NULL EXACT CONTROLLABILITY OF THE PARABOLIC EQUATIONS WITH EQUIVALUED SURFACE BOUNDARY CONDITION

ZHONGQI YIN

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This paper is devoted to showing the null exact controllability for a class of parabolic equations with equivalued surface boundary condition. Our method is based on the duality argument and global Carleman-type estimate for a parabolic operator.

1. Introduction

Let $T>0$, $\Omega\subset\mathbb{R}^n$ $(n\in\mathbb{N})$ be a given bounded domain, $\partial\Omega=\Gamma_0\cup\Gamma_1$ $(\Gamma_1\neq\emptyset)$ (where Γ_0 is the interior boundary and Γ_1 the outer boundary), $\Gamma_0\cap\Gamma_1=\emptyset$. For simplicity, we assume that $\Gamma_0,\Gamma_1\in C^\infty$ and $\omega\neq\emptyset$ is a given subdomain of Ω. Denote the characteristic function of ω by χ_ω, and the unit outward normal vector of Ω by $(n_1,\dots,n_n)$. Put $Q=\Omega\times(0,T)$, $Q^\omega=\omega\times(0,T)$, and $\Sigma=\partial\Omega\times(0,T)$. Let $a_{ij}(x)\in C^2(\overline{\Omega})$ satisfy $a_{ij}=a_{ji}$, and for some $\Lambda>0$, it holds that

$$\sum_{i,j}a_{ij}\xi_i\xi_j\geq\Lambda|\xi|^2,\quad\forall(x,\xi)\in\Omega\times\mathbb{R}^n.\tag{1.1}$$

Here and henceforth, we denote $\sum_{i,j=1}^n$ simply by $\sum_{i,j}$.

We consider the following controlled parabolic equation with equivalued surface boundary condition:

$$\begin{gathered}\frac{\partial y}{\partial t}-\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right)=\chi_\omega(x)b\quad\text{in }Q,\\ y|_{\Gamma_1}=0,\qquad y|_{\Gamma_0}=m(t),\\ \int_{\Gamma_0}\frac{\partial y}{\partial n_A}ds=0,\qquad y(x,0)=y_0(x)\quad\text{in }\Omega,\end{gathered}\tag{1.2}$$

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where $y = y(x,t)$ is the *state*, $b = b(x,t)$ is the *control*, $m(t) \in L^2(0,T)$ is an unknown function which depends only on the time variable t, y_0 is the initial state, and

$$\frac{\partial y}{\partial n_A} = \sum_{i,j} a_{ij}(x)\frac{\partial y}{\partial x_j} n_i. \tag{1.3}$$

In system (1.2), the *state space* is chosen as $L^2(\Omega)$, and the *control space* is $L^2(\omega)$. Let

$$Y \triangleq \Big\{ y \in C([0,T];L^2(\Omega)) \bigcap L^2(0,T;H^1(\Omega)) \mid y|_{\Gamma_1} = 0,\ y|_{\Gamma_0} = m(\cdot) \in L^2(0,T) \Big\}. \tag{1.4}$$

It can be shown that for any $y_0 \in L^2(\Omega)$ and $b \in L^2(\omega \times (0,T))$, system (1.2) admits one and only one weak solution $y \in Y$ (cf. [5, 6, 8]).

The null exact controllability problem of (1.2) is formulated as follows: for any given $y_0 \in L^2(\Omega)$, find a control $b(x,t) \in L^2(\omega \times (0,T))$ (if possible) such that the weak solution $y(\cdot) \in Y$ satisfies $y(T) = 0$.

There are many concrete physical backgrounds for problem (1.2), for example, the problem of resistivity well logging, the unstable temperature field around an underground electric cable, and so on (cf. [5, 6]).

In recent years, great progress has been made in the exact controllability problem of the linear and semilinear partial differential equations with Dirichlet or Neumann boundary condition, or other sorts of pointwise boundary value conditions ([1–4, 7, 9], and the references cited therein). However, to the author's best knowledge, there is no reference devoted to the same problem for the parabolic equations but with a spatial nonlocal boundary condition. In this paper, we will show the null exact controllability for system (1.2). By duality, the problem is reduced to the obtention of an observability inequality for the corresponding adjoint equation, which in turn is derived by means of a global Carleman-type estimate. Our method is stimulated by that in [4].

The rest of this paper is organized as follows. In Section 2, we state some preliminary results and our main results. The final section, Section 3, is devoted to the proof of our main theorem.

2. Main result

Throughout this paper, C denotes a positive constant depending only on λ, μ, Ω, T, and ω, which may change from line to line.

