CHOVER-TYPE LAWS OF THE ITERATED LOGARITHM FOR WEIGHTED SUMS OF ρ^*-MIXING SEQUENCES

GUANG-HUI CAI

Received 24 September 2004; Revised 31 May 2005; Accepted 31 May 2005

To derive a Baum-Katz-type result, we establish a Chover-type law of the iterated logarithm for the weighted sums of ρ^*-mixing and identically distributed random variables with a distribution in the domain of a stable law. Our result obtained not only generalizes the main results of Peng and Qi (2003) and Qi and Cheng (1996) to ρ^*-mixing sequences of random variables, but also improves them.

1. Introduction

Let $\{X_i,\ i \geq 1\}$ be independent and identically distributed (i.i.d.) with symmetric stable distributions, which belong to the domain of normal attraction and nongeneration. So, their characteristic functions are of the forms:

$$E\exp(itX_i) = \exp(-|t|^\alpha), \quad t \in R,\ i \geq 1. \tag{1.1}$$

Chover [4] has obtained that

$$\limsup_{n\to\infty}\left(\frac{|\sum_{i=1}^n X_i|}{n^{1/\alpha}}\right)^{1/\log\log n} = e^{1/\alpha} \quad \text{a.s.} \tag{1.2}$$

We call this a Chover-type LIL (laws of the iterated logarithm). This type LIL has been established by Vasudeva and Divanji [13], Zinchenko [14] for delayed sums, by Chen and Huang [3] for geometric weighted sums, and by Chen [2] for weighted sums. Qi and Cheng [11] extended the Chover-type law of the iterated logarithm for the partial sums to the case where the underlying distribution is in the domain of attraction of a nonsymmetric stable distribution (see below for details).

Let L_α denote a stable distribution with exponent $\alpha \in (0,2)$. Recall that the distribution of X is said to be *in the domain of attraction of* L_α if there exist some constants $A_n \in R$

Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2006, Article ID 65023, Pages 1–7
DOI 10.1155/JAMSA/2006/65023

and $B_n > 0$ such that

$$\frac{S_n - A_n}{B_n} \xrightarrow{d} L_\alpha. \tag{1.3}$$

Under (1.3), Qi and Cheng [11] and Peng and Qi [10] showed that

$$\limsup_{n\to\infty}\left(\frac{\left|\sum_{i=1}^{n} X_i - A_n\right|}{B_n}\right)^{1/\log\log n} = e^{1/\alpha} \quad \text{a.s.} \tag{1.4}$$

It is well known that (1.3) holds if and only if

$$1 - F(x) = \frac{C_1(x)l(x)}{x^\alpha}, \quad F(-x) = \frac{C_2(x)l(x)}{x^\alpha}, \quad \text{for } x > 0, \tag{1.5}$$

where, for $x > 0$, $C_i(x) \geq 0$, $\lim_{x\to\infty} C_i(x) = C_i$, $i = 1,2$, $C_1 + C_2 > 0$, and $l(x) \geq 0$ is slowly varying in the sense of Karamata function, that is,

$$\lim_{t\to\infty}\frac{l(tx)}{l(t)} = 1, \quad \text{for } x > 0. \tag{1.6}$$

By Lin et al. [6, page 76, Exercise 21], we have $B_n = (nl(n))^{1/\alpha}$.

For nonempty sets $S, T \subset \mathcal{N}$, we define $\mathcal{F}_S = \sigma(X_k, k \in S)$. And we define the maximal correlation coefficient $\rho_n^* = \sup \operatorname{corr}(f, g)$ where the supremum is taken over all (S,T) with $\operatorname{dist}(S,T) \geq n$ and for all $f \in L_2(\mathcal{F}_S)$, $g \in L_2(\mathcal{F}_T)$, and $\operatorname{dist}(S,T) = \inf_{x\in S,\, y\in T}|x - y|$.

