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Research Article

On a Class of Measure-Dependent Stochastic Evolution Equations Driven by fBm

Eduardo Hernandez, David N. Keck, and Mark A. McKibben

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We investigate a class of abstract stochastic evolution equations driven by a fractional Brownian motion (fBm) dependent upon a family of probability measures in a real separable Hilbert space. We establish the existence and uniqueness of a mild solution, a continuous dependence estimate, and various convergence and approximation results. Finally, the analysis of three examples is provided to illustrate the applicability of the general theory.

1. Introduction

The focus of this investigation is the class of abstract measure-dependent stochastic evolution equations driven by fractional Brownian motion (fBm) of the general form

$$\begin{aligned} dx(t) &= \big(Ax(t) + f\big(t, x(t), \mu(t)\big)\big)dt + g(t)dB^H(t), \quad 0 \le t \le T, \\ x(0) &= x_0, \\ \mu(t) &= \text{probability distribution of } x(t) \end{aligned} \tag{1.1}$$

in a real separable Hilbert space U. (By the probability distribution of $x(t)$, we mean $\mu(t)(A) = P(\{\omega \in \Omega : x(t,\omega) \in A\})$ for each $A \in \mathfrak{B}(U)$, where $\mathfrak{B}(U)$ stands for the Borel class on U.) Here, $A : D(A) \subset U \to U$ is a linear (possibly unbounded) operator which generates a strongly continuous semigroup $\{S(t) : t \ge 0\}$ on U; $f : [0,T] \times U \times \wp_{\lambda^2}(U) \to U$

(where $\wp_{\lambda^2}(U)$ denotes a particular subset of probability measures on U) is a given mapping; $g : [0,T] \to \mathfrak{L}(V,U)$ is a bounded, measurable mapping (where V is a real separable Hilbert space and $\mathfrak{L}(V,U)$ denotes the space of Hilbert-Schmidt operators from V into U with norm $\|\cdot\|_{\mathfrak{L}(V,U)}$); $\{B^H(t) : t \geq 0\}$ is a V-valued fBm with Hurst parameter $H \in (1/2,1)$; and $x_0 \in L^2(\Omega;U)$.

Stochastic partial functional differential equations naturally arise in the mathematical modeling of phenomena in the natural sciences (see [1–8]). It has been shown that some applications, such as communication networks and certain financial models, exhibit a self-similarity property in the sense that the processes $\{x(\alpha t) : 0 \leq t \leq T\}$ and $\{\alpha^H x(t) : 0 \leq t \leq T\}$ have the same law (see [4, 6]). Indeed, while the case when $H = 1/2$ generates a standard Brownian motion, concrete data from a variety of applications have exhibited other values of H, and it seems that this difference enters in a non-negligible way in the modeling of this phenomena. In fact, since $B^H(t)$ is not a semimartingale unless $H = 1/2$, the standard stochastic calculus involving the Itó integral cannot be used in the analysis of related stochastic evolution equations. There have been several papers devoted to the formulation of stochastic calculus for fBm [9–11] and differential/evolution equations driven by fBm [12–15] published in the past decade. We provide an outline of only the necessary concomitant technical details concerning the construction of the stochastic integral driven by an fBm in Section 2.

Often times, a more accurate model of such phenomena can be formulated by allowing the nonlinear perturbations to depend, in addition, on the probability distribution of the state process at time t. A prototypical example in the finite-dimensional setting would be an interacting N-particle system in which (1.1) describes the dynamics of the particles $x_1,\dots,x_N$ moving in a space U in which the probability measure μ is taken to be the empirical measure $\mu_N(t) = (1/N)\sum_{k=1}^{N}\delta_{x_k(t)}$, where $\delta_{x_k(t)}$ denotes the dirac measure. Researchers have investigated related models concerning diffusion processes in the finite-dimensional case (see [16–18]). Related infinite-dimensional problems in a Hilbert space setting have recently been examined (see [19–21]).

The purpose of this work is to study the class of abstract stochastic evolution equations obtained by accounting for more general nonlinear perturbations (in the sense of McKean-Vlasov equations, as described in [19]) in the mathematical description of phenomena involving an fBm. In particular, the existence and convergence results we present constitute generalizations of the theory governing standard models arising in the mathematical modeling of nonlinear diffusion processes [1, 16–19, 22], communication networks [4], Sobolev-type equations arising in the study of consolidation of clay [8], shear in second-order fluids [23], and fluid flow through fissured rocks [24]. As a part of our general discussion, we establish an approximation result concerning the effect of the dependence of the nonlinearity on the probability law of the state process, as well as the noise arising from the stochastic integral, for a special case of (1.1) arising often in applications.

The remainder of the paper is organized as follows. First, we make precise the necessary notation and function spaces, and gather certain preliminary results in Section 2. The main results are stated in Section 3 while their proofs form the contents of Section 4. Finally, we conclude the paper with a discussion of three concrete examples in Section 5.

2. Preliminaries

For details of this section, we refer the reader to [12, 25–29] and the references therein. Throughout this paper, U is a real separable Hilbert space with norm $\|\cdot\|_U$ and inner product $\langle\cdot,\cdot\rangle_U$ equipped with a complete orthonormal basis $\{e_j \mid j = 1,2,\ldots\}$. Also, $(\Omega,\mathcal{F},P)$ is a complete probability space. For brevity, we suppress the dependence of all random variables on ω throughout the manuscript.

