \infty)$ are continuous. As a result, our nonlinearity may be singular at $u = 0$ and may change sign.

By a solution u of the boundary value problem (1.1), we mean $u : [0, T+1] \rightarrow \mathbb{R}$, u satisfies the difference equation (1.1) on $[1, T]$ and the stated boundary data.

We will let $C[0, T+1]$ denote the class of map u continuous on $[0, T+1]$ (discrete topology), with norm $\|u\|_\infty = \max_{k \in [0, T+1]} |u(k)|$.

2. Main results

The main result of the paper is the following.

2 Singular discrete BVPs

THEOREM 2.1. *Suppose the following conditions hold:*

(G) *there exist $g_i : [1, T] \times (0, \infty) \rightarrow (0, \infty)$ ($i = 1, 2$) continuous functions such that*

$$\begin{aligned} &g_i(k, \cdot) \text{ is strictly decreasing for } k \in [1, T], \\ &-g_1(k, u) \leq g(k, u) \leq g_2(k, u) \quad \text{for } (k, u) \in [1, T] \times (0, \infty), \\ &\int_0^1 g_1(k, s) ds < \infty \quad \text{for } k \in [1, T], \\ &\forall s_0 > 0, \quad \sup_{s_0 \leq s} \left| \frac{\partial}{\partial s} g_2(\cdot, s) \right| \in C[1, T]; \end{aligned} \quad (2.1)$$

(H) *there exist $h_i : [1, T] \times [0, \infty) \rightarrow (0, \infty)$ ($i = 1, 2$) continuous functions such that*

$$\begin{aligned} &h_i(k, \cdot) \text{ increasing for } k \in [1, T], \\ &h_1(k, u) \leq h(k, u) \leq h_2(k, u) \quad \text{for } (k, u) \in [1, T] \times (0, \infty), \\ &\lim_{u \rightarrow \infty} \frac{h_2(k, u)}{u} = 0 \quad \text{for } k \in [1, T], \\ &\text{there exists } \bar{s} > 0 \text{ such that } h_1(k, \bar{s}) > 0 \text{ for all } k \in [1, T]. \end{aligned} \quad (2.2)$$

Then there exists $\lambda_0 \geq 0$ such that for every $\lambda \geq \lambda_0$, problem (1.1) has at least one solution $u \in C[0, T+1]$ and $u(k) > 0$ for $k \in [1, T]$. Moreover, there exists $c_i = c_i(\lambda, g, h, \phi_1) > 0$ ($i = 1, 2$) such that

$$c_1 \phi_1(k) \leq u(k) \leq c_2 (\phi_1(k) + 1) \quad \text{for } k \in [0, T+1], \quad (2.3)$$

where ϕ_1 is defined in Lemma 2.2.

It is worth remarking here that an estimate for λ_0 will be given in the proof of Lemma 2.11.

We first give some lemmas which will help us to prove Theorem 2.1.

LEMMA 2.2 [1]. *Consider the following eigenvalue problem:*

$$\begin{aligned} -\Delta^2 u(k-1) &= \lambda u(k), \quad k \in [1, T], \\ u(0) &= u(T+1) = 0. \end{aligned} \quad (2.4)$$

Then the eigenvalues are

$$\lambda_m = 4 \sin^2 \frac{m\pi}{2(T+1)}, \quad 1 \leq m \leq T, \quad (2.5)$$

and the corresponding eigenfunctions are

$$\phi_m(k) = \sin \frac{mk\pi}{T+1} \quad \text{for } k \in [0, T+1], \quad 1 \leq m \leq T. \quad (2.6)$$

LEMMA 2.3 [3]. *Let $G_a(k, l)$ be Green's function of the BVP*

$$\begin{aligned} -\Delta^2 u(k-1) + a(t)u(t) &= 0 \quad \text{for } t \in [1, T], \\ u(0) &= 0, \quad u(T+1) = 0. \end{aligned} \quad (2.7)$$

Then

$$0 < G_a(k, l) \leq G_a(l, l) \quad \text{for every } (k, l) \in [1, T] \times [1, T], \quad (2.8)$$

where $a \in C[1, T]$ and $a(k) \geq 0$ for $k \in [1, T]$.

Remark 2.4. If $a(k) \equiv 0$ for $k \in [1, T]$, then

$$G_0(k, l) = \frac{1}{T+1} \begin{cases} l(T+1-k), & l \in [0, k-1], \\ k(T+1-l), & l \in [k, T+1], \end{cases} \quad \text{for } k \in [1, T]. \quad (2.9)$$

Next we consider the boundary value problem

$$\begin{aligned} -\Delta^2 u(k-1) + a(k)u(k) &= f(k), \quad k \in [1, T], \\ u(0) &= 0 = u(T+1), \end{aligned} \quad (2.10)$$

where $a, f \in C[1, T]$ and $a(k) \geq 0$ for $k \in [1, T]$.

Let $A : C[1, T] \rightarrow C[1, T]$ be the operator defined by

$$Au(k) := \sum_{l=1}^T G_a(k, l)u(l). \quad (2.11)$$

It is easy to see that A is a completely continuous operator (see [3]).

