MIXED QUASI-EQUILIBRIUM-LIKE PROBLEMS

MUHAMMAD ASLAM NOOR

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We use the auxiliary principle technique in conjunction with the Bregman function to suggest and analyze a three-step predictor-corrector method for solving mixed quasi-equilibrium-like problems. We also study the convergence criteria of this new method under some mild conditions. As special cases, we obtain various new and known methods for solving variational-like inequalities and related optimization problems.

1. Introduction

Equilibrium problems, which were introduced and studied by Blum and Oettli [1] and Noor and Oettli [14], are being used to study a wide class of diverse unrelated problems arising in various branches of pure and applied sciences in a unified framework. Various generalizations and extensions of equilibrium problems have been considered in different directions using novel and innovative techniques. A useful and important generalization of the equilibrium problems is called the invex equilibrium (equilibrium-like) problem, which has been studied and investigated by Noor [13] recently. It can be shown that the equilibrium-like problems include the variational-like inequalities as a special case, which have been studied extensively. It is known that the variational-like inequalities are closely related to the concept of the invex and preinvex functions, which generalize the notion of convexity of functions. In fact, Yang and Chen [17] and Noor [7, 8] have shown that the minimum of the differentiable preinvex (invex) functions on the invex sets can be characterized by variational-like inequalities. This shows that the variational-like inequalities are only defined on the invex set with respect to function $\eta(\cdot,\cdot)$. We emphasize the fact that the function $\eta(\cdot,\cdot)$ plays a significant and crucial part in the definitions of invex, preinvex functions, and invex sets. Ironically, we note that all the results in variational-like inequalities are being obtained under the assumptions of standard convexity concepts. No attempt has been made to utilize the concept of invexity theory. Note that the preinvex functions and invex sets may not be convex functions and convex sets, respectively.

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We would like to emphasize the fact that the variational-like inequalities are well defined only in the invexity setting.

There are a substantial number of numerical methods including projection technique and its variant forms, Wiener-Hopf equations, auxiliary principle, and resolvent equations methods for solving variational inequalities. However, it is known that projection, Wiener-Hopf equations, and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving equilibrium problems. This fact has motivated to use the auxiliary principle technique which is due to Glowinski, Lions, and Trémolières [4]. In this paper, we again use the auxiliary principle technique in conjunction with the Bregman function to suggest and analyze a three-step iterative algorithm for solving a new class of equilibrium problems, which is called the mixed quasi-equilibrium-like problems. It is shown that the convergence of this method requires partially relaxed strongly η-monotonicity, which is a weaker condition than η-monotonicity. Our results can be considered as a novel and important application of the auxiliary principle technique. Since mixed quasi-equilibrium-like problems include several classes of equilibrium problems, variational-like inequalities, and related optimization problems as special cases, results obtained in this paper continue to hold for these problems.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty closed set in H. Let $f : K \to H$ and $\eta(\cdot,\cdot) : K \times K \to H$ be functions. First of all, we recall the following well-known results and concepts; see [5, 7, 11, 16].

Definition 2.1. Let $u \in K$. Then the set K is said to be invex at u with respect to $\eta(\cdot,\cdot)$ if

$$u + t\eta(v,u) \in K, \quad \forall u,v \in K,\ t \in [0,1]. \tag{2.1}$$

K is said to be an invex set with respect to η if K is invex at each $u \in K$. The invex set K is also called η-connected set.

From now onward, K is a nonempty closed invex set in H with respect to $\eta(\cdot,\cdot)$ unless otherwise specified.

Definition 2.2. The function $f : K \to H$ is said to be preinvex with respect to η if

$$f(u + t\eta(v,u)) \le (1-t)f(u) + tf(v), \quad \forall u,v \in K,\ t \in [0,1]. \tag{2.2}$$

The function $f : K \to H$ is said to be preconcave if and only if f is preinvex.