To begin with, we fix ω_0 to be a nonempty open subset of Ω such that $\overline{\omega}_0 \subset \omega$. Let $\psi \in C^\infty(\overline{\Omega})$ satisfy $\psi > 0$ in Ω, $\psi = 0$ on $\partial\Omega$, and $|\nabla\psi(x)| > 0$ for all $x \in \Omega_0 = \Omega \backslash \omega_0$. The existence of function ψ was proved in [7]. In this paper, we further assume that the following technical condition holds:

$$\left(\frac{\partial \psi}{\partial n}\right)^2 \sum_{i,j} a_{ij} n_i n_j |_{\Gamma_0} = \text{Const}. \tag{2.1}$$

This technical condition admits several interesting cases such as $\Omega = \{x \in \mathbb{R}^n \mid r < |x| < R\}$

for some $0 < r < R < \infty$, $a_{ij}(x) = \delta_i^j$, and $\partial\psi/\partial n = \text{Const}$ on Γ_0. In this case, one may choose $\psi(x) = (|x|^2 - r^2)(R^2 - |x|^2)$.

The main result in this paper is stated as follows.

THEOREM 2.1. *Under the assumption (2.1), system (1.2) is null exact controllable.*

By means of the usual duality argument (see, e.g., [3, 9]), the proof of Theorem 2.1 is easily reduced to the obtention of an observability inequality for the following adjoint system of system (1.2):

$$
\begin{gathered}
-Lu \equiv -\frac{\partial u}{\partial t} - \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0 \quad \text{in } Q, \\
u|_{\Gamma_1} = 0, \qquad u|_{\Gamma_0} = c(t), \\
\int_{\Gamma_0} \frac{\partial u}{\partial n_A} ds = 0, \qquad u(x,T) = u(T) \quad \text{in } \Omega,
\end{gathered}
\tag{2.2}
$$

where $u(T) \in L^2(\Omega)$, and similarly to system (1.2), $c(\cdot) \in L^2(0,T)$ is an unknown function. More precisely, we need to show the following.

THEOREM 2.2. *Under the assumption (2.1), there is a constant $C > 0$ such that solutions $u \in Y$ of system (2.2) satisfy*

$$
||u(0)||_{L^2(\Omega)} \le C||u(x,t)||_{L^2(\omega\times(0,T))}. \tag{2.3}
$$

Remark 2.3. It would be quite interesting to drop the technical condition (2.1). But this is by now an unsolved problem.

3. Proof of the main theorem

It suffices to prove Theorem 2.2. To this end, for any given parameters λ and μ, we set

$$
\begin{gathered}
\alpha(x,t) = \big(t(T-t)\big)^{-1}\big(e^{\mu\psi(x)} - e^{2\mu\|\psi\|_{C(\overline{\Omega})}}\big), \\
\varphi(x,t) = \big(t(T-t)\big)^{-1}e^{\mu\psi(x)}, \qquad \theta(x,t) = e^{\lambda\alpha(x,t)}.
\end{gathered}
\tag{3.1}
$$

Clearly, Theorem 2.2 is an easy consequence of the following global Carleman-type estimate for solutions of (2.2).

THEOREM 3.1. *Let (2.1) hold. Then there exist a constant μ_1 and a function $\lambda_1 : \mathbb{R}^+ \to (1,\infty)$ such that for any $\mu > \mu_1$ and $\lambda > \lambda_1(\mu)$, solutions $u \in Y$ of system (2.2) satisfy*

$$
\begin{aligned}
\lambda^3 \int_Q \theta^2\varphi^3 u^2 dx\,dt &+ \lambda\int_Q \theta^2\varphi|\nabla u|^2 dx\,dt + \lambda^{-1}\int_Q \theta^2\varphi^{-1}\big[u_t^2 + (\Delta u)^2\big]dx\,dt \\
&\le C\lambda^3 \int_{Q^\omega} \theta^2\varphi^3 u^2 dx\,dt.
\end{aligned}
\tag{3.2}
$$

The rest of this section is devoted to prove Theorem 3.1. For this, we need the following pointwise estimate for parabolic operator Lu, which is a special case of [4, Lemma 3.1].

Lemma 3.2. *Let $u,\alpha \in C^2(\mathbb{R}^{n+1})$ and $\lambda > 0$ be given. Put $\theta = e^{\lambda\alpha}$ and $v = \theta u$. Then it holds that*