Note that if $u \in C[0, T+1]$, $u(0) = u(T+1) = 0$, and

$$u(k) = A(f)(k) \quad \text{for } k \in [1, T], \quad (2.12)$$

then u is a solution of (2.10).

From Lemma 2.3, we have the following lemma.

LEMMA 2.5. *The following statements hold:*

- (i) *for any $f \in C[1, T]$, (2.10) is uniquely solvable and $u = A(f)$;*
- (ii) *if $f(k) \geq 0$ for $k \in [1, T]$, then the solution of (2.10) is nonnegative.*

COROLLARY 2.6. *If $f_1(k) \leq f_2(k)$ for $k \in [1, T]$, then $A(f_1)(k) \leq A(f_2)(k)$ for $k \in [1, T]$.*

LEMMA 2.7. *Suppose (G) and (H) hold. Let $n_0 \in \mathbb{N}$. Assume that for every $n > n_0$, there exist $a_n \in C[1, T]$, $0 \leq a_n$, and there exist $\bar{u}, \bar{u}_n, \hat{u}_n, \hat{u} \in C[0, T+1]$ such that*

$$0 < \bar{u}(k) \leq \bar{u}_n(k) \leq \hat{u}_n(k) \leq \hat{u}(k) \quad \text{for } k \in [1, T], \quad (2.13)$$

and $\hat{u}(0) = \hat{u}(T+1) = 0$. If

$$\begin{aligned} -\Delta^2 \bar{u}_n(k-1) + a_n(k)\bar{u}_n(k) \\ \leq g\left(k, \frac{1}{n} + v(k)\right) + \lambda h(k, v(k)) + a_n(k)v(k) \quad \text{for } k \in [1, T], \end{aligned} \quad (2.14)$$

$$\begin{aligned} -\Delta^2 \hat{u}_n(k-1) + a_n(k)\hat{u}_n(k) \\ \geq g\left(k, \frac{1}{n} + v(k)\right) + \lambda h(k, v(k)) + a_n(k)v(k) \quad \text{for } k \in [1, T], \end{aligned} \quad (2.15)$$

4 Singular discrete BVPs

where $\lambda \geq 0$ and $v \in [\bar{u}_n, \hat{u}_n] = \{u \in C[0, T+1], \bar{u}_n(k) \leq u(k) \leq \hat{u}_n(k) \text{ for } k \in [0, T+1]\}$, then problem (1.1) has a solution $u \in C[0, T+1]$ such that $\bar{u}(k) \leq u(k) \leq \hat{u}(k)$ for $k \in [0, T+1]$.

Proof. Fix $v \in [\bar{u}, \hat{u}]$. From Lemma 2.5, there exists $\Psi(v) \in C[0, T+1]$ such that

$$\begin{aligned} & -\Delta^2 \Psi(v)(k-1) + a_n(k) \Psi(v)(k) \\ & = g\left(k, \frac{1}{n} + v(k)\right) + \lambda h(k, v(k)) + a_n(k) v(k) \quad \text{for } k \in [1, T], \\ & \Psi(v)(0) = \Psi(v)(T+1) = 0. \end{aligned} \quad (2.16)$$

Then

$$\Psi(v)(k) = A\left(g\left(\cdot, \frac{1}{n} + v\right) + \lambda h(\cdot, v) + a_n v\right)(k) \quad \text{for } k \in [1, T]. \quad (2.17)$$

Note also that $\Psi : C[0, T+1] \rightarrow C[0, T+1]$ is a completely continuous operator. By Corollary 2.6, we have

$$\bar{u}_n(k) \leq \Psi(v)(k) \leq \hat{u}_n(k) \quad \text{for } k \in [0, T+1]. \quad (2.18)$$

From Schauder's fixed point theorem (note that $\Psi z : [\bar{u}, \hat{u}] \rightarrow [\bar{u}, \hat{u}]$), there exists $u_n \in C[0, T+1]$ such that $\bar{u}_n(k) \leq u_n(k) \leq \hat{u}_n(k)$ and $\Psi(u_n)(k) = u_n(k)$ for $k \in [1, T]$. Note that

$$\begin{aligned} -\Delta^2 u_n(k-1) & = g\left(k, \frac{1}{n} + u_n(k)\right) + \lambda h(k, u_n(k)) \quad \text{for } k \in [1, T], \\ u_n(0) & = u_n(T+1) = 0. \end{aligned} \quad (2.19)$$

Let $m := \min\{\bar{u}(k) : k \in [1, T]\} > 0$ and $M := \max\{\hat{u}(k) : k \in [1, T]\}$. Then

$$m \leq u_n(k) \leq M \quad \text{for } k \in [1, T], \quad n = 1, 2, \dots, \quad (2.20)$$

and for $k \in [1, T]$, we have

$$\left| g\left(k, \frac{1}{n} + u_n(k)\right) + \lambda h(k, u_n(k)) \right| \leq g_2(k, m) + \lambda h_2(k, M). \quad (2.21)$$

From the Arzela-Ascoli theorem, there exist a $u \in C[0, T+1]$ and a subsequence $\{u_{n_m}\}_{m \in \mathbb{N}}$ converging to u in $C[0, T+1]$, and of course

$$u(k) = \lim_{m \rightarrow \infty} u_{n_m}(k) \quad \text{for } k \in [0, T+1]. \quad (2.22)$$

