ON NONDENSELY DEFINED SEMILINEAR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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The nonlinear alternative of Leray-Schauder type is used to investigate the existence of solutions for first-order semilinear stochastic functional differential equations in Hilbert spaces.

1. Introduction

This paper is concerned with the existence of integral solutions for initial value problems for first-order stochastic semilinear functional differential equations with nonlocal conditions in Hilbert spaces. More precisely in Section 3, we consider first-order stochastic semilinear functional differential equations of the form

$$
\begin{gathered}
y'(t) = Ay(t) + f(t, y_t)\frac{dw(t)}{dt}, \quad t \in J := [0, b], \\
y(t) + h_t(y) = \phi(t), \quad t \in [-r, 0],
\end{gathered} \tag{1.1}
$$

where $f : J \times \widehat{M}_2([-r,0],H) \to H$ is a given function, $A : D(A) \subset H \to H$ is a nondensely defined closed linear operator on H, the function $w(t)$ is a Hilbert space Q-valued Wiener process, $\phi \in \widehat{M}_2([-r,0],D(A))$, $0 < r < \infty$, is a suitable initial random function independent of $w(t)$, $h : \widehat{M}_2([-r,0],D(A)) \to D(A)$, H a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$, and $\widehat{M}_2$ is a class of H-valued stochastic processes that will be specified later (see Section 2). Here $y_t(\cdot)$ represents the history of stochastic processes state from time $t-r$, up to the present time t. The nonlocal conditions were initiated by Byszewski. We refer the readers to [4] and the references cited therein for motivation regarding the nonlocal initial conditions. The nonlocal condition can be applied in physics

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with better effect than the classical initial condition $y(0)=y_0$. For example, $h_t(y)$ may be given by

$$h_t(y)=\sum_{i=1}^{p}c_i y(t_i+t),\quad t\in[-r,0],\tag{1.2}$$

where c_i, $i=1,\ldots,p$, are given constants and $0<t_1<\cdots<t_p\leq b$. At time $t=0$, we have

$$h_0(y)=\sum_{i=1}^{p}c_i y(t_i).\tag{1.3}$$

Random differential and integral equations play an important role in characterizing many social, physical, biological, and engineering problems; see, for instance, the monographs of Da Prato and Zabczyk [6] and Sobczyk [14]. For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [16] to be a closed system with a simplified heat, one organ or capillary bed, and recirculation of blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. The basic theory concerning stochastic differential equations can be found in the monographs of Bharucha-Reid [3], Da Prato and Zabczyk [6], and Tsokos and Padgett [16]. For recent results, we refer to the papers of Liu [11], McKibben [12, 13], and Taniguchi [15].

Recently, Balasubramaniam and Ntouyas [2] studied the semilinear stochastic evolution delay equations with nonlocal conditions, where A is a densely defined linear operator. Our goal here is to extend the results of Balasubramaniam and Ntouyas [2], where A is nondensely defined. These results can be seen as a contribution to the literature.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let K be another real separable Hilbert space and let $w(t)$, $t\geq 0$, be a K-valued Wiener process with mean zero and covariance operator Q with $\operatorname{tr}Q<\infty$ ($\operatorname{tr}Q$ denotes the trace of the operator Q) defined by