$$
\begin{aligned}
\theta^2|Lu|^2 \geq & \Big[-\lambda\alpha_t v^2 - \sum_{i,j}\Big(a_{ij}v_{x_i}v_{x_j} + \lambda(a_{ij})_{x_j}\alpha_{x_i}v^2 + \lambda a_{ij}\alpha_{x_ix_j}v^2 - \lambda^2 a_{ij}\alpha_{x_i}\alpha_{x_j}v^2\Big)\Big]_t \\
& + \sum_j \Big[2\sum_i (a_{ij}v_{x_i}v_t + \lambda^2 a_{ij}\alpha_{x_i}\alpha_t v^2) \\
& \qquad + 2\lambda\sum_i \Big(a_{ij}a_{\ell m}\alpha_{x_i}v_{x_\ell}v_{x_m} + a_{ij}(a_{\ell m}\alpha_{x_\ell x_m})_{x_i}v^2 - 2a_{ij}a_{\ell m}\alpha_{x_\ell}v_{x_i}v_{x_m} \\
& \qquad\qquad - 2a_{ij}a_{\ell m}\alpha_{x_\ell x_m}vv_{x_i} + \lambda a_{ij}a_{\ell m}\alpha_{x_i}\alpha_{x_\ell x_m}v^2 \\
& \qquad\qquad + \lambda a_{ij}(a_{\ell m})_{x_m}\alpha_{x_\ell}\alpha_{x_m}v^2 - \lambda^2 a_{ij}a_{\ell m}\alpha_{x_\ell}\alpha_{x_m}v^2\Big)\Big]_{x_j} \\
& + \Big\{\lambda\alpha_{tt} - \lambda\sum_{i,j}\big[((a_{ij})_{x_j}\alpha_{x_i})_t + (a_{ij}\alpha_{x_ix_j})_t - \lambda(\alpha_{ij}\alpha_{x_i}\alpha_{x_j})_t \\
& \qquad\qquad - 2\lambda(a_{ij}\alpha_{x_i}\alpha_t)_{x_j} + 4\lambda a_{ij}\alpha_{x_ix_j}\alpha_t\big] \\
& \quad + 2\lambda\sum_{i,j,\ell,m}\big[2\lambda(a_{ij})_{x_j}a_{\ell m}\alpha_{x_i}\alpha_{x_\ell x_m} - (a_{ij}(a_{\ell m}\alpha_{x_\ell x_m})_{x_j})_{x_i} - \lambda((a_{\ell m})_{x_m}a_{ij}\alpha_{x_i}\alpha_{x_\ell})_{x_j} \\
& \qquad\qquad - \lambda(a_{ij}a_{\ell m}\alpha_{x_ix_j}\alpha_{x_\ell})_{x_m} - 2\lambda a_{ij}a_{\ell m}\alpha_{x_ix_j}\alpha_{x_\ell x_m} \\
& \qquad\qquad + \lambda^2(a_{ij}a_{\ell m}\alpha_{x_i}\alpha_{x_j}\alpha_{x_\ell})_{x_m} - 2\lambda^2 a_{ij}a_{\ell m}\alpha_{x_i}\alpha_{x_j}\alpha_{x_\ell x_m}\big]\Big\}v^2 \\
& + 2\lambda\sum_{i,j,\ell,m}\big[2a_{ij}(a_{\ell m}\alpha_{x_\ell})_{x_j}v_{x_i}v_{x_m} - (a_{ij}a_{\ell m})_{x_m}\alpha_{x_\ell}v_{x_i}v_{x_j} + a_{ij}a_{\ell m}\alpha_{x_\ell x_m}v_{x_i}v_{x_j}\big].
\end{aligned}
\tag{3.3}
$$

Proof of Theorem 3.1. The main idea of our proof is to use the pointwise estimate in Lemma 3.2. The proof is divided into several steps.
Step 1. Recall that $\theta = e^{\lambda\alpha}$, $v = \theta u$. We claim that

$$
\begin{aligned}
&\int_Q \theta^2(Lu)^2dx\,dt - \int_Q \operatorname{div} V\,dx\,dt + C\lambda^3\mu^3\int_Q \varphi^3v^2dx\,dt + C\lambda\mu^2\int_Q\varphi|\nabla v|^2dx\,dt \\
&\quad \geq 2\Lambda^2\lambda^3\mu^4\int_Q\varphi^3|\nabla\psi|^4v^2dx\,dt + 2\Lambda^2\lambda\mu^2\int_Q\varphi|\nabla v|^2|\nabla\psi|^2dx\,dt,
\end{aligned}
\tag{3.4}
$$

where $V = (V_1, V_2, \dots, V_n)$, and

$$
\begin{aligned}
V_j = & -2\sum_i (a_{ij}v_{x_i}v_t + \lambda^2 a_{ij}\alpha_{x_i}\alpha_t v^2) \\
& + 2\lambda\sum_{i,\ell,m}\big(a_{ij}a_{\ell m}\alpha_{x_i}v_{x_\ell}v_{x_m} - 2a_{ij}a_{\ell m}\alpha_{x_\ell}v_{x_i}v_{x_m} + \lambda a_{ij}(a_{\ell m})_{x_m}\alpha_{x_i}\alpha_{x_m}v^2 \\
& \qquad - \lambda^2 a_{ij}a_{\ell m}\alpha_{x_i}\alpha_{x_\ell}\alpha_{x_m}v^2 + \lambda a_{ij}a_{\ell m}\alpha_{x_i}\alpha_{x_\ell x_m}v^2 \\
& \qquad - 2a_{ij}a_{\ell m}\alpha_{x_\ell x_m}vv_{x_i} + a_{ij}(a_{\ell m}\alpha_{x_\ell x_m})_{x_i}v^2\big).
\end{aligned}
\tag{3.5}
$$