Observe that $u_{n_m} \in [\bar{u}, \hat{u}]$, so $u(0) = u(T+1) = 0$ and $u \in C[0, T+1]$ with $u > 0$ in $[1, T]$. Also,

$$\begin{aligned} u(k) &= \lim_{m \rightarrow \infty} \sum_{l=1}^T G_0(k, l) \left[g\left(l, \frac{1}{n} + u_{n_m}(l)\right) + \lambda h(l, u_{n_m}(l)) \right] \\ &= \sum_{l=1}^T G_0(k, l) [g(l, u(l)) + \lambda h(l, u(l))]. \end{aligned} \quad (2.23)$$

As a result

$$\begin{aligned} -\Delta^2 u(k-1) &= g(k, u(k)) + \lambda h(k, u(k)) \quad \text{for } k \in [1, T], \\ u(0) &= u(T+1) = 0. \end{aligned} \quad (2.24)$$

□

LEMMA 2.8. *Let $\psi : [1, T] \times (0, \infty) \rightarrow (0, \infty)$ be a continuous function with $\psi(k, \cdot)$ strictly decreasing. Then the problem*

$$\begin{aligned} -\Delta^2 \omega(k-1) &= \psi\left(k, \omega + \frac{1}{n}\right) \quad \text{for } k \in [0, T], \\ \omega(0) &= \omega(T+1) = 0 \end{aligned} \quad (2.25)$$

has a solution $\omega_n \in C[0, T+1]$ such that

$$\omega_n(k) \leq \omega_{n+1}(k) \leq 1 + \omega_1(k) \quad \text{for } k \in [0, T+1], \quad n \in \mathbb{N}. \quad (2.26)$$

If $\omega(k) = \lim_{n \rightarrow \infty} \omega_n(k)$ for $k \in [0, T+1]$, then

$$\begin{aligned} \omega &\in C[0, T+1], \quad \omega(k) > 0, \quad \text{for } k \in [1, T], \\ -\Delta^2 \omega(k-1) &= \psi(k, \omega) \quad \text{for } k \in [1, T], \\ \omega(0) &= \omega(T+1) = 0. \end{aligned} \quad (2.27)$$

Proof. There exists $\chi_1 \in C[0, T+1]$ such that

$$\begin{aligned} -\Delta^2 \chi_1(k-1) &= \psi(k, 1), \\ \chi_1(0) &= \chi_1(T+1) = 0, \\ \chi_1(k) &> 0 \quad \text{for } k \in [1, T]. \end{aligned} \quad (2.28)$$

Notice that

$$\begin{aligned} -\Delta^2 \chi_1(k-1) &= \psi(k, 1) \geq \psi(k, 1 + \chi_1(k)), \\ 0 &\leq \psi(k, 1 + 0). \end{aligned} \quad (2.29)$$

By a standard upper-lower solution method [2, page 264], there exists $\omega_1 \in C[0, T+1]$ such that

$$\begin{aligned} -\Delta^2 \omega_1(k-1) &= \psi(k, 1 + \omega_1(k)) \quad \text{for } k \in [1, T], \\ \omega_1(0) &= \omega_1(T+1) = 0. \end{aligned} \quad (2.30)$$

6 Singular discrete BVPs

Suppose that there exists $\omega_n \in C[0, T+1]$ such that

$$\begin{aligned} -\Delta^2 \omega_n(k-1) &= \psi\left(k, \frac{1}{n} + \omega_n(k)\right), \\ \omega_n(0) &= \omega_n(T+1) = 0, \\ \omega_n(k) &> 0 \quad \text{for } k \in [1, T]. \end{aligned} \quad (2.31)$$

We know that there exist $\chi_{n+1} \in C[0, T+1]$ such that

$$\begin{aligned} -\Delta^2 \chi_{n+1}(k-1) &= \psi\left(k, \frac{1}{n+1}\right), \\ \chi_{n+1}(0) &= \chi_{n+1}(T+1) = 0, \\ \chi_{n+1}(k) &> 0 \quad \text{for } k \in [1, T]. \end{aligned} \quad (2.32)$$

Then

$$\begin{aligned} -\Delta^2 \chi_{n+1}(k-1) &= \psi\left(k, \frac{1}{n+1}\right) \geq \psi\left(k, \frac{1}{n+1} + \chi_{n+1}(k)\right), \\ -\Delta^2 \omega_n(k-1) &= \psi\left(k, \frac{1}{n} + \omega_n(k)\right) \leq \psi\left(k, \frac{1}{n+1} + \omega_n(k)\right) \quad \text{for } k \in [1, T], \\ \omega_n(0) &= \omega_n(T+1) = 0, \\ \omega_n(k) &= \sum_{l=1}^T G_0(k, l) \psi\left(l, \frac{1}{n} + \omega_n(l)\right) \leq \sum_{l=1}^T G_0(k, l) \psi\left(l, \frac{1}{n+1}\right) = \chi_{n+1}(k) \quad \text{for } k \in [1, T]. \end{aligned} \quad (2.33)$$