From Definition 2.2, it follows that the minimum of the differentiable preinvex function f on the invex set K in H can be characterized by the inequality of the type

$$\langle f'(u), \eta(v,u) \rangle \ge 0, \quad \forall v \in K, \tag{2.3}$$

which is known as the variational-like inequality; see [7, 8, 17]. Here $f'(u)$ is the differential of a preinvex (invex) function $f(u)$ at $u \in K$. From this formulation, it is clear that the set K involved in the variational-like inequality is an invex set, otherwise the variational-like inequality problem is not well defined.

Definition 2.3. A function f is said to be strongly preinvex function on K with respect to the function $\eta(\cdot,\cdot)$ with modulus μ if

$$f(u+t\eta(v,u)) \leq (1-t)f(u)+tf(v)-t(1-t)\mu\|\eta(v,u)\|^2, \quad \forall u,v \in K,\ t\in[0,1]. \tag{2.4}$$

Clearly the differentiable strongly preinvex function f is a strongly invex function with module constant μ, that is,

$$f(v)-f(u) \geq \langle f'(u),\eta(v,u)\rangle + \mu\|\eta(v,u)\|^2, \quad \forall u,v\in K, \tag{2.5}$$

and the converse is also true under certain conditions; see [11].

Let K be a nonempty closed and invex set in H. For given continuous trifunction $F(\cdot,\cdot,\cdot): K\times K\times K \to H$ and continuous bifunction $\varphi(\cdot,\cdot): K\times K \to \mathbb{R}\cup\{\infty\}$, we consider the problem of finding $u\in K$ such that

$$F(u,Tu,\eta(v,u))+\varphi(v,u)+\varphi(u,u)\geq 0, \quad \forall v\in K. \tag{2.6}$$

Problems of type (2.6) are called the *mixed quasi-equilibrium problems.*

We note that if $F(u,Tu,\eta(v,u)) = \langle Tu,\eta(v,u)\rangle$, then the problem (2.6) is equivalent to finding $u\in K$ such that

$$\langle Tu,\eta(v,u)\rangle+\varphi(v,u)+\varphi(u,u)\geq 0, \quad \forall v\in K, \tag{2.7}$$

which is known as the mixed quasivariational-like inequality. It has been shown that a wide class of problems arising in elasticity, fluid flow through porous media, and nonconvex optimization can be studied in the general framework of problems (2.6) and (2.7).

In particular, if the function $\varphi(\cdot,\cdot)=\varphi(\cdot)$ is an indicator function of an invex closed set K in H, then problem (2.6) is equivalent to finding $u\in K$ such that

$$F(u,Tu,\eta(v,u))\geq 0, \quad \forall v\in K, \tag{2.8}$$

which is called the equilibrium-like problem.

If $F(u,Tu,\eta(v,u)) = \langle Tu,\eta(v,u)\rangle$, then problem (2.7) is equivalent to finding $u\in K$ such that

$$\langle Tu,\eta(v,u)\rangle\geq 0, \quad \forall v\in K, \tag{2.9}$$

which is known as the variational-like inequality and has been studied extensively in recent years. It has been shown in [7, 8, 17] that the minimum of the differentiable preinvex (invex) functions $f(u)$ on the invex sets in the normed spaces can be characterized by a class of variational-like inequalities (2.8) with $Tu = f'(u)$, where $f'(u)$ is the differential of the preinvex function $f(u)$. This shows that the concept of variational-like inequalities is closely related to the concept of invexity. For suitable and appropriate choice of the operators T, $\varphi(\cdot,\cdot)$, $\eta(\cdot,\cdot)$, and spaces H, one can obtain several classes of variational-like inequalities and variational inequalities as special cases of problem (2.6); see [1–7, 14, 8–13, 15–18].