$$E\langle w(t),g\rangle\langle w(s),h\rangle=(t\wedge s)\langle Qg,h\rangle\quad\text{for every } g,h\in K,\tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product and E stands for integration with respect to probability measure P. Let $L(K,H)$ denote the space of bounded linear operators from K into H. For $g_1,g_2\in L(K,H)$, we define $\langle\langle g_1,g_2\rangle\rangle=\operatorname{tr}(g_1Qg_2^*)$, where g_2^* is the adjoint of the operator g_2 and Q is the nuclear operator associated with the Brownian motion, where $Q\in L_n^+(K)$, the space of positive nuclear operator in K. Let $L(K_Q,H)$ denote the completion of $L(K,H)$ with respect to the topology induced by the norm $\|\cdot\|_2$, where $\|g\|_2^2=\langle\langle g,g\rangle\rangle$. Let $(\Omega,\mathcal{F},\mathcal{F}_t,P,H)$ be a complete probability space furnished with a complete family of right continuous increasing σ-algebras $\{\mathcal{F}_t,\ t \in [0,T]\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. Let $L^2(\Omega,\mathcal{F},\mathcal{F}_t,P,H)$ be a space of all square random variables with values in H that are measurable with respect to $\{\mathcal{F}_t,\ t \in [0,b]\}$. Let $\widehat{M}_2([-r,b],H)$ denote the classes of H-valued stochastic processes $\{\xi(t) : t \in [-r,b]\}$ which are $\mathcal{F}_t$-adapted and have finite second moments, that is,

$$\|\xi\|_{\widehat{M}_2} = \sup_{t\in[-r,b]} \left(E\left|\xi(t)\right|^2\right)^{1/2} < \infty. \tag{2.2}$$

It is easy to verify that $\widehat{M}_2$, furnished with the norm topology as defined above, is a Banach space. White noise is usually regarded as informal time derivative $w'(t)$ of Brownian motion or Wiener process $w(t)$. In the Itô theory of stochastic integration, an integral with respect to $w'(t)$ is rewritten as one with respect to $dw(t)$, that is,

$$\int_a^b \psi(t)dw(t) = \int_a^b \psi(t)w'(t)dt. \tag{2.3}$$

The Itô integral $\int_a^b \psi(t)dw(t)$ is defined for any process $\psi(t)$ which satisfies the following conditions:

(1) ψ is nonanticipating,

(2) almost all sample paths of ψ belong to $L^2([a,b])$. Moreover, $\int_a^b \psi(t)dw(t) \in L^2(\Omega)$ if and only if $\psi \in L^2([a,b]\times\Omega)$. In fact the following equality holds:

$$E\left|\int_a^b \psi(t)dw(t)\right|^2 = E\int_a^b \left|\psi(t)\right|^2 dt. \tag{2.4}$$

For more details on Brownian motion and white noise, we refer the reader to the books of Hida [8] and Hida et al. [9].

$B(H)$ denotes the Banach space of bounded linear operators from H into H with norm

$$\|N\|_{B(H)} = \sup\{\left|N(y)\right| : |y| = 1\}. \tag{2.5}$$

Definition 2.1 (see [1]). Let E be a Banach space. An integrated semigroup is a family of operators $(S(t))_{t\ge 0}$ of bounded linear operators $S(t)$ on E with the following properties:

(i) $S(0) = 0$;

(ii) $t \to S(t)$ is strongly continuous;

(iii) $S(s)S(t) = \int_0^s (S(t+r) - S(r))dr$, for all $t,s \ge 0$.

Definition 2.2 (see [10]). An operator A is called a generator of an integrated semigroup if there exists $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \rho(A)$ ($\rho(A)$ is the resolvent set of A) and there exists a strongly continuous exponentially bounded family $(S(t))_{t\ge 0}$ of bounded operators such that $S(0) = 0$ and $R(\lambda,A) := (\lambda I - A)^{-1} = \lambda\int_0^\infty e^{-\lambda t}S(t)dt$ exists for all λ with $\lambda > \omega$.

Proposition 2.3 (see [1]). *Let A be the generator of an integrated semigroup $(S(t))_{t\ge 0}$. Then for all $x \in E$ and $t \ge 0$,*

$$\int_0^t S(s)x\,ds \in D(A), \qquad S(t)x = A\int_0^t S(s)x\,ds + tx. \tag{2.6}$$

Definition 2.4 (see [10]). (i) An integrated semigroup $(S(t))_{t\geq 0}$ is called locally Lipschitz continuous if, for all $\tau > 0$, there exists a constant L such that

$$|S(t) - S(s)| \leq L|t - s|, \quad t,s \in [0,\tau]. \tag{2.7}$$

(ii) An integrated semigroup $(S(t))_{t\geq 0}$ is called nondegenerate if $S(t)x = 0$, for all $t \geq 0$, implies that $x = 0$.