By the definition of α and φ in (3.1), it is easy to see that

$$\int_Q \Big[-\lambda\alpha_t v^2 + \sum_{i,j} \big(a_{ij} v_{x_i} v_{x_j} + \lambda (a_{ij})_{x_j} \alpha_{x_i} v^2 - \lambda^2 a_{ij} \alpha_{x_i} \alpha_{x_j}^2 + \lambda a_{ij} \alpha_{x_i x_j} v^2\big) \Big]_t dt = 0. \tag{3.6}$$

Let us estimate the last three "energy" terms of first order in the right-hand side of (3.3). First,

$$\begin{aligned} &\Big| \int_Q 4\lambda \sum_{i,j,\ell,m} a_{ij} (a_{\ell m}\alpha_{x_\ell})_{x_j} v_{x_i} v_{x_m} dx\,dt \Big| \\ &\quad = \Big| \int_Q 4\lambda \sum_{i,j,\ell,m} \big(a_{ij}(a_{\ell m})_{x_j} \mu\psi_{x_\ell}\varphi v_{x_i} v_{x_m} + a_{ij} a_{\ell m} (\mu\psi_{x_\ell x_j}\varphi + \mu^2 \psi_{x_\ell}\psi_{x_j}\varphi) v_{x_i} v_{x_m}\big) dx\,dt \Big| \\ &\quad \le C\lambda\mu \int_Q \varphi |\nabla v|^2 dx\,dt + C\lambda\mu^2 \int_Q \varphi |\nabla v|^2 dx\,dt, \end{aligned} \tag{3.7}$$

where C is a positive constant for λ large enough.

Next, it is easy to see that

$$\begin{aligned} &\Big| -\int_Q 2\lambda \sum_{i,j,\ell,m} (a_{ij} a_{\ell m})_{x_m} \alpha_{x_\ell} v_{x_i} v_{x_j} dx\,dt \Big| \le C\lambda\mu \int_Q \varphi |\nabla v|^2 dx\,dt, \\ &\int_Q 2\lambda \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \alpha_{x_\ell x_m} v_{x_i} v_{x_j} dx\,dt \\ &\quad = \int_Q 2\lambda \sum_{i,j,\ell,m} \big(\mu a_{ij} a_{\ell m} \psi_{x_\ell x_m} \varphi v_{x_i} v_{x_j} + \mu^2 a_{ij} a_{\ell m} \psi_{x_\ell}\psi_{x_m} \varphi v_{x_i} v_{x_j}\big) dx\,dt \\ &\quad \ge -C\lambda\mu \int_Q \varphi |\nabla v|^2 dx\,dt + 2\lambda\mu^2\Lambda^2 \int_Q \varphi |\nabla\psi|^2 |\nabla v|^2 dx\,dt. \end{aligned} \tag{3.8}$$

It remains to deal with the "energy" term of zero order, that is, $\int_Q \{\cdots\} v^2 dx\,dt$, in the right-hand side of (3.3). Similarly to [4], we have

$$\begin{aligned} &-2\int_{\mathcal{Q}} \lambda^3 \sum_{i,j,\ell,m} \big(2a_{ij} a_{\ell m} \alpha_{x_i}\alpha_{x_j}\alpha_{x_\ell x_m} v^2 - (a_{ij} a_{\ell m} \alpha_{x_i}\alpha_{x_j}\alpha_{x_\ell})_{x_m} v^2\big) dx\,dt \\ &\quad = -\int_{\mathcal{Q}} 2\lambda^3\mu^3 \sum_{i,j,\ell,m} \big(2a_{ij} a_{\ell m} \psi_{x_i}\psi_{x_j}\psi_{x_\ell x_m} \varphi^3 v^2 + 2\mu a_{ij} a_{\ell m} \psi_{x_i}\psi_{x_j}\psi_{x_\ell}\psi_{x_m} \varphi^3 v^2 \\ &\qquad\qquad - (a_{ij} a_{\ell m} \psi_{x_i}\psi_{x_j}\psi_{x_\ell})_{x_m} \varphi^3 v^2 - 3\mu a_{ij} a_{\ell m} \psi_{x_i}\psi_{x_j}\psi_{x_\ell}\psi_{x_m} \varphi^3 v^2\big) dx\,dt \\ &\quad \ge -C\lambda^3\mu^3 \int_{\mathcal{Q}} \varphi^3 v^2 dx\,dt + 2\lambda^3\mu^4\Lambda^2 \int_{\mathcal{Q}} \varphi^3 |\nabla\psi|^4 v^2 dx\,dt. \end{aligned} \tag{3.9}$$