Definition 2.4. The trifunction $F(\cdot,\cdot,\cdot)$ and the operator $T : K \to H$ are said to be

(i) *jointly η-monotone* if

$$F(u,Tu,\eta(v,u)) + F(v,Tv,\eta(u,v)) \leq 0, \quad \forall u,v \in K; \tag{2.10}$$

(ii) *partially relaxed strongly jointly η-monotone* if there exists a constant $\alpha > 0$ such that

$$F(u,Tu,\eta(v,u)) + (v,Tv,\eta(z,v)) \leq \alpha\|\eta(z,u)\|^2, \quad \forall u,v,z \in K. \tag{2.11}$$

Note that for $z = u$ partially relaxed strongly η-monotonicity reduces to η-monotonicity of the operator T.

Definition 2.5. The bifunction $\varphi(\cdot,\cdot) : H \times H \to \mathbb{R} \cup \{+\infty\}$ is called *skew-symmetric* if and only if

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) - \varphi(v,v) \geq 0, \quad \forall u,v \in H. \tag{2.12}$$

Clearly if the skew-symmetric bifunction $\varphi(\cdot,\cdot)$ is bilinear, then

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) = \varphi(u-v,u-v) \geq 0, \quad \forall u,v \in H. \tag{2.13}$$

We also need the following assumption about the functions $\eta(\cdot,\cdot) : K \times K \to H$, which plays an important part in obtaining our results.

Assumption 2.6. The operator $\eta : K \times K \to H$ satisfies the condition

$$\eta(u,v) = \eta(u,z) + \eta(z,v), \quad \forall u,v,z \in K. \tag{2.14}$$

In particular, it follows that $\eta(u,v) = 0$ if and only if $u = v$, for all $u,v \in K$. Assumption 2.6 has been used to suggest and analyze some iterative methods for various classes of equilibrium problems and variational-like inequalities; see [9, 10, 13].

3. Main results

In this section, we use the auxiliary principle technique to suggest and analyze a three-step iterative algorithm for solving mixed quasivariational-like inequalities (2.6).

For a given $u \in K$, consider the problem of finding $z \in K$ such that

$$\rho F(u, Tu, \eta(v,u)) + \langle E'(z) - E'(u), \eta(v,z)\rangle \geq \rho\varphi(z,z) - \rho\varphi(v,z), \quad \forall v \in K, \tag{3.1}$$

where $E'(u)$ is the differential of a strongly preinvex function $E(u)$ and $\rho > 0$ is a constant. Problem (3.1) has a unique solution due to the strongly preinvexity of the function $E(u)$.

Remark 3.1. The function $B(w,u) = E(w) - E(u) - \langle E'(u), \eta(w,u)\rangle$ associated with the preinvex function $E(u)$ is called the generalized Bregman function. By the strongly preinvexity of the function $E(u)$, the Bregman function $B(\cdot,\cdot)$ is nonnegative and $B(w,u) = 0$ if and only if $u = w$, for all $u, w \in K$. We note that if $\eta(z,u) = z - u$, then $B(z,u) = E(z) - E(u) - \langle E'(u), z - u\rangle$ is the well-known Bregman function. For the applications of the Bregman function in solving variational inequalities and complementarity problems, see [7–13, 18].

We remark that if $z = u$, then z is a solution of the equilibrium-like problems (2.6). On the basis of this observation, we suggest and analyze the following iterative algorithm for solving (2.6) as long as (3.1) is easier to solve than (2.6).

Algorithm 3.2. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} &\rho F(w_n, Tw_n, \eta(v,w_n)) + \langle E'(u_{n+1}) - E'(w_n), \eta(v,u_{n+1})\rangle \\ &\qquad \geq \rho\varphi(u_{n+1},u_{n+1}) - \rho\varphi(v,u_{n+1}), \quad \forall v \in K, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\nu F(y_n, Ty_n, \eta(v,w_n)) + \langle E'(w_n) - E'(y_n), \eta(v,w_n)\rangle \\ &\qquad \geq \nu\varphi(w_n,w_n) - \nu\varphi(v,w_n), \quad \forall v \in K, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\mu F(u_n, Tu_n, \eta(v,y_n)) + \langle E'(y_n) - E'(u_n), \eta(v,y_n)\rangle \\ &\qquad \geq \mu\varphi(y_n,y_n) - \mu\varphi(v,y_n), \quad \forall v \in K, \end{aligned} \tag{3.4}$$

where E' is the differential of a strongly preinvex function E. Here $\rho > 0$, $\nu > 0$, and $\mu > 0$ are constants. Algorithm 3.2 is called the three-step predictor-corrector iterative method for solving the mixed quasivariational-like inequalities (2.6).