Definition 2.5. We say that the linear operator A satisfies the Hille-Yosida condition if there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \rho(A)$ and

$$\sup\left\{(\lambda - \omega)^n \left|(\lambda I - A)^{-n}\right| : n \in \mathbb{N},\, \lambda > \omega\right\} \leq M. \tag{2.8}$$

THEOREM 2.6 (see [10]). *The following assertions are equivalent:*

(H0) *A is the generator of a nondegenerate, locally Lipschitz continuous integrated semigroup;*

(H1) *A satisfies the Hille-Yosida condition.*

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from [1], $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$ and $(S'(t))_{t\geq 0}$ is a C_0 semigroup on $\overline{D(A)}$.

Definition 2.7. A map $f : J \times \widehat{M}_2([-r,0],H) \to H$ is said to be L^2-Carathéodory if

(i) $t \mapsto f(t,u)$ is measurable for each $u \in \widehat{M}_2([-r,0],H)$;

(ii) $u \mapsto f(t,u)$ is continuous for almost all $t \in J$;

(iii) for each $q > 0$, there exists $h_q \in L^1(J,\mathbb{R}_+)$ such that

$$\left|f(t,u)\right|^2 \leq h_q(t) \quad \forall \|u\|^2_{\widehat{M}_2} \leq q \text{ and for almost all } t \in J. \tag{2.9}$$

In what follows, we will assume that f is an L^2-Carathéodory function.

3. Main result

The aim of this section is to study the existence of integral solutions for the nonlocal problem (1.1).

Definition 3.1. For any H-valued $\mathcal{F}_0$-measurable stochastic processes ϕ satisfying the condition $E\|\phi(t)\|^2 < \infty$ for every $t \in [-r,0]$, an element $y \in \widehat{M}_2$ is said to be an integral solution of (1.1) if

(i) $y(t) + h_t(y) = \phi(t)$, $t \in [-r,0]$,

(ii) $\int_0^t y(s)ds \in D(A)$, $t \in J$,

(iii) $y(t) = S'(t)[\phi(0) - h_0(y)] + A\int_0^t y(s)ds + \int_0^t f(s,y_s)dw(s)$, $t \in J$.

From the definition it follows that $y(t) \in \overline{D(A)}$, $t \geq 0$. Moreover, y satisfies the following variation of constant formula:

$$y(t) = S'(t)[\phi(0) - h_t(y)] + \frac{d}{dt}\int_0^t S(t-s)f(s,y_s)dw(s), \quad t \in J. \tag{3.1}$$

We are now in a position to state and prove our existence result for the problem (1.1).

Theorem 3.2. *Assume (H1) and*

(H2) *w is an H-valued Wiener process defined on Hilbert space K;*

(H3) *$S'(t)$, $t > 0$, is compact and there exist $M > 0$, $\omega \in \mathbb{R}$ such that*

$$\|S'(t)\|^2_{B(H)} \leq Me^{\omega t}, \quad t \geq 0; \tag{3.2}$$

(H4) *the function h is continuous with respect to t and there exists a constant $\beta > 0$ such that*

$$|h_t(u)|^2 \leq \beta, \quad u \in \widehat{M}_2([-r,0],H), \tag{3.3}$$

and for each $k > 0$, the set

$$\{\phi(0) - h_0(y) : y \in \widehat{M}_2([-r,0],H),\ \|y\|_{\widehat{M}_2} \leq k\} \tag{3.4}$$

is precompact in H;

(H5) *there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and $p \in L^1([0,b],\mathbb{R}_+)$ such that*

$$E|f(t,u)|^2 \leq p(t)\psi\left(E\|u\|^2_{\widehat{M}_2}\right) \quad \text{for a.e. } t \in [0,b] \text{ and each } u \in \widehat{M}_2([-r,0],\overline{D(A)}),$$
$$\int_0^b p_*(s)ds < \int_c^\infty \frac{dx}{x+\psi(x)}, \tag{3.5}$$

where $p_*(t) = \max(|\omega|, 2Mp(t))$ and $c = 4ME(|\phi(0)|^2 + \beta)$ are satisfied. Then the problem (1.1) has at least one integral solution on $[-r,b]$.