Therefore, for large λ, the following estimate holds:

$$\begin{aligned}\int_Q \Big\{&\lambda\alpha_{tt} - \lambda\sum_{i,j}\big[((a_{ij})_{x_j}\alpha_{x_i})_t - \lambda(a_{ij}\alpha_{x_i}\alpha_{x_j})_t + (a_{ij}\alpha_{x_ix_j})_t - 2\lambda(a_{ij}\alpha_{x_i}\alpha_t)_{x_j} + 4\lambda a_{ij}\alpha_{x_ix_j}\alpha_t\big]\\ &+2\lambda\sum_{i,j,\ell,m}\Big[\lambda^2(a_{ij}a_{\ell m}\alpha_{x_i}\alpha_{x_j}\alpha_{x_\ell})_{x_m} - \lambda(a_{ij}a_{\ell m}\alpha_{x_ix_j}\alpha_{x_\ell})_{x_m} - \lambda((a_{\ell m})_{x_m}a_{ij}\alpha_{x_i}\alpha_{x_j})_{x_j}\\ &\qquad + 2\lambda(a_{ij})_{x_j}a_{\ell m}\alpha_{x_i}\alpha_{x_\ell x_m} - 2\lambda^2 a_{ij}a_{\ell m}\alpha_{x_i}\alpha_{x_j}\alpha_{x_\ell x_m}\\ &\qquad - 2\lambda a_{ij}a_{\ell m}\alpha_{x_ix_j}\alpha_{x_\ell x_m} - \Big(a_{ij}(a_{\ell m}\alpha_{x_\ell x_m})_{x_j}\Big)_{x_i}\Big]\Big\}v^2\,dx\,dt\\ \geq{}& -C\lambda^3\mu^3\int_{\mathcal{Q}}\varphi^3v^2\,dx\,dt + 2\lambda^3\mu^4\Lambda^2\int_Q\varphi^3|\nabla\psi|^4v^2\,dx\,dt.\end{aligned} \tag{3.10}$$

Step 2. From (3.4), one finds

$$\begin{aligned}C\Big(\int_Q\theta^2(Lu)^2\,dx\,dt - \int_Q\operatorname{div}V\,dx\,dt + \lambda^3\mu^3\int_Q\varphi^3v^2\,dx\,dt + \lambda\mu^2\int_Q\varphi|\nabla v|^2\,dx\,dt\Big)\\ \geq \lambda^3\mu^4\int_Q\varphi^3|\nabla\psi|^4v^2\,dx\,dt + \lambda\mu^2\int_Q\varphi|\nabla\psi|^2|\nabla v|^2\,dx\,dt.\end{aligned} \tag{3.11}$$

Set

$$Q^{\omega_0} = \omega_0\times(0,T). \tag{3.12}$$

Noting that $|\nabla\psi(x)| > 0$ for all $x\in\Omega_0 = \overline{\Omega}\setminus\omega_0$, by (3.11), it is easy to see that

$$\begin{aligned}C\Big(\int_Q\theta^2(Lu)^2\,dx\,dt - \int_Q\operatorname{div}V\,dx\,dt + \lambda^3\mu^3\int_{Q^{\omega_0}}\varphi^3v^2\,dx\,dt + \lambda\mu^2\int_{Q^{\omega_0}}\varphi|\nabla v|^2\,dx\,dt\Big)\\ \geq \lambda^3\mu^4\int_Q\varphi^3v^2\,dx\,dt + \lambda\mu^2\int_Q\varphi|\nabla v|^2\,dx\,dt.\end{aligned} \tag{3.13}$$

Returning v to $e^{\lambda\alpha}u$ in (3.13), we arrive at

$$\begin{aligned}C\Big(\int_Q\theta^2(Lu)^2\,dx\,dt - \int_Q\operatorname{div}V\,dx\,dt + \lambda^3\int_{Q^{\omega_0}}\theta^2\varphi^3u^2\,dx\,dt + \lambda\int_{Q^{\omega_0}}\theta^2\varphi|\nabla u|^2\,dx\,dt\Big)\\ \geq \lambda^3\int_Q\theta^2\varphi^3u^2\,dx\,dt + \lambda\int_Q\theta^2\varphi|\nabla u|^2\,dx\,dt.\end{aligned} \tag{3.14}$$

Step 3. The purpose of this step is to get rid of the fourth term in the left-hand side of (3.14). To this end, we multiply Lu by $\tilde{\chi}\theta^2\varphi u$ and then integrate it over Q, where $\tilde{\chi}\in C_0^\infty(\omega)$, $\tilde{\chi}=1$ in ω_0, and $\tilde{\chi}=0$ in $\Omega\backslash\omega$. Then, we obtain

$$\int_Q\frac{\partial u}{\partial t}\tilde{\chi}\theta^2\varphi u\,dx\,dt + \int_Q\sum_{i,j}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big)\tilde{\chi}\theta^2\varphi u\,dx\,dt = \int_Q(Lu)\tilde{\chi}\theta^2\varphi u\,dx\,dt. \tag{3.15}$$