We begin by making precise the definition of a U-valued fBm and related stochastic integral used in this manuscript. The approach we use coincides with the one formulated and analyzed in [10, 12]. Let $\{B_j^H(t)|t \geq 0\}_{j=1}^{\infty}$ be a sequence of independent, one-dimensional fBms with Hurst parameter $H \in (1/2,1)$ such that for all $j = 1,2,\ldots$ the following hold:

(i) $B_j^H(0) = 0$,

(ii) $E[B_j^H(t) - B_j^H(s)]^2 = |t-s|^{2H}\nu_j$,

(iii) $E[B_j^H(1)]^2 = \nu_j > 0$,

(iv) $\sum_{j=1}^{\infty}\nu_j < \infty$.

In such case, $\sum_{j=1}^{\infty} E\|B_j^H(t)e_j\|_U^2 = t^{2H}\sum_{j=1}^{\infty}\nu_j < \infty$, so that the following definition is meaningful.

Definition 2.1. For every $t \geq 0$, $B^H(t) = \sum_{j=1}^{\infty} B_j^H(t)e_j$ is a U-valued fBm, where the convergence is understood to be in the mean-square sense.

It has been shown in [12] that the covariance operator of $\{B^H(t) : t \geq 0\}$ is a positive nuclear operator Q such that

$$\operatorname{tr} Q(t,s) = \frac{1}{2}\sum_{j=1}^{\infty}\nu_j\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]. \tag{2.1}$$

Next, we outline the discussion leading to the definition of the stochastic integral associated with $\{B^H(t) : t \geq 0\}$ for bounded, measurable functions, as presented in [10, 12]. To begin, assume that $g : [0,T] \to \mathfrak{L}(V,U)$ is a simple function, that is, there exists $\{g_i : i = 1,\ldots,n\} \subseteq \mathbb{R}$ such that

$$g(t) = g_i \quad \forall\, t_{i-1} \leq t \leq t_i, \tag{2.2}$$

where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ and $\max_{1\leq i\leq n}\|g_i\|_{\mathfrak{L}(V,U)} = K$.

Definition 2.2. The U-valued stochastic integral $\int_0^T g(t)dB^H(t)$ is defined by

$$\int_0^T g(t)dB^H(t) = \sum_{j=1}^{\infty}\left(\int_0^T g(t)dB_j^H(t)\right)e_j = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{n} g_i\left[B_j^H(t_i) - B_j^H(t_{i-1})\right]\right)e_j. \tag{2.3}$$

As argued in [12, Lemma 2.2], this integral is well defined since

$$E\left\|\int_0^T g(t)dB^H(t)\right\|_U^2 \leq K^2T^{2H}\sum_{j=1}^{\infty}\nu_j < \infty. \tag{2.4}$$

Since the set of simple functions is dense in the space of bounded, measurable $\mathfrak{L}(V,U)$-valued functions, a standard density argument can be used to extend Definition 2.2 to the case of a general bounded, measurable integrand.

We make use of several different function spaces throughout this paper. For one, $\mathfrak{BL}(U)$ is the space of all bounded linear operators on U while $L^2(\Omega;U)$ stands for the space of all U-valued random variables y for which $E\|y\|_U^2 < \infty$. Also, $C([0,T];U)$ stands for the space of L^2-continuous U-valued random variables $y:[0,T]\to U$ such that

$$\|y\|^2_{C([0,T];U)} \equiv \sup_{0\le t\le T} E\|y(t)\|_U^2 < \infty. \tag{2.5}$$

The remaining function spaces coincide with those used in [19]; we recall them here for convenience. First, $\mathfrak{B}(U)$ stands for the Borel class on U and $\wp(U)$ represents the space of all probability measures defined on $\mathfrak{B}(U)$ equipped with the weak convergence topology. Define $\lambda : U \to \mathbb{R}^+$ by $\lambda(x) = 1 + \|x\|_U$, $x \in U$, and consider the space

$$\begin{aligned}\mathscr{C}(U) = \Big\{&\varphi : U \longrightarrow U \mid \varphi \text{ is continuous and}\\ &\|\varphi\|_{\mathscr{C}} = \sup_{x\in U}\frac{\|\varphi(x)\|_U}{\lambda^2(x)} + \sup_{x\neq y\in U}\frac{\|\varphi(x)-\varphi(y)\|_U}{\|x-y\|_U} < \infty\Big\}.\end{aligned} \tag{2.6}$$

For $p \ge 1$, we let

$$\begin{aligned}\wp^s_{\lambda^p}(U) = \Big\{&m : U \longrightarrow \mathbb{R} \mid m \text{ is a signed measure on } U \text{ such that}\\ &\|m\|_{\lambda^p} = \int_U \lambda^p(x)|m|(dx) < \infty\Big\},\end{aligned} \tag{2.7}$$

where $m = m^+ - m^-$ is the Jordan decomposition of m, and $|m| = m^+ + m^-$. Then, define the space $\wp_{\lambda^2}(U) = \wp^s_{\lambda^2}(U) \cap \wp(U)$ equipped with the metric ρ given by

$$\rho(\varsigma_1,\varsigma_2) = \sup\Big\{\int_U \varphi(x)(\varsigma_1-\varsigma_2)(dx) : \|\varphi\|_{\mathscr{C}} \le 1\Big\}. \tag{2.8}$$

It is known that $(\wp_{\lambda^2}(U),\rho)$ is a complete metric space. The space of all continuous $\wp_{\lambda^2}(U)$-valued functions defined on $[0,T]$, denoted by $\mathscr{C}_{\lambda^2} = \mathscr{C}_{\lambda^2}([0,T];(\wp_{\lambda^2}(U),\rho))$, is complete when equipped with the metric

$$D_T(\varsigma_1,\varsigma_2) = \sup_{t\in[0,T]} \rho(\varsigma_1(