Proof. We transform the problem (1.1) into a fixed-point problem. Consider the operator $N : \widehat{M}_2([-r,b],\overline{D(A)}) \to \widehat{M}_2([-r,b],\overline{D(A)})$ defined by

$$N(y)(t) := \begin{cases} \phi(t) - h_t(y) & \text{if } t \in [-r,0], \\ S'(t)[\phi(0) - h_0(y)] + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)f(s,y_s)dw(s) & \text{if } t \in [0,b]. \end{cases} \tag{3.6}$$

Remark 3.3. It is clear that the fixed points of N are integral solutions to (1.1).

In order to use the Leray-Schauder alternative, we will obtain a priori estimates for the solutions of the integral equation

$$y(t) = \lambda\left[S'(t)[\phi(t) - h_0(y)] + \frac{d}{dt}\int_0^t S(t-s)f(s,y_s)ds\right], \tag{3.7}$$

and $y(t) = \lambda[\phi(t) - h_t(y)]$, $t \in [-r,0]$, $\lambda \in (0,1)$. Hence

$$\begin{aligned} |y(t)|^2 &= \lambda^2\left|S'(t)[\phi(0) - h_0(y)] + \frac{d}{dt}\int_0^t S(t-s)f(s,y_s)dw(s)\right|^2 \\ &\leq 2|S'(t)[\phi(0) - h_0(y)]|^2 + 2\left|\frac{d}{dt}\int_0^t S(t-s)f(s,y_s)dw(s)\right|^2. \end{aligned} \tag{3.8}$$

Thus by (H3), (H4), and (H5), we have

$$E\left(|y(t)|^2\right) \le 4Me^{\omega t}E\left(|\phi(0)|^2+\beta\right)+2Me^{\omega t}\int_0^t e^{-\omega s}p(s)\psi\left(E\left(\|y_s\|^2\right)\right)ds. \tag{3.9}$$

We consider the function μ defined by

$$\mu(t)=\sup\{|y(s)|:0\le s\le t\},\quad 0\le t\le b. \tag{3.10}$$

Let $t_* \in [0,t]\subset[0,b]$ be such that $\mu(t)=|y(t_*)|$. By the previous inequality, we have for $t\in[0,b]$,

$$e^{-\omega t}E(\mu(t)^2)\le 4ME(|\phi(0)|^2+\beta)+2M\int_0^t e^{-\omega s}p(s)\psi(E(\mu(s)^2))ds. \tag{3.11}$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\begin{aligned} E(\mu(t)^2) &\le e^{\omega t}v(t)\quad \forall t\in[0,b],\\ c:=v(0)&=4ME(|\phi(0)|^2+\beta),\\ v'(t)&=2Me^{-\omega t}p(t)\psi(E(\mu(t)^2))\quad \text{a.e. } t\in[0,b]. \end{aligned} \tag{3.12}$$

Using the increasing character of ψ, we get

$$v'(t)\le 2Me^{-\omega t}p(t)\psi(e^{\omega t}v(t))\quad \text{a.e. } t\in[0,b]. \tag{3.13}$$

We remark that

$$\begin{aligned} [e^{\omega t}v(t)]' &= \omega e^{\omega t}v(t)+e^{\omega t}v'(t)\\ &\le |\omega|e^{\omega t}v(t)+2Mp(t)\psi(e^{\omega t}v(t))\\ &\le p_*(t)[e^{\omega t}v(t)+\psi(e^{\omega t}v(t))]. \end{aligned} \tag{3.14}$$

Thus

$$\int_{v(0)}^{e^{\omega t}v(t)}\frac{dx}{x+\psi(x)}\le\int_0^b p_*(s)ds<\int_c^\infty\frac{dx}{x+\psi(x)}. \tag{3.15}$$

From (H5), there exists a constant K_* such that $e^{\omega t}v(t)\le K_*$, $t\in[0,b]$, and there exists M_* such that $\|y\|_{\widehat{M}_2}\le M_*$.