A sequence of random variables $\{X_n,\ n \geq 1\}$ on a probability space $\{\Omega, \mathcal{F}, P\}$ is called ρ^*-mixing if

$$\lim_{n\to\infty}\rho_n^* = 0. \tag{1.7}$$

As for ρ^*-mixing sequences of random variables, one can refer to Bryc and Smolenski [1], who established bounds for the moments of partial sums for a sequence of random variables satisfying

$$\lim_{n\to\infty}\rho_n^* < 1. \tag{1.8}$$

Peligrad [7] established a CLT. Peligrad [8] established an invariance principle. Peligrad and Gut [9] established Rosenthal-type maximal inequalities and Baum-Katz-type results. Utev and Peligrad [12] established an invariance principle of nonstationary sequences.

To derive a Baum-Katz-type result, the main purpose of this paper is to establish a Chover-type law of the iterated logarithm for the weighted sums of ρ^*-mixing and identically distributed random variables with a distribution in the domain of a stable law. Our result not only generalizes the main results of Peng and Qi [10] and Qi and Cheng [11] to ρ^*-mixing sequences of random variables, but also improves them.

Throughout this paper, let $h \in B[0,1]$ denote that the function h is bounded on [0,1]. C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leq Cb_n$.

2. The main results

In order to prove our results, we need the following lemma and definition.

Lemma 2.1 (Utev and Peligrad [12]). *Let $\{X_i,\ i \ge 1\}$ be a ρ^*-mixing sequence of random variables, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \ge 2$ and for every $i \ge 1$. Then there exists $C = C(p)$, such that*

$$E \max_{1 \le k \le n} \left| \sum_{i=1}^{k} X_i \right|^p \le C \left\{ \sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2 \right)^{p/2} \right\}. \tag{2.1}$$

Definition 2.2 (Lin and Lu [5]). *A function $f(x) > 0$ $(x > 0)$ is said to be quasimonotone nondecreasing, if*

$$\limsup_{x \to \infty} \sup_{0 \le t \le x} \frac{f(t)}{f(x)} < \infty. \tag{2.2}$$

Here are our main results.

Theorem 2.3. *Let $\{X, X_i,\ i \ge 1\}$ be a ρ^*-mixing sequence of identically distributed random variables. Let h be a bounded function on $[0,1]$, continuous at $x_0 \in (0,1)$. Let $S_n = \sum_{i=1}^{n} h(i/n) X_i$, $EX = 0$, when $\alpha > 1$. Let $f(x) > 0$ be quasimonotone nondecreasing and $\int_1^\infty (1/x f(x)) dx < \infty$. Then under condition (1.3), for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \le j \le n} |S_j| > \varepsilon (n f(n) l(n))^{1/\alpha} \right) < \infty. \tag{2.3}$$

Proof of Theorem 2.3. For any $i \ge 1$, define $X_i^{(n)} = X_i I(|X_i| \le a_n)$, $S_j^{(n)} = \sum_{i=1}^{j} (h(i/n) X_i^{(n)} - Eh(i/n) X_i^{(n)})$, where $a_n = (n f(n) l(n))^{1/\alpha}$. Then for any $\varepsilon > 0$,

$$\begin{aligned} &P\left(\max_{1 \le j \le n} |S_j| > \varepsilon a_n \right) \\ &\quad \le P\left(\max_{1 \le j \le n} |X_j| > a_n \right) + P\left(\max_{1 \le j \le n} |S_j^{(n)}| > \varepsilon a_n - \max_{1 \le j \le n} \left| \sum_{i=1}^{j} Eh\left(\frac{i}{n}\right) X_i^{(n)} \right| \right). \end{aligned} \tag{2.4}$$

First we show that

$$\frac{1}{a_n} \max_{1 \le j \le n} \left| \sum_{i=1}^{j} Eh\left(\frac{i}{n}\right) X_i^{(n)} \right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.5}$$