By a standard upper-lower solution method, there exist $\omega_{n+1} \in C[0, T+1]$ such that

$$\begin{aligned} -\Delta^2 \omega_{n+1}(k-1) &= \psi\left(k, \frac{1}{n+1} + \omega_{n+1}\right) \quad \text{for } k \in [1, T], \\ \omega_{n+1}(0) &= \omega_{n+1}(T+1) = 0, \\ \omega_n(k) &\leq \omega_{n+1}(k) \quad \text{for } k \in [0, T+1]. \end{aligned} \quad (2.34)$$

Next we prove

$$\omega_{n+1}(k) + \frac{1}{n+1} \leq \omega_n(k) + \frac{1}{n} \quad \text{for } k \in [0, T+1]. \quad (2.35)$$

To see this, we consider the problem

$$\begin{aligned} -\Delta^2 v(k-1) &= \psi(k, v) \quad \text{for } k \in [1, T], \\ v(0) &= v(T+1) = \frac{1}{n}. \end{aligned} \quad (2.36)_n$$

Then $v_n(k) = 1/n + \omega_n(k)$ for $k \in [0, T+1]$ is a solution of $(2.36)_n$. We next prove

$$v_{n+1}(k) \leq v_n(k) \quad \text{for } k \in [0, T+1]. \quad (2.37)$$

Since $v_{n+1}(0) = 1/(n+1) < 1/n = v_n(0)$, $v_{n+1}(1) = 1/(n+1) < 1/n = v_n(1)$, we need only to prove that

$$v_{n+1}(k) \leq v_n(k) \quad \text{for } k \in [1, T]. \quad (2.38)$$

If this is not true, then there exist $m \in [1, T]$ with $v_{n+1}(m) > v_n(m) > 0$. Let σ be the point where $v_{n+1}(k) - v_n(k)$ assumes its maximum over $[1, T]$. Certainly, $v_{n+1}(\sigma) - v_n(\sigma) > 0$. Let $y(k) = v_{n+1}(k) - v_n(k)$. Now $y(\sigma) \geq y(\sigma+1)$ and $y(\sigma) \geq y(\sigma-1)$ imply that

$$2y(\sigma) \geq y(\sigma+1) + y(\sigma-1), \quad (2.39)$$

that is,

$$y(\sigma+1) + y(\sigma-1) - 2y(\sigma) \leq 0. \quad (2.40)$$

Thus

$$\Delta^2 y(\sigma-1) \leq 0. \quad (2.41)$$

On the other hand, since $v_{n+1}(\sigma) > v_n(\sigma)$, we have

$$\begin{aligned} \Delta^2 y(\sigma-1) &= \Delta^2 v_{n+1}(\sigma-1) - \Delta^2 v_n(\sigma-1) \\ &= -\psi(\sigma, v_{n+1}(\sigma)) + \psi(\sigma, v_n(\sigma)) \\ &= \psi(\sigma, v_n(\sigma)) - \psi(\sigma, v_{n+1}(\sigma)) > 0. \end{aligned} \quad (2.42)$$

This is a contradiction. Thus $v_{n+1}(k) \leq v_n(k)$ for $k \in [1, T]$, and so

$$0 < \frac{1}{n+1} + \omega_{n+1} \leq \omega_n + \frac{1}{n}. \quad (2.43)$$

Also notice that

$$\omega_1(k) \leq \omega_n(k) \leq \omega_{n+1}(k) \leq 1 + \omega_1(k) \quad \text{for } k \in [0, T+1], \quad n \in \mathbb{N}. \quad (2.44)$$

Now with

$$\omega(k) = \lim_{n \rightarrow \infty} \omega_n(k) = \sup_{n \in \mathbb{N}} \omega_n(k) \quad \text{for } k \in [0, T+1], \quad (2.45)$$

we have

$$\begin{aligned} 0 < \omega_1(k) &\leq \omega(k) \leq 1 + \omega_1(k) \quad \text{for } k \in [1, T], \\ \omega(0) &= \omega(T+1) = 0. \end{aligned} \quad (2.46)$$

8 Singular discrete BVPs

Also for $k \in [1, T]$, we have

$$\begin{aligned}\omega(k) &= \lim_{n \rightarrow \infty} \omega_n(k) \\ &= \lim_{n \rightarrow \infty} \sum_{l=1}^T G(k, l) \psi\left(l, \frac{1}{n} + \omega_n(l)\right) \\ &= \sum_{l=1}^T G(k, l) \psi(l, \omega(l)),\end{aligned}\tag{2.47}$$

so

$$\begin{aligned}-\Delta^2 \omega(k-1) &= \psi(k, \omega) \quad \text{for } k \in [0, T], \\ \omega(0) &= \omega(T+1) = 0.\end{aligned}\tag{2.48}$$

□

LEMMA 2.9. Suppose that $m : [1, T] \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that

$$\begin{aligned}m(k, \cdot) &\text{ is increasing,} \\ \lim_{u \rightarrow +\infty} \frac{m(k, u)}{u} &= 0 \quad \text{for } k \in [1, T].\end{aligned}\tag{2.49}$$