In the next steps, we will prove that N is continuous and completely continuous.

Step 1. N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $\widehat{M}_2([-r,b],\overline{D(A)})$. Then for each $t \in [0,b]$,

$$\begin{aligned}
&|N(y_n)(t) - N(y)(t)|^2 \\
&\quad = \left| S'(t)[h_0(y_n) - h_0(y)] + \frac{d}{dt}\int_0^t S(t-s)[f(s,y_{ns}) - f(s,y_s)]dw(s) \right|^2 \\
&\quad \le 2|S'(t)[h_0(y_n) - h_0(y)]|^2 + 2\left| \frac{d}{dt}\int_0^t S(t-s)[f(s,y_{n_s}) - f(s,y_s)]dw(s) \right|^2 \\
&\quad \le 2Me^{\omega t}|h_0(y_n) - h_0(y)|^2 + 2\left| \frac{d}{dt}\int_0^t S(t-s)[f(s,y_{ns}) - f(s,y_s)]dw(s) \right|^2 \\
&\quad \le 2Me^{\omega t}|h_0(y_n) - h_0(y)|^2 + 2M\left| \int_0^t |f(s,y_{n_s}) - f(s,y_s)|dw(s) \right|^2 \\
&\quad \le 2M\max(e^{\omega b},1)|h_0(y_n) - h_0(y)|^2 + 2\left| \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s,y_{n_s}) - f(s,y_s)dw(s) \right|^2 .
\end{aligned} \tag{3.16}$$

Then

$$\begin{aligned}
E\left(|N(y_n)(t) - N(y)(t)|^2\right) &\le E\left(2M\max(e^{\omega b},1)|h_0(y_n) - h_0(y)|^2\right) \\
&\quad + E\left(2\left| \lim_{\lambda\to\infty}\int_0^t S'(t-s)[f(s,y_{n_s}) - f(s,y_s)]dw(s) \right|^2\right) \\
&\le 2M\max(e^{\omega b},1)E\left(|h_0(y_n) - h_0(y)|^2\right) \\
&\quad + 2M\max(e^{|\omega|b},1)\int_0^b E\left(|f(s,y_{n_s}) - f(s,y_s)|^2\right)ds.
\end{aligned} \tag{3.17}$$

Thus

$$\begin{aligned}
&\|N(y_n) - N(y)\|_{\widehat{M}_2} \\
&\quad \le \sqrt{2M\max(e^{\omega b},1)b}\,|h_0(y_n) - h_0(y)| \\
&\qquad + \sqrt{2Mb\max(e^{|\omega|b},1)}\|f(\cdot,y_n) - f(\cdot,y)\|_{\widehat{M}_2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.18}$$

Step 2. N maps bounded sets into bounded sets in $\widehat{M}_2([-r,b],H)$.

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $y \in \mathcal{B}_q = \{y \in \widehat{M}_2([-r,b],H) : \|y\|^2_{\widehat{M}_2} \le q\}$, one has $\|N(y)\|_{\widehat{M}_2} \le \ell$.