If $F(u, Tu, \eta(v,u)) = \langle Tu, \eta(v,u)\rangle$, then Algorithm 3.2 reduces to the following algorithm.

Algorithm 3.3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}
\langle \rho T w_n + E'(u_{n+1}) - E'(w_n), \eta(v, u_{n+1})\rangle &\geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}), \quad \forall v \in K,\\
\langle \nu T y_n + E'(w_n) - E'(y_n), \eta(v, w_n)\rangle &\geq \nu\varphi(w_n, w_n) - \nu\varphi(v, w_n), \quad \forall v \in K,\\
\langle \mu T u_n + E'(y_n) - E'(u_n), \eta(v, y_n)\rangle &\geq \mu\varphi(y_n, y_n) - \mu\varphi(v, y_n), \quad \forall v \in K,
\end{aligned} \tag{3.5}$$

where E' is the differential of a strongly preinvex function E. Algorithm 3.3 is known as the three-step iterative method for solving variational-like inequalities (2.7) and appears to be a new one. For appropriate and suitable choice of the operators T, $\eta(\cdot,\cdot)$, $\varphi(\cdot,\cdot)$, and the space H, one can obtain several new and known three-step, two-step, and one-step iterative methods for solving various classes of variational inequalities and related optimization problems.

We now study the convergence analysis of Algorithm 3.2.

Theorem 3.4. *Let E be strongly differentiable preinvex function with modulus β. Let Assumption 2.6 hold and let the bifunction $\varphi(\cdot,\cdot)$ be skew-symmetric. If the function $F(\cdot,\cdot,\cdot)$ and the operator T are partially relaxed strongly jointly η-monotone with constant $\alpha > 0$ and (2.14) holds, then the approximate solution obtained from Algorithm 3.2 converges to a solution $u \in K$ of (2.6) for $\rho < \beta/\alpha$, $\nu < \beta/\alpha$, and $\mu < \beta/\alpha$.*

Proof. Let $u \in K$ be a solution of (2.6). Then

$$\begin{aligned}
\rho\{F(u, Tu, \eta(v, u)) + \varphi(v, u) - \varphi(u, u)\} &\geq 0, \quad \forall v \in K,\\
\nu\{F(u, Tu, \eta(v, u)) + \varphi(v, u) - \varphi(u, u)\} &\geq 0, \quad \forall v \in K,\\
\mu\{F(u, Tu, \eta(v, u)) + \varphi(v, u) - \varphi(u, u)\} &\geq 0, \quad \forall v \in K,
\end{aligned} \tag{3.6}$$

where $\rho > 0$, $\mu > 0$, and $\nu > 0$ are constants. Taking $v = u_{n+1}$ in (3.6) and $v = u$ in (3.2), we have

$$\begin{aligned}
&F(u, Tu, \eta(u_{n+1}, u)) + \varphi(u_{n+1}, u) - \varphi(u, u) \geq 0,\\
&\rho F(w_n, Tw_n, \eta(u, u_{n+1})) + E'(u_{n+1}) - E'(w_n), \eta(u, u_{n+1}))\rangle\\
&\qquad \geq \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})\}.
\end{aligned} \tag{3.7}$$

Consider the Bregman function [10, 18]

$$B(u, z) = E(u) - E(z) - \langle E'(z), \eta(u, z)\rangle \geq \beta\|\eta(u, z)\|^2 \tag{3.8}$$

since the function $E(u)$ is strongly preimvex.