Using integration by parts, it is easy to deduce from (3.15) that

$$\lambda \int_{Q^{\omega_0}} \theta^2 \varphi |\nabla u|^2 dx\,dt \le C\left(\int_Q \theta^2 (Lu)^2 dx\,dt + \lambda^3 \int_{Q^{\omega}} \theta^2 \varphi^3 u^2 dx\,dt \right). \tag{3.16}$$

Combining (3.13) and (3.16), we end up with

$$\begin{aligned} &C\left(\int_Q \theta^2 (Lu)^2 dx\,dt + \lambda^3 \int_{Q^{\omega}} \theta^2 \varphi^3 u^2 dx\,dt - \int_Q \operatorname{div} V\,dx\,dt \right) \\ &\quad \ge \lambda^3 \int_Q \theta^2 \varphi^3 u^2 dx\,dt + \lambda \int_Q \theta^2 \varphi |\nabla u|^2 dx\,dt. \end{aligned} \tag{3.17}$$

Step 4. This step is to estimate the "divergence" term $\operatorname{div} V$. Denote the terms on the right-hand side of (3.5) by I_i, $i = 1, 2, \ldots, 9$. First,

$$\begin{aligned} I_1 + I_2 &= \int_Q \sum_j \left(2 \sum_i \left(a_{ij} v_{x_i} v_t + \lambda^2 a_{ij} \alpha_{x_i} \alpha_t v^2 \right) \right)_{x_j} \\ &= \lambda^2 \mu \int_0^T \theta^2 \varphi_t c^2(t) dt \int_{\Gamma_0} \sum_{i,j} a_{ij} \psi_{x_i} n_j ds - 2\lambda^2 \mu \int_0^T \theta^2 \varphi \alpha_t c^2(t) dt \int_{\Gamma_0} \sum_{i,j} a_{ij} \psi_{x_i} n_j ds. \end{aligned} \tag{3.18}$$

Next,

$$\begin{aligned} I_3 + I_5 + I_9 &= \int_Q \sum_j \left(2\lambda \sum_{i,\ell,m} \left(a_{ij} a_{\ell m} \alpha_{x_i} v_{x_\ell} v_{x_m} - 2 a_{ij} a_{\ell m} \alpha_{x_\ell} v_{x_i} v_{x_m} - \lambda^2 a_{ij} a_{\ell m} \alpha_{x_i} \alpha_{x_m} v^2 \right) \right)_{x_j} dx\,dt \\ &= -4\lambda^3 \mu^3 \int_0^T \varphi^3 c^2(t) \theta^2 dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_i} \psi_{x_\ell} \psi_{x_m} n_j ds \\ &\quad - 4\lambda^2 \mu^2 \int_0^T \varphi^2 \theta^2 c(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_\ell} \psi_{x_i} u_{x_m} n_j ds \\ &\quad - 2 \int_0^T \lambda \mu \varphi \theta^2 dt \int_{\Gamma} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_\ell} u_{x_i} u_{x_m} n_j ds. \end{aligned} \tag{3.19}$$

Further,

$$\begin{aligned} I_4 &= \int_Q \sum_j \left(2\lambda \sum_i a_{ij} \left(a_{\ell m} \alpha_{x_\ell x_m} \right)_{x_i} v^2 \right)_{x_j} dx\,dt \\ &= 2\lambda \mu^2 \int_0^T \theta^2 \varphi c^2(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} \left(a_{\ell m} \right)_{x_i} \psi_{x_\ell} \psi_{x_m} n_j ds \\ &\quad + 2\lambda \mu \int_0^T \theta^2 \varphi c^2(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} \left(a_{\ell m} \right)_{x_i} \psi_{x_\ell x_m} n_j ds \\ &\quad + 2\lambda \mu^3 \int_0^T \theta^2 \varphi c^2(t) dt \int_{\Gamma_0} \sum_{i,j,\ell,m} a_{ij} a_{\ell m} \psi_{x_i} \psi_{x_\ell} \psi_{x_m} n_j ds \end{aligned}$$

$$
\begin{aligned}
&+2\lambda\mu^2\int_0^T\theta^2\varphi c^2(t)dt\int_{\Gamma_0}\sum_{i,j,\ell,m}a_{ij}a_{\ell m}\psi_{x_ix_\ell}\psi_{x_m}n_j ds\\
&+2\lambda\mu^2\int_0^T\theta^2\varphi c^2(t)dt\int_{\Gamma_0}\sum_{i,j,\ell,m}a_{ij}a_{\ell m}\psi_{x_\ell}\psi_{x_ix_m}n_j ds\\
&+2\lambda\mu\int_0^T\theta^2\varphi c^2(t)dt\int_{\Gamma_0}\sum_{i,j,\ell,m}a_{ij}a_{\ell m}\psi_{x_ix_\ell x_\ell}n_j ds\\
&\geq C\lambda\mu^3\int_0^T\theta^2\varphi c^2(t)dt\int_{\Gamma_0}\sum_{i,j,\ell,m}a_{ij}a_{\ell m}\psi_{x_i}\psi_{x_\ell}\psi_{x_m}n_j ds.
\end{aligned}
\tag{3.20}
$$