There exist $R_0 > 0$ and $\tilde{v} \in C[0, T+1]$ with $0 \leq \tilde{v} \leq R_0 \phi_1$ and

$$\begin{aligned}-\Delta^2 \tilde{v}(k-1) &= m(k, \tilde{v}) \quad \text{for } k \in [1, T], \\ \tilde{v}(0) &= \tilde{v}(T+1) = 0.\end{aligned}\tag{2.50}$$

Proof. We first prove that

$$\lim_{R \rightarrow \infty} \frac{\sum_{l=1}^T G_0(k, l) m(l, v(l))}{R \phi_1(k)} = 0 \quad \text{for } k \in [1, T],\tag{2.51}$$

for all $v \in C[0, T+1]$ with $0 \leq v(i) \leq R \phi_1(i)$ for $i \in [0, T+1]$.

From (2.49), for all $\sigma > 0$, there exist $s_\sigma > 0$ such that

$$m(k, s) \leq \sigma s \quad \text{for } k \in [1, T] \text{ and } s_\sigma \leq s.\tag{2.52}$$

As a result,

$$m(k, v(k)) \big|_{0 \leq v(k) \leq R \phi_1(k)} \leq m(k, s_\sigma) + \sigma v(k) \leq m(k, s_\sigma) + \sigma R \phi_1(k) \quad \text{for } k \in [1, T],\tag{2.53}$$

so

$$\begin{aligned}\frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) m(l, v(l)) &\leq \frac{1}{\phi_1(k)} \left[\sum_{l=1}^T G_0(k, l) m(l, s_\sigma) + R \sigma \sum_{l=1}^T G_0(k, l) \phi_1(l) \right] \\ &= \frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) m(l, s_\sigma) + \frac{R \sigma}{\lambda_1},\end{aligned}\tag{2.54}$$

and consequently

$$\frac{1}{R\phi_1(k)} \sum_{l=1}^T G_0(k, l) m(l, v(l)) \leq \frac{1}{R\phi_1(k)} \sum_{l=1}^T G_0(k, l) m(l, s_\sigma) + \frac{\sigma}{\lambda_1}, \quad (2.55)$$

so (2.51) follows. Thus there exist $R_0 > 0$ such that if $v \in C[0, T+1]$ and $0 \leq v(i) \leq R_0\phi_1(i)$ for $i \in [0, T+1]$, then

$$\frac{1}{R_0\phi_1(k)} \sum_{l=1}^T G_0(k, l) m(l, v(l)) \leq 1 \quad \text{for } k \in [1, T], \quad (2.56)$$

and so

$$0 \leq \sum_{l=1}^T G_0(k, l) m(l, v(l)) \leq R_0\phi_1(k) \quad \text{for } k \in [1, T]. \quad (2.57)$$

Let $\Phi : C[1, T] \rightarrow C[1, T]$ be the operator defined by

$$(\Phi v)(k) := \sum_{l=1}^T G_0(k, l) m(l, v(l)) \quad \text{for } v \in C[1, T], k \in [1, T]. \quad (2.58)$$

It is easy to see that Φ is a completely continuous operator. Also if $v \in C[0, T+1]$ and $0 \leq v(k) \leq R_0\phi_1(k)$ for $k \in [1, T]$, then $0 \leq \Phi(v)(k) \leq R_0\phi_1(k)$ for $k \in [1, T]$, so Schauder's fixed point theorem guarantees that there exists $\tilde{v} \in [0, R_0\phi_1]$ with $\Phi(\tilde{v}) = \tilde{v}$, that is,

$$-\Delta^2 \tilde{v}(k-1) = m(k, \tilde{v}), \quad \tilde{v}(0) = \tilde{v}(T+1) = 0. \quad (2.59)$$

□

COROLLARY 2.10. *Let $\psi(k, s)$, $m(k, s)$, $(\omega_n)_{n \in \mathbb{N}}$, and $R_0 > 0$ be as in Lemmas 2.8 and 2.9. Then there exist $\{\tilde{v}_n\}_{n \in \mathbb{N}} \subset C[0, T+1]$ and $0 \leq \tilde{v}_n \leq R_0\phi_1$ such that*

$$\begin{aligned} -\Delta^2 \tilde{v}_n(k-1) &= m(k, \omega_n + \tilde{v}_n) \quad \text{for } k \in [1, T], \\ \tilde{v}_n(0) &= \tilde{v}_n(T+1) = 0, \end{aligned} \quad (2.60)$$

$$-\Delta^2 (w_n + \tilde{v}_n)(k-1) \geq \psi\left(k, \frac{1}{n} + \omega_n + \tilde{v}_n\right) + m(k, \omega_n + \tilde{v}_n) \quad \text{for } k \in [1, T].$$