Let $y \in B_q$, then for each $t \in [0,b]$, we have

$$\begin{aligned}
|N(y)(t)|^2 &= \left| S'(t)[\phi(0) - h_0(y)] + \frac{d}{dt}\int_0^t S(t-s)f(s,y_s)\,dw(s)\right|^2 \\
&\le 2\left|S'(t)[\phi(0) - h_0(y)]\right|^2 + 2\left|\frac{d}{dt}\int_0^t S(t-s)f(s,y_s)\,dw(s)\right|^2 \\
&\le 4M\max\left(e^{\omega b},1\right)\left[\left|h_0(y)\right|^2 + \left|\phi(0)\right|^2\right] \\
&\quad + 2\left|\lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s,y_s)\,dw(s)\right|^2.
\end{aligned} \tag{3.19}$$

Thus

$$\begin{aligned}
\|N(y)\|_{\widehat{M}_2} &\le 4M\max\left(e^{\omega b},1\right)b\left[\left|h_0(y)\right|^2 + \left|\phi(0)\right|^2\right] \\
&\quad + 2Mb\max\left(e^{|\omega| b},1\right)\|h_q\|_{L^2} := \ell.
\end{aligned} \tag{3.20}$$

Step 3. N maps bounded sets into equicontinuous sets in $\widehat{M}_2([-r,b],H)$.

Let $t_1, t_2 \in J$, $t_1 < t_2$, B_q be a bounded set of $\widehat{M}_2([-r,b],H)$ as in Step 2 and let $y \in B_q$. Then

$$\begin{aligned}
\left|N(y)(t_1) - N(y)(t_2)\right|^2 &= \Bigg| [S'(t_2) - S'(t_1)][\phi(0) - h_0(y)] + \lim_{\lambda\to\infty}\int_{t_1}^{t_2} S'(t_2-s)f(s,y_s)\,dw(s) \\
&\quad + \lim_{\lambda\to\infty}\int_0^{t_1}[S'(t_2-s) - S'(t_1-s)]f(s,y_s)\,dw(s)\Bigg|^2 \\
&\le 4\left|S'(t_2) - S'(t_1)\right|^2\left[\left|h_0(y)\right|^2 + \left|\phi(0)\right|^2\right] \\
&\quad + 4\left|\lim_{\lambda\to\infty}\int_{t_1}^{t_2} S'(t_2-s)B_\lambda f(s,y_s)\,dw(s)\right|^2 \\
&\quad + 4\left|\lim_{\lambda\to\infty}\int_0^{t_1}[S'(t_2-s) - S'(t_1-s)]B_\lambda f(s,y_s)\,dw(s)\right|^2.
\end{aligned} \tag{3.21}$$

Hence

$$\begin{aligned}
E\left(|N(y)(t_1)-N(y)(t_2)|^2\right) &\le E\left(4|S'(t_2)-S'(t_1)|^2\left[|h_0(y)|^2+|\phi(0)|^2\right]\right)\\
&\quad+E\left(4\left|\lim_{\lambda\to\infty}\int_{t_1}^{t_2}S'(t_2-s)B_\lambda f(s,y_s)\,dw(s)\right|^2\right)\\
&\quad+E\left(4\left|\lim_{\lambda\to\infty}\int_0^{t_1}[S'(t_2-s)-S'(t_1-s)]B_\lambda f(s,y_s)\,dw(s)\right|^2\right)\\
&\le 4b|S'(t_2)-S'(t_1)|^2\left[|h_0(y)|^2+|\phi(0)|^2\right]\\
&\quad+4b\int_{t_1}^{t_2}|S'(t_2-s)|^2p(s)\psi(E(q))\,ds\\
&\quad+4b\int_0^{t_1}|S'(t_2-s)-S'(t_1-s)|^2h_q(s)\,ds.
\end{aligned}\tag{3.22}$$

The right-hand side tends to zero as $t_2-t_1\to 0$. Now we will show that $N\mathcal{B}_q(t)$ is relatively compact for every $t\in[0,b]$. In the case where $t=0$, we have $N\mathcal{B}_q(0)=\{\phi(0)-h_0(y)\}$ which is precompact from (H4). Let $0<t\le b$ and $\epsilon<t\le b$. For $y\in\mathcal{B}_q$,

$$\begin{aligned}
N_\epsilon(y)(t)&=S'(t)[\phi(0)-h_0(y)]+\lim_{\lambda\to\infty}\int_{t-\epsilon}^{t}S'(t-s)B_\lambda f(s,y_s)\,dw(s)\\
&\quad+\lim_{\lambda\to\infty}S'(\epsilon)\int_0^{t-\epsilon}S'(t-\epsilon-s)B_\lambda f(s,y_s)\,dw(s).
\end{aligned}\tag{3.23}$$