Further,

$$
\begin{aligned}
I_6 &= -\int_Q\sum_j\left(4\lambda\sum_{i,\ell,m}a_{ij}a_{\ell m}\alpha_{x_\ell x_m}vv_{x_i}\right)_{x_j}dx\,dt\\
&= -4\lambda^2\mu^3\int_0^T\theta^2\varphi^2c^2(t)dt\int_{\Gamma_0}\sum_{i,j,\ell,m}a_{ij}a_{\ell m}\psi_{x_i}\psi_{x_\ell}\psi_{x_m}n_j ds\\
&\quad -4\lambda\mu^2\int_0^T\theta^2\varphi^2c^2(t)dt\int_{\Gamma_0}\sum_{i,j,\ell,m}a_{ij}a_{\ell m}\psi_{x_i}\psi_{x_\ell x_m}n_j ds\\
&\quad -4\lambda\mu^2\int_0^T\theta^2\varphi c(t)dt\int_{\Gamma_0}\sum_{i,j,\ell,m}a_{ij}a_{\ell m}\psi_{x_\ell}\psi_{x_m}u_{x_i}n_j ds\\
&\quad -4\lambda\mu\int_0^T\theta^2\varphi c(t)dt\int_{\Gamma_0}\sum_{i,j,\ell,m}a_{ij}a_{\ell m}\psi_{x_\ell x_m}u_{x_i}n_j ds.
\end{aligned}
\tag{3.21}
$$

Finally,

$$
\begin{aligned}
I_7+I_8 &= \int_Q\sum_j\left(2\lambda^2\sum_{i,\ell,m}\left(a_{ij}a_{\ell m}\alpha_{x_i}\alpha_{x_\ell x_m}v^2+a_{ij}\left(a_{\ell m}\right)_{x_m}\alpha_{x_i}\alpha_{x_m}v^2\right)\right)_{x_j}dx\,dt\\
&= 2\lambda^2\mu^3\int_0^T\theta^2\varphi^2c^2(t)dt\int_{\Gamma_0}\sum_{i,j,\ell,m}a_{ij}a_{\ell m}\psi_{x_i}\psi_{x_\ell}\psi_{x_m}n_j ds\\
&\quad +2\lambda^2\mu^2\int_0^T\theta^2\varphi^2c^2(t)dt\int_{\Gamma_0}\sum_{i,j,\ell,m}a_{ij}a_{\ell m}\psi_{x_i}\psi_{x_\ell x_m}n_j ds\\
&\quad +2\lambda^2\mu^2\int_0^T\theta^2\varphi^2c^2(t)dt\int_{\Gamma_0}\sum_{i,j,\ell,m}a_{ij}\left(a_{\ell m}\right)_{x_m}\psi_{x_i}\psi_{x_m}n_j ds.
\end{aligned}
\tag{3.22}
$$

Now, combining (3.18)–(3.22) and using the technical condition (2.1), we conclude that, for large λ and μ, it holds that

$$
\int_Q\operatorname{div}V\,dx\,dt\geq C\lambda^3\mu^3\int_0^T\theta^2\varphi^3c^2(t)dt-4\lambda\mu\Lambda\int_0^T\theta^2\varphi dt\int_\Gamma|\nabla u|^2\sum_{i,j}a_{ij}\psi_{x_i}n_j ds.
\tag{3.23}
$$

Step 5. It remains to estimate $\int_Q (\lambda\varphi)^{-1}\theta^2(u_t^2 + \sum_{i,j}(a_{ij}u_{x_i})_{x_j})^2 dx\,dt$. For this, we observe that

$$\left(v_t + \sum_{i,j}(a_{ij}v_{x_i})_{x_j}\right)^2 \le C\Big[\lambda^2\alpha_t^2 v^2 + \sum_{i,j}(\lambda^2 (a_{ij})_{x_j}^2\alpha_{x_i}^2 v^2 + \lambda^2 a_{ij}^2\alpha_{x_ix_j}^2 v^2 \\ + \lambda^4 a_{ij}^2\alpha_{x_i}^2\alpha_{x_j}^2 v^2 + \lambda^2 a_{ij}^2\alpha_{ij}^2\alpha_{x_i}^2 v_{x_j}^2 + (a_{ij})_{x_j}^2 v_{x_i}^2)\Big]. \tag{3.24}$$