Proof. Let $n \in \mathbb{N}$ be fixed. Then $m(k, \omega_n + s)$ satisfies the conditions of Lemma 2.9, so there exists $\tilde{v}_n \in C[0, T+1]$ with $0 \leq \tilde{v}_n \leq R_0\phi_1$ such that (2.60) holds and

$$\begin{aligned} -\Delta^2 (w_n + \tilde{v}_n)(k-1) &= -\Delta^2 w_n(k-1) - \Delta^2 \tilde{v}_n(k-1) = \psi\left(k, \frac{1}{n} + \omega_n\right) + m(k, \omega_n + \tilde{v}_n) \\ &\geq \psi\left(k, \frac{1}{n} + \omega_n + \tilde{v}_n\right) + m(k, \omega_n + \tilde{v}_n) \quad \text{for } k \in [1, T]. \end{aligned} \quad (2.61)$$

□

LEMMA 2.11. Suppose (G) and (H) hold. Then there exist $\lambda_0 \geq 0$, $c > 0$ such that for all $\lambda \geq \lambda_0$, there exist $R_c > c$, $\bar{u} \in C([0, T+1])$ with $c\phi_1(k) \leq \bar{u}(k) \leq R_c\phi_1(k)$ and

$$\begin{aligned} -\Delta^2 \bar{u}(k-1) &= -g_1(k, \bar{u}) + \lambda h_1(k, \bar{u}) \quad \text{for } k \in [1, T], \\ \bar{u}(0) &= \bar{u}(T+1) = 0. \end{aligned} \quad (2.62)$$

Proof. Let us consider the operator $T_\lambda : C[1, T] \rightarrow C[1, T]$ given by

$$T_\lambda(v)(k) := \frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) [-g_1(l, v(l)\phi_1(l)) + \lambda h_1(l, v(l)\phi_1(l))] \quad \text{for } k \in [1, T]. \quad (2.63)$$

By (H), there exists $\bar{s} \geq 0$ such that $0 < h_1(k, \bar{s})$ for $k \in [1, T]$. We let

$$c = 2 \frac{\bar{s} + 1}{|\phi_1|_\infty}, \quad \Theta = \left\{ k \in [1, T] : \frac{|\phi_1|_\infty}{2} < \phi_1(k) \right\}. \quad (2.64)$$

Note that Θ is nonempty. If $k \in \Theta$, $v \in C[0, T+1]$, and $c \leq v$, we have

$$\bar{s} = \frac{c|\phi_1|_\infty}{2} - 1 \leq \frac{c|\phi_1|_\infty}{2} \leq c\phi_1(k) \leq v(k)\phi_1(k), \quad (2.65)$$

so

$$h_1(k, \bar{s}) \leq h_1(k, v(k)\phi_1(k)), \quad (2.66)$$

for all $v \in C[0, T+1]$ with $c \leq v$. Let

$$\rho = \min_{k \in [1, T]} \frac{1}{\phi_1(k)} \sum_{l \in \Theta} G_0(k, l) h_1(l, \bar{s}) > 0, \quad (2.67)$$

and note for $v \in C[0, T+1]$ with $c \leq v$ that

$$\begin{aligned} \frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) h_1(l, v(l)\phi_1(l)) &\geq \frac{1}{\phi_1(k)} \sum_{l \in \Theta} G_0(k, l) h_1(l, v(l)\phi_1(l)) \\ &\geq \frac{1}{\phi_1(k)} \sum_{l \in \Theta} G_0(k, l) h_1(l, \bar{s}) \quad (\text{see (2.66)}) \\ &\geq \min_{k \in [1, T]} \frac{1}{\phi_1(k)} \sum_{l \in \Theta} G_0(k, l) h_1(l, \bar{s}) \\ &= \rho \quad \forall k \in [1, T], \end{aligned} \quad (2.68)$$

that is,

$$\frac{\phi_1(k)}{\sum_{l=1}^T G_0(k, l) h_1(l, v(l)\phi_1(l))} \leq \frac{1}{\rho}. \quad (2.69)$$

On the other hand, for all $v \in C[0, T+1]$ with $v \geq c$, we have

$$\begin{aligned} c + \frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) g_1(l, v(l) \phi_1(l)) \\ \leq c + \frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) g_1(l, c \phi_1(l)) \leq c + \frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) g_1(l, c \mu), \end{aligned} \quad (2.70)$$

where $\mu = \min_{1 \leq l \leq T} \phi_1(l)$. Thus, for all $v \in C[0, T+1]$ with $v(k) \geq c$, we have

$$\begin{aligned} \frac{c + (1/\phi_1(k)) \sum_{l=1}^T G_0(k, l) g_1(l, v(l) \phi_1(l))}{\left(\sum_{l=1}^T G_0(k, l) h_1(l, v(l) \phi_1(l)) \right) / \phi_1(k)} \\ \leq \frac{1}{\rho} \left(c + \frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) g_1(l, c \mu) \right) \quad \text{for } k \in [1, T]. \end{aligned} \quad (2.71)$$