Since $S'(t)$ is a compact operator, the set $H_\epsilon(t)=\{N_\epsilon(y)(t):y\in\mathcal{B}_q\}$ is precompact in $\overline{D(A)}$ for every ϵ, $0<\epsilon<t$. Moreover, for every $y\in\mathcal{B}_q$, we have

$$|N_\epsilon(y)(t)-N(y)(t)|^2\le\left|\lim_{\lambda\to\infty}\int_{t-\epsilon}^{t}S'(t-s)B_\lambda f(s,y_s)\,dw(s)\right|^2\tag{3.24}$$

Then

$$\begin{aligned}
E\left(|N_\epsilon(y)(t)-N(y)(t)|^2\right)&\le E\left(\left|\lim_{\lambda\to\infty}\int_{t-\epsilon}^{t}S'(t-s)B_\lambda f(s,y_s)\,dw(s)\right|^2\right)\\
&\le b\int_{t-\epsilon}^{t}\|S'(t-s)\|_{B(H)}^2h_q(s)\,ds.
\end{aligned}\tag{3.25}$$

Therefore, there are precompact sets arbitrarily close to the set $\{N_\epsilon(y)(t):y\in\mathcal{B}_q\}$. Hence the set $\{N_\epsilon(y)(t):y\in\mathcal{B}_q\}$ is precompact in $\overline{D(A)}$.

The cases when $t_1, t_2 < 0$ or $t_1 < 0 < t_2$ are obvious.
Set

$$U = \{z \in \widehat{M}_2([-r,b],H) : \|y\|_{\widehat{M}_2} < M_* + 1\}. \tag{3.26}$$

From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N(y)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [7], we deduce that N has a fixed point y in U which is an integral solution of the problem (1.1). □

Remark 3.4. We can replace (H5) by the following condition.

(H5)* There exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$, $p \in L^1([0,b],\mathbb{R}_+)$, and nonnegative number $M_* > 0$ such that

$$\begin{gathered} E\left(|f(t,u)|^2\right) \le p(t)\psi\left(E\|u\|^2_{\widehat{M}_2}\right) \quad \text{for each } u \in \widehat{M}_2([-r,0],H), \\ \frac{M_*}{4ME\left(|\phi(0)|^2 + \beta\right) + 2M\max\left(e^{|\omega|b},1\right)\psi(M_*)\int_0^b p(s)\,ds} > 1. \end{gathered} \tag{3.27}$$

Then the step on a priori estimates will be modified as follows.
Let y be solution of the problem (1.1), then we have

$$\begin{aligned} |y(t)|^2 &= \left|S'(t)[\phi(0) - h_0(y)] + \frac{d}{dt}\int_0^t S(t-s)f(s,y_s)\,dw(s)\right|^2 \\ &\le 2\left|S'(t)[\phi(0) - h_0(y)]\right|^2 + 2\left|\frac{d}{dt}\int_0^t S(t-s)f(s,y_s)\,dw(s)\right|^2 \\ &\le 4M\max\left(e^{\omega b},1\right)\left[|\phi(0)|^2 + |h_0(y)|^2\right] \\ &\quad + 2\left|\lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s,y_s)\right|dw(s)^2. \end{aligned} \tag{3.28}$$