In fact, (i) when $0 < \alpha \le 1$, $h \in B[0,1]$. For any positive integers n, N,

$$\begin{aligned}\frac{1}{a_n}\max_{1\le j\le n}&\left|\sum_{i=1}^{j} Eh\left(\frac{i}{n}\right)X_i^{(n)}\right|\\ &\le \frac{1}{a_n}\sum_{i=1}^{n}E\left|h\left(\frac{i}{n}\right)X_i^{(n)}\right| \le \frac{Cn}{a_n}\int_{|x|\le a_n}|x|\,dF(x)\\ &\le \frac{Cn}{a_n}a_N + \frac{Cn}{a_n}\int_{a_N<|x|\le a_n}|x|\,dF(x) =: C(A+B).\end{aligned} \tag{2.6}$$

Since $f(x) > 0$ is a quasimonotone nondecreasing and by (1.5), we have, for $n \ge N$, N large enough,

$$\begin{aligned}B &= \frac{n}{a_n}\sum_{k=N+1}^{n}\int_{a_{k-1}<|x|\le a_k}|x|\,dF(x) \le \frac{n}{a_n}\sum_{k=N+1}^{n}a_kP(a_{k-1}<|X|\le a_k)\\ &\le C\sum_{k=N+1}^{n}kP(a_{k-1}<|X|\le a_k) \le CNP(|X|\ge a_N) + C\sum_{k=N}^{\infty}P(|X|\ge a_k)\\ &\le C\frac{1}{f(N)} + C\sum_{k=N}^{\infty}\frac{1}{kf(k)} \le C\frac{1}{f(N)} + C\int_N^{\infty}\frac{dx}{kf(k)} < \frac{\varepsilon}{4}.\end{aligned} \tag{2.7}$$

It is obvious that for each given N,

$$A \le C\frac{a_N}{(f(n))^{1/\alpha}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.8}$$

So, for $0 < \alpha \le 1$, we have (2.5).

(ii) When $1 < \alpha < 2$, using $EX_i = 0$, $h \in B[0,1]$, and (1.5), when $n \to \infty$, we have

$$\begin{aligned}\frac{1}{a_n}&\max_{1\le j\le n}\left|\sum_{i=1}^{j}Eh\left(\frac{i}{n}\right)X_i^{(n)}\right|\\ &= \frac{1}{a_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j}Eh\left(\frac{i}{n}\right)X_iI(|X_i|>a_n)\right| \le \frac{1}{a_n}\sum_{i=1}^{n}E\left|h\left(\frac{i}{n}\right)X_i\right|I(|X_i|>a_n)\\ &\le \frac{Cn}{a_n}E|X|I(|X|>a_n) = \frac{Cn}{a_n}\int_{a_n}^{\infty}P(|X|\ge x)\,dx = \frac{Cn}{a_n}\int_{a_n}^{\infty}\frac{Cl(n)}{x^{\alpha}}dx\\ &= \frac{n}{a_n}Ca_n^{1-\alpha} = \frac{C}{f(n)} < \frac{\varepsilon}{2}.\end{aligned} \tag{2.9}$$

So, for $1 < \alpha < 2$, we also have (2.5). Hence (2.5) holds for $0 < \alpha < 2$.

By (2.4) and (2.5), we have that

$$P\left(\max_{1\le j\le n}|S_j| > \varepsilon a_n\right) \le \sum_{j=1}^{n}P(|X_j|>a_n) + P\left(\max_{1\le j\le n}|S_j^{(n)}| > \frac{\varepsilon}{2}a_n\right), \tag{2.10}$$

for n large enough. Hence we need only to prove

$$I =: \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P(|X_j| > a_n) < \infty,$$
$$II =: \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\le j\le n} |S_j^{(n)}| > \frac{\varepsilon}{2} a_n \Big) < \infty. \tag{2.11}$$

From (1.5), it is easily seen that

$$I = \sum_{n=1}^{\infty} P(|X| > a_n) \le \sum_{n=1}^{\infty} \frac{C}{nf(n)} \le C\int_1^{\infty} \frac{dx}{xf(x)} < \infty. \tag{2.12}$$