Let

$$\lambda_0 := \sup \left\{ \left| \frac{c + (1/\phi_1(k)) \sum_{l=1}^T G_0(k, l) g_1(l, v(l) \phi_1(l))}{\left(\sum_{l=1}^T G_0(k, l) h_1(l, v(l) \phi_1(l)) \right) / \phi_1(k)} \right|_* : v \in C[0, T+1], c \leq v \right\} < \infty, \quad (2.72)$$

where $|u|_* = \max[1, T] |u(k)|$. Then, for all $\lambda \geq \lambda_0$, $v \in C[0, T+1]$, and $c \leq v$, we have for $k \in [1, T]$ that

$$\frac{c + (1/\phi_1(k)) \sum_{l=1}^T G_0(k, l) g_1(l, v(l) \phi_1(l))}{\left(\sum_{l=1}^T G_0(k, l) h_1(l, v(l) \phi_1(l)) \right) / \phi_1(k)} \leq \lambda, \quad (2.73)$$

that is,

$$c + \frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) g_1(l, v(l) \phi_1(l)) \leq \frac{\lambda}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) h_1(l, v(l) \phi_1(l)), \quad (2.74)$$

so

$$\begin{aligned} c &\leq \frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) (-g_1(l, v(l) \phi_1(l)) + \lambda h_1(l, v(l) \phi_1(l))) \\ &= T_\lambda(v)(k) \quad \text{for } k \in [1, T]. \end{aligned} \quad (2.75)$$

On the other hand, for all $v \in C[0, T+1]$ with $v \geq c$, we have

$$\begin{aligned} \frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) g_1(l, v(l) \phi_1(l)) &\leq \frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) g_1(l, c \phi_1(l)) \\ &\leq \max_{k \in [1, T]} \frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) g_1(l, c \phi_1(l)), \end{aligned} \quad (2.76)$$

12 Singular discrete BVPs

so

$$\lim_{R \rightarrow \infty} \frac{1}{R} \left[\frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) g_1(l, v(l) \phi_1(l)) \right] = 0, \quad (2.77)$$

for all $v \in C[0, T+1]$ with $v \geq c$ and $k \in [1, T]$. Essentially the same reasoning as in the proof of (2.51) yields (note that $\lim_{u \rightarrow \infty} (h_1(k, u)/u) = 0$ for $k \in [1, T]$)

$$\lim_{R \rightarrow \infty} \frac{1}{R} \left[\frac{1}{\phi_1(k)} \sum_{l=1}^T G_0(k, l) h_1(l, v(l) \phi_1(l)) \right] = 0 \quad (2.78)$$

for all $v \in C[0, T+1]$ with $0 \leq v(i) \leq R$ and $i \in [1, T]$. Thus if $\lambda \geq \lambda_0$, there exists $R_c > c$ with $T_\lambda([c, R_c]) \subset [c, R_c]$.

It is easy to see that T_λ is a completely continuous operator, so Schauder's fixed point theorem guarantees that there exists $\bar{v} \in [c, R_c]$ with $T_\lambda(\bar{v}) = \bar{v}$, that is,

$$\bar{v}(k) \phi_1(k) = \sum_{l=1}^T G_0(k, l) (-g_1(l, \bar{v}(l) \phi_1(l)) + \lambda h_1(l, \bar{v}(l) \phi_1(l))). \quad (2.79)$$

The function $\bar{u} = \phi_1 \bar{v}$ satisfies (2.62). □

Proof of Theorem 2.1. Let $\lambda_0 > 0$, $c > 0$, and $\bar{u} \in (C[0, T+1])$ be defined as in Lemma 2.11. Also let

$$\begin{aligned} \psi(k, s) &= g_2(k, s) + \lambda h_1(k, \bar{u}(k)) \quad \text{for } k \in [1, T], \\ m(k, s) &= \lambda h_2(k, s), \end{aligned} \quad (2.80)$$

where $\lambda \geq \lambda_0$.

From (G), we notice that ψ satisfies the assumptions of Lemma 2.8. As a result, there exist $\omega, \omega_n \in C[0, T+1]$ such that

$$\begin{aligned} -\Delta^2 \omega_n(k-1) &= g_2\left(k, \frac{1}{n} + \omega_n\right) + \lambda h_1(k, \bar{u}(k)) \quad \text{for } k \in [1, T], \\ \omega_n(0) &= \omega_n(T+1) = 0, \\ \omega(k) &= \lim_{n \rightarrow \infty} \omega_n(k) \quad \text{for } k \in [0, T+1]. \end{aligned} \quad (2.81)$$

From (H), we notice that m satisfies the assumptions of Lemma 2.9. As a result from Corollary 2.10, there exist $R_0 > 0$ and $\tilde{v}_n \in C([0, T+1])$, $0 \leq \tilde{v}_n(k) \leq R_0 \phi_1(k)$ for $k \in [0, T+1]$ such that

$$\begin{aligned} -\Delta^2 \tilde{v}_n(k-1) &= \lambda h_2(k, \omega_n + \tilde{v}_n) \quad \text{for } k \in [1, T], \\ \tilde{v}_n(0) &= \tilde{v}_n(T+1) = 0, \\ -\Delta^2 (\omega_n + \tilde{v}_n)(k-1) &\geq g_2\left(k, \frac{1}{n} + \omega_n + \tilde{v}_n\right) + \lambda h_1(k, \bar{u}(k)) + \lambda h_2(k, \omega_n + \tilde{v}_n) \quad \text{for } k \in [1, T]. \end{aligned} \quad (2.82)$$

Let

$$\hat{u}_n(k) = \omega_n(k) + \tilde{v}_n(k) \quad \text{for } k \in [0, T+1]. \quad (2.83)$$

Then, $\hat{u}_n \in C[0, T+1]$, $\hat{u}_n(1) = \hat{u}_n(T+1) = 0$.