Thus using (H5)*, instead of (H5), we have

$$\begin{aligned} E\left(|y(t)|^2\right) &\le E\left(4M\max\left(e^{\omega b},1\right)\left[|\phi(0)|^2 + \beta\right]\right) \\ &\quad + E\left(2\left|\lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s,y_s)\,dw(s)\right|^2\right) \\ &\le 4M\max\left(e^{\omega b},1\right)E\left(|\phi(0)|^2 + \beta\right) \\ &\quad + 2M\max\left(e^{\omega b},1\right)b\int_0^t e^{-\omega s}E\left(|f(s,y_s)|^2\right)ds \\ &\le 4M\max\left(e^{\omega b},1\right)E\left(|\phi(0)|^2 + \beta\right) \\ &\quad + 2M\max\left(e^{\omega b},1\right)\int_0^t e^{-\omega s}p(s)\psi\left(E\left(\|y_s\|^2\right)\right)ds. \end{aligned} \tag{3.29}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : 0 \le s \le t\}, \quad 0 \le t \le b. \tag{3.30}$$

Let $t_* \in [0,t] \subset [0,b]$ be such that $\mu(t) = |y(t_*)|$. By the previous inequality, we have for $t \in [0,b]$,

$$\begin{aligned} E(\mu^2(t)) \le{}& 4M\max(e^{|\omega|b},1)E\Big(|\phi(0)|^2+\beta\Big) \\ &+ 2M\max(e^{|\omega|b},1)\int_0^b p(s)\psi(E(\mu^2(s)))\,ds. \end{aligned} \tag{3.31}$$

Consequently,

$$\frac{\|y\|_{\widehat{M}_2}}{4M\max(e^{|\omega|b},1)E(|\phi(0)|^2+\beta)+2M\psi(\|y\|_{\widehat{M}_2})}\max(e^{|\omega|b},1)\int_0^b p(s)ds \le 1. \tag{3.32}$$

Then by (H5)*, there exists M_* such that $\|y\|_{\widehat{M}_2} \neq M_*$.
Set

$$U = \{y \in \widehat{M}_2([-r,b],\mathbb{R}) : \|y\|_{\widehat{M}_2} < M_*\} \tag{3.33}$$

and proceed as in Theorem 3.2.

4. An example

To apply the previous result, we consider the following partial stochastic differential equation:

$$\begin{gathered} \frac{\partial}{\partial t}u(t,x) = \triangle u(t,x) + f(t,u(t-r,x))\frac{dw(t)}{dt}, \quad 0 \le t \le b,\ x \in \Omega, \\ u(t,x) = 0, \quad 0 \le t \le b,\ x \in \partial\Omega, \\ u(t,x) + h_t(x) = v_0(x) \quad t \in [-r,0],\ x \in \Omega, \end{gathered} \tag{4.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with regular boundary $\partial\Omega$, $v_0 \in C(\Omega,\mathbb{R}^n)$, $f : [0,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function, and $\triangle = \sum_{k=1}^n(\partial^2/\partial x_k^2)$. Consider $E = C(\overline{\Omega})$, the Banach space of continuous function on $\overline{\Omega}$ with values in $\mathbb{R}^n$. Define the linear operator A on E by

$$Az = \triangle z \quad \text{in } D(A) = \{z \in C(\overline{\Omega}) : z = 0 \text{ on } \partial\Omega,\ \triangle z \in C(\overline{\Omega})\}. \tag{4.2}$$

Now, we have

$$\overline{D(A)} = C_0(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\} \neq C(\overline{\Omega}). \tag{4.3}$$

It is well known from [5] that A is sectorial, $(0,+\infty) \subseteq \rho(A)$, and for $\lambda > 0$,

$$|R(\lambda, \triangle)| \leq \frac{1}{\lambda}. \tag{4.4}$$

It follows that A generates an integrated semigroup $(S(t))_{t\geq 0}$ and that $|S'(t)| \leq e^{-\mu t}$ for $t \geq 0$ for some constant $\mu > 0$. The partial stochastic differential equation (4.1) can be reformulated as the abstract semilinear stochastic differential equation (1.1) in E, where $F : [0,b] \times D(A) \to E$ is the Nemyskii operator given by

$$F(t,u)(x) = f(t, u(t-r,x)). \tag{4.5}$$

If we assume that f is an L^2-Carathéodory function satisfying (H5) and the conditions (H1), (H4) are satisfied, then the integral solution of (1.1) exists by Theorem 3.2.

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