Using (2.14), (3.7), and (3.8), we have

$$\begin{aligned}
B(u,w_n) - B(u,u_{n+1}) &= E(u_{n+1}) - E(w_n) - \langle E'(w_n), \eta(u,u_n)\rangle + \langle E'(u_{n+1}), \eta(u,u_{n+1})\rangle \\
&= E(u_{n+1}) - E(u_n) - \langle E'(w_n) - E'(u_{n+1}), \eta(u,u_{n+1})\rangle \\
&\quad - \langle E'(w_n), \eta(u_{n+1},u_n)\rangle \\
&\geq \beta||\eta(u_{n+1},u_n)||^2 + \langle E'(u_{n+1}) - E'(w_n), \eta(u,u_{n+1})\rangle \\
&\geq \beta||\eta(u_{n+1},w_n)||^2 - \rho F(w_n, Tw_n, \eta(u,w_n)) \\
&\quad + \rho\{\varphi(u_{n+1},u_{n+1}) - \rho\varphi(u,u_{n+1})\} \geq \beta||\eta(u_{n+1},w_n)||^2 \\
&\quad + \rho\{\varphi(u_{n+1},u_{n+1}) - \rho\varphi(u,u_{n+1}) - \varphi(u_{n+1},u) + \varphi(u,u)\} \\
&\quad - \rho\{F(w_n,Tw_n,\eta(u,w_n)) + F(u,Tu,\eta(u_{n+1},u))\} \\
&\geq \beta||\eta(u_{n+1},w_n)||^2 - \alpha\rho||\eta(u_{n+1},w_n)||^2 \\
&= \{\beta - \rho\alpha\}||\eta(u_{n+1},w_n)||^2,
\end{aligned} \tag{3.9}$$

where we have used the fact that the bifunction $\varphi(\cdot,\cdot)$ is skew-symmetric and the function $F(\cdot,\cdot,\cdot)$ and the operator T are partially relaxed strongly jointly η-monotone with constant $\alpha > 0$.

In a similar way, we have

$$\begin{aligned}
B(u,y_n) - B(u,w_n) &\geq \{\beta - \nu\alpha\}||\eta(w_n,y_n)||^2, \\
B(u,u_n) - B(u,y_n) &\geq \{\beta - \mu\alpha\}||\eta(y_n,u_n)||^2.
\end{aligned} \tag{3.10}$$

If $u_{n+1} = w_n = u_n$, then clearly u_n is a solution of the equilibrium-like problems (2.6). Otherwise, for $\rho < \beta/\alpha$, $\nu < \beta/\alpha$, and $\mu < \beta/\alpha$, the sequences $B(u,w_n) - B(u,u_{n+1})$, $B(u,y_n) - B(u,w_n)$, and $B(u,u_n) - B(u,w_n)$ are nonnegative, and we must have

$$\begin{aligned}
\lim_{n\to\infty} ||\eta(u_{n+1},w_n)|| &= 0, \\
\lim_{n\to\infty} ||\eta(w_n,y_n)|| &= 0, \\
\lim_{n\to\infty} ||\eta(y_n,u_n)|| &= 0.
\end{aligned} \tag{3.11}$$

Thus

$$\lim_{n\to\infty} ||\eta(u_{n+1},u_n)|| = \lim_{n\to\infty} ||\eta(u_{n+1},w_n)|| + \lim_{n\to\infty} ||\eta(w_n,y_n)|| + \lim_{n\to\infty} ||\eta(y_n,u_n)|| = 0. \tag{3.12}$$

From (3.12), it follows that the the sequence $\{u_n\}$ is bounded. Let $\overline{u}$ be a cluster point of the subsequence $\{u_{n_i}\}$, and let $\{u_{n_i}\}$ be a subsequence converging toward $\overline{u}$. Now by using the technique of Zhu and Marcotte [18], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\overline{u}$ satisfying the equilibrium-like problems (2.6). □

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Muhammad Aslam Noor: Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan
E-mail address: noormaslam@hotmail.com