We let

$$\hat{u}(k) = \omega(k) + R_0\phi_1(k) \quad \text{for } k \in [0, T+1], \quad (2.84)$$

so

$$0 \leq \hat{u}_n(k) \leq \hat{u}(k) \quad \text{for } k \in [0, T+1]. \quad (2.85)$$

From Lemma 2.11, we have

$$\begin{aligned} -\Delta^2 \bar{u}(k-1) &= -g_1(k, \bar{u}(k)) + \lambda h_1(k, \bar{u}(k)) \\ &\leq \lambda h_1(k, \bar{u}(k)) \\ &\leq \lambda h_1(k, \bar{u}(k)) + g_2\left(k, \frac{1}{n} + \hat{u}_n(k)\right) + \lambda h_2(k, \hat{u}_n(k)) \\ &\leq -\Delta^2 \hat{u}_n(k-1) \quad \text{for } k \in [1, T], \end{aligned} \quad (2.86)$$

that is,

$$-\Delta^2(\bar{u} - \hat{u}_n)(k-1) \leq 0. \quad (2.87)$$

A standard argument (see the argument to show (2.35)) yields

$$\bar{u}(k) \leq \hat{u}_n(k) \quad \text{for } k \in [1, T]. \quad (2.88)$$

Let

$$a_n(k) = \sup \left\{ \left| \frac{\partial}{\partial s} g_2\left(k, \frac{1}{n} + s\right) \right| : 0 < s \right\}, \quad (2.89)$$

and notice that $s \rightarrow g_2(k, 1/n + s) + a(k)s$ is increasing. Let $\bar{u}_n = \bar{u}$. From (2.85) and (2.88), we have

$$\bar{u}(k) = \bar{u}_n(k) \leq \hat{u}_n(k) \leq \hat{u}(k) \quad \text{for } k \in [0, T+1]. \quad (2.90)$$

Also for $v \in C[1, T]$ with $\bar{u}_n(k) \leq v(k) \leq \hat{u}_n(k)$, $k \in [1, T]$, we have

$$\begin{aligned} -\Delta^2 \bar{u}_n(k-1) + a_n(k)\bar{u}_n(k) &= -g_1(k, \bar{u}_n(k)) + \lambda h_1(k, \bar{u}_n(k)) + a_n(k)\bar{u}_n(k) \\ &\leq -g_1(k, v(k)) + \lambda h_1(k, v(k)) + a_n(k)v(k) \\ &\leq -g_1\left(k, \frac{1}{n} + v(k)\right) + \lambda h_1(k, v(k)) + a_n(k)v(k) \\ &\leq g\left(k, \frac{1}{n} + v(k)\right) + \lambda h(k, v) + a_n(k)v(k) \quad \text{for } k \in [1, T], \end{aligned} \quad (2.91)$$

so (2.14) holds.

Also for $v \in C[1, T]$ with $\bar{u}_n(k) \leq v(k) \leq \hat{u}_n(k)$, $k \in [1, T]$, we have

$$\begin{aligned}
& -\Delta^2 \hat{u}_n(k-1) + a_n(k) \hat{u}_n(k) \\
& \geq g_2 \left(k, \frac{1}{n} + \hat{u}_n(k) \right) + \lambda h_1(k, \bar{u}(k)) + \lambda h_2(k, \hat{u}_n(k)) + a_n(k) \hat{u}_n(k) \\
& \geq g_2 \left(k, \frac{1}{n} + \hat{u}_n(k) \right) + a_n(k) \hat{u}_n(k) + \lambda h_2(k, \hat{u}_n(k)) \\
& \geq g_2 \left(k, \frac{1}{n} + v(k) \right) + a_n(k) v(k) + \lambda h_2(k, v(k)) \\
& \geq g \left(k, \frac{1}{n} + v(k) \right) + \lambda h(k, v(k)) + a_n(k) v(k) \quad \text{for } k \in [1, T],
\end{aligned} \tag{2.92}$$

so (2.15) holds. Lemma 2.7 guarantees that there exists a solution $u \in C[0, T+1]$ to (1.1) with

$$\bar{u}(k) \leq u(k) \leq \hat{u}(k) \quad \text{for } k \in [0, T+1]. \tag{2.93}$$

Moreover, because $\hat{u}(k) \leq |\omega|_\infty + R_0 \phi_1(k) \leq (|\omega|_\infty + R_0)(1 + \phi_1(k))$ and $c\phi_1(k) \leq \bar{u}(k)$ (see Lemma 2.11), the estimates asserted in the theorem follow. $\square$

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References

- [1] R. P. Agarwal, *Difference Equations and Inequalities*, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 228, Marcel Dekker, New York, 2000.
- [2] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Dordrecht, 1999.
- [3] F. Merdivenci Atici, *Existence of positive solutions of nonlinear discrete Sturm-Liouville problems*, Mathematical and Computer Modelling **32** (2000), no. 5-6, 599–607.

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