This implies

$$\begin{aligned}
&(\lambda\varphi)^{-1}\left(v_t + \sum_{i,j}(a_{ij}v_{x_i})_{x_j}\right)^2 \\
&\quad\le C(\lambda\varphi)^{-1}\Big[\lambda^2\alpha_t^2 v^2 + \sum_{i,j}\Big(\lambda^2 (a_{ij})_{x_j}^2\alpha_{x_i}^2 v^2 + \lambda^2 a_{ij}^2\alpha_{x_ix_j}^2 v^2 \\
&\qquad + \lambda^4 a_{ij}^2\alpha_{x_i}^2\alpha_{x_j}^2 v^2 + \lambda^2 a_{ij}^2\alpha_{ij}^2\alpha_{x_i}^2 v_{x_j}^2 + (a_{ij})_{x_j}^2 v_{x_i}^2\Big)\Big] \\
&\quad\le C(\lambda^3\mu^4\varphi^3 v^2 + \lambda\mu\varphi|\nabla v|^2).
\end{aligned} \tag{3.25}$$

Noting that

$$\begin{aligned}
&2\int_Q \sum_{i,j}(\lambda\varphi)^{-1}v_t(a_{ij}v_{x_i})_{x_j}\,dx\,dt \\
&\quad= 2\sum_{i,j}\Big(\int_Q (a_{ij}(\lambda\varphi)^{-1}v_{x_i}v_t)_{x_j}\,dx\,dt - 2\int_Q a_{ij}(\lambda\varphi)_{x_j}^{-1}v_{x_i}v_t\,dx\,dt + \int_Q a_{ij}(\lambda\varphi)_t^{-1}v_{x_i}v_{x_j}\,dx\,dt \\
&\qquad + \int_Q (a_{ij})_t(\lambda\varphi)^{-1}v_{x_i}v_{x_j}\,dx\,dt + \int_Q a_{ij}((\lambda\varphi)^{-1}v_t)_{x_i}v_{x_j}\,dx\,dt\Big),
\end{aligned} \tag{3.26}$$

we get

$$\begin{aligned}
2\int_Q (\lambda\varphi)^{-1}\sum_{i,j}(a_{ij}v_{x_i})_{x_j}v_t\,dx\,dt &\le \frac{1}{2}\int_Q (\lambda\varphi)^{-1}v_t^2\,dx\,dt + C\lambda\int_Q \varphi|\nabla v|^2\,dx\,dt \\
&\quad + \lambda\mu\int_0^T \theta\alpha_t c^2(t)\,dt\int_{\Gamma_0}\sum_{i,j}a_{ij}\psi_{x_i}n_j\,ds.
\end{aligned} \tag{3.27}$$

By (3.24)–(3.27), we obtain that

$$\begin{aligned}
&\int_Q (\lambda\varphi)^{-1}\left[v_t^2 + (v_{x_ix_j})^2\right]dx\,dt \\
&\quad\le C\Big(\int_Q (Lu)^2\theta^2\,dx\,dt + \lambda^3\int_{Q^{\omega_0}}\mu^3\varphi^3 v^2\,dx\,dt + \lambda\int_{Q^{\omega_0}}\mu\varphi|\nabla v|^2\,dx\,dt\Big) \\
&\qquad + \lambda\mu\int_0^T \theta\alpha_t c^2(t)\,dt\int_{\Gamma_0}\sum_{i,j}a_{ij}\psi_{x_i}n_j\,ds.
\end{aligned} \tag{3.28}$$

This gives

$$\begin{aligned}
&\int_Q (\lambda\varphi)^{-1}[v_t^2+(\Delta v)^2]\,dx\,dt \\
&\quad \le C\Big(\int_Q (Lu)^2\theta^2\,dx\,dt+\lambda^3\int_{Q^{\omega_0}}\mu^3\varphi^3v^2\,dx\,dt+\lambda\int_{Q^{\omega_0}}\mu\varphi|\nabla v|^2\,dx\,dt\Big) \\
&\qquad +\lambda\mu\int_0^T\theta\alpha_t c^2(t)\,dt\int_{\Gamma_0}\sum_{i,j}a_{ij}\psi_{x_i}n_j\,ds.
\end{aligned} \tag{3.29}$$

It is easy to see that the last term of the above inequality can be absorbed by (3.23) for large λ and μ. Finally, replacing v by $e^{\lambda\alpha}u$ and combining (2.2), (3.16), (3.17), (3.23), and (3.29), we get the desired estimate (3.2). This completes the proof of Theorem 3.1. □

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Zhongqi Yin: Department of Engineering and Technology of Caotang School,
Sichuan Normal University, Chengdu 610072, China
E-mail address: zhongqiyin@sohu.com


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