By Lemma 2.1 and the fact that $h \in B[0,1]$, it follows that

$$\begin{aligned} II &\le C\sum_{n=1}^{\infty} n^{-1} E \max_{1\le j\le n} |S_j^{(n)}|^2 \frac{1}{a_n^2} \le C\sum_{n=1}^{\infty} n^{-1} \frac{1}{a_n^2} \Big(\sum_{i=1}^{n} E\Big| h\Big(\frac{i}{n}\Big) X_i^{(n)} \Big|^2 \Big) \\ &\le C\sum_{n=1}^{\infty} \frac{1}{a_n^2} E|X|^2 I(|X| \le a_n) = C\sum_{n=1}^{\infty} \frac{1}{a_n^2} \int_{|x|\le a_n} x^2 dF(x) \\ &= C\sum_{n=1}^{\infty} \frac{1}{a_n^2} \sum_{k=1}^{n} \int_{a_{k-1}<|x|\le a_k} x^2 dF(x) \le C\sum_{k=1}^{\infty} a_k^2 P(a_{k-1} < |X| \le a_k) \sum_{n=k}^{\infty} \frac{1}{a_n^2} \\ &\le C\sum_{k=1}^{\infty} kP(a_{k-1} < |X| \le a_k) \le C\int_1^{\infty} \frac{dx}{xf(x)} < \infty, \end{aligned} \tag{2.13}$$

which completes the proof of Theorem 2.3. □

Corollary 2.4. *Under the conditions of Theorem 2.3,*

$$\limsup_{n\to\infty} \Big(\frac{|S_n|}{B_n} \Big)^{1/\log\log n} \le e^{1/\alpha} \quad a.s. \tag{2.14}$$

Proof of Corollary 2.4. Notice that for any positive integer n, there exists an nonnegative integer k, such that $2^k \le n < 2^{k+1}$. And there exists a $t \in [0,1)$, such that $n = 2^{k+t}$. By (2.3), we have

$$\sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} (2^{k+1}-1)^{-1} P\Big(\max_{1\le j\le 2^{k+t}} |S_j| > \varepsilon \big(2^{k+1} f(2^{k+t}) l(2^{k+t})\big)^{1/\alpha} \Big) < \infty. \tag{2.15}$$

Then

$$\sum_{k=0}^{\infty} P\Big(\max_{1\le j\le 2^{k+t}} |S_j| > \varepsilon \big(2^{k+1} f(2^{k+t}) l(2^{k+t})\big)^{1/\alpha} \Big) < \infty. \tag{2.16}$$

Then

$$\frac{\max_{1\le j\le 2^{k+t}}|S_j|}{\left(2^{k+1}f(2^{k+t})l(2^{k+t})\right)^{1/\alpha}}\longrightarrow 0 \quad \text{a.s.} \tag{2.17}$$

So

$$\begin{aligned}\frac{|S_n|}{\left(nf(n)l(n)\right)^{1/\alpha}}&\le\frac{\max_{1\le j\le 2^{k+t}}|S_j|}{\left(2^{k+1}f(2^{k+t})l(2^{k+t})\right)^{1/\alpha}}\frac{\left(2^{k+1}f(2^{k+t})l(2^{k+t})\right)^{1/\alpha}}{\left(nf(n)\right)^{1/\alpha}}\\&\le 2^{1/\alpha}\frac{\max_{1\le j\le 2^{k+t}}|S_j|}{\left(2^{k+1}f(2^{k+t})\right)^{1/\alpha}}\longrightarrow 0 \quad \text{a.s.}\end{aligned} \tag{2.18}$$

Then

$$\limsup_{n\to\infty}\frac{|S_n|}{\left(nf(n)l(n)\right)^{1/\alpha}}=0 \quad \text{a.s.} \tag{2.19}$$

Given $\varepsilon>0$, let $f(x)=\log^{1+\varepsilon}x$. It is obvious that $\int_1^\infty(1/xf(x))dx<\infty$. By (2.19), we have

$$\limsup_{n\to\infty}\frac{|S_n|}{\left(nl(n)\log^{1+\varepsilon}n\right)^{1/\alpha}}=0 \quad \text{a.s.} \tag{2.20}$$

Then

$$\limsup_{n\to\infty}\left(\frac{|S_n|}{B(n)}\right)^{1/\log\log n}\le e^{(1+\varepsilon)/\alpha} \quad \text{a.s.} \tag{2.21}$$

Therefore

$$\limsup_{n\to\infty}\left(\frac{|S_n|}{B(n)}\right)^{1/\log\log n}\le e^{1/\alpha} \quad \text{a.s.,} \tag{2.22}$$

which completes the proof of (2.14). □

Remark 2.5. Corollary 2.4 generalizes the estimate

$$\limsup_{n\to\infty}\left(\frac{|S_n|}{B_n}\right)^{1/\log\log n}\le e^{1/\alpha} \quad \text{a.s.} \tag{2.23}$$

obtained in Peng and Qi [10, Theorem 2.1] to ρ^*-mixing sequences of random variables.

Corollary 2.6. *Under the conditions of Corollary 2.4, letting $h(x)\equiv 1$, yields*

$$\limsup_{n\to\infty}\left(\frac{|\sum_{i=1}^n X_i|}{B_n}\right)^{1/\log\log n}\le e^{1/\alpha} \quad \textit{a.s.} \tag{2.24}$$

Remark 2.7. Corollary 2.6 generalizes in Qi and Cheng [11, Theorem 1.1] to ρ^*-mixing sequences of random variables.

References

[1] W. Bryc and W. Smoleński, *Moment conditions for almost sure convergence of weakly correlated random variables*, Proceedings of the American Mathematical Society **119** (1993), no. 2, 629–635.

[2] P. Y. Chen, *Limiting behavior of weighted sums with stable distributions*, Statistics & Probability Letters **60** (2002), no. 4, 367–375.

[3] P. Y. Chen and L. H. Huang, *The Chover law of the iterated logarithm for random geometric series of stable distribution*, Acta Mathematica Sinica **43** (2000), no. 6, 1063–1070 (Chinese).

[4] J. Chover, *A law of the iterated logarithm for stable summands*, Proceedings of the American Mathematical Society **17** (1966), no. 2, 441–443.

[5] Z. Y. Lin and C. R. Lu, *Limit Theory for Mixing Dependent Random Variables*, Mathematics and Its Applications, vol. 378, Kluwer Academic, Dordrecht; Science Press, New York, 1996.

[6] Z. Y. Lin, C. R. Lu, and Z. G. Su, *Foundation of Probability Limit Theory*, Higher Education Press, Beijing, 1999.

[7] M. Peligrad, *On the asymptotic normality of sequences of weak dependent random variables*, Journal of Theoretical Probability **9** (1996), no. 3, 703–715.

[8] ———, *Maximum of partial sums and an invariance principle for a class of weak dependent random variables*, Proceedings of the American Mathematical Society **126** (1998), no. 4, 1181–1189.

[9] M. Peligrad and A. Gut, *Almost-sure results for a class of dependent random variables*, Journal of Theoretical Probability **12** (1999), no. 1, 87–104.

[10] L. Peng and Y. C. Qi, *Chover-type laws of the iterated logarithm for weighted sums*, Statistics & Probability Letters **65** (2003), no. 4, 401–410.

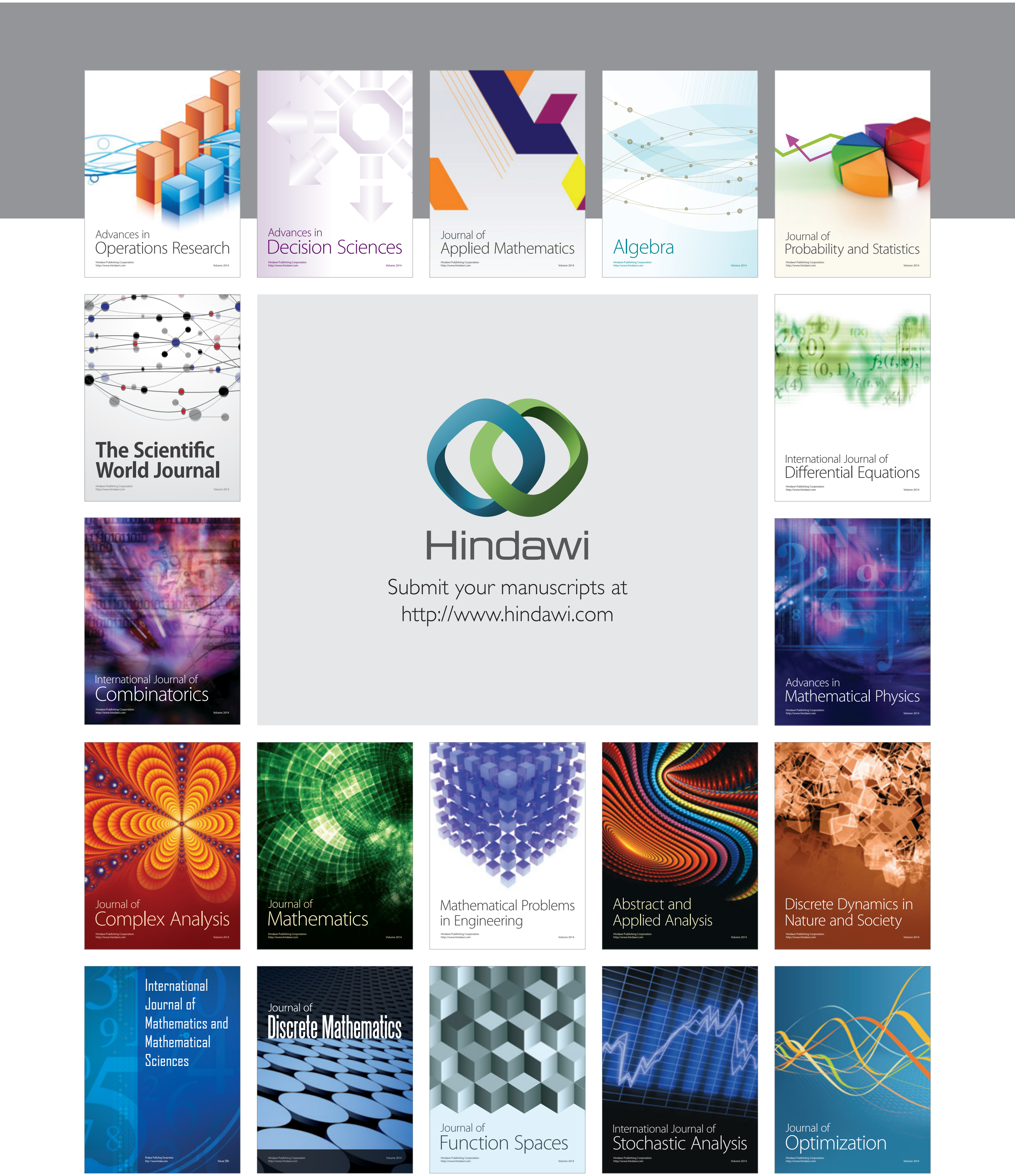
[11] Y. C. Qi and P. Cheng, *A law of the iterated logarithm for partial sums in the field of attraction of a stable law*, Chinese Annals of Mathematics. Series A. Shuxue Niankan. A Ji **17** (1996), no. 2, 195–206 (Chinese).

[12] S. Utev and M. Peligrad, *Maximal inequalities and an invariance principle for a class of weakly dependent random variables*, Journal of Theoretical Probability **16** (2003), no. 1, 101–115.

[13] K. Vasudeva and G. Divanji, *LIL for delayed sums under a non-identically distributed setup*, Teoriya Veroyatnosteĭ i ee Primeneniya **37** (1992), no. 3, 534–542 (Russian), translation in Theory Probab. Appl. **37** (1992), no. 3, 497–506.

[14] N. M. Zinchenko, *A modified law of iterated logarithm for stable random variables*, Theory of Probability and Mathematical Statistics **49** (1994), 69–76 (1995).

Guang-hui Cai: Department of Mathematics and Statistics, Zhejiang Gongshang University,
Hangzhou 310035, China
E-mail address: cghzju@163.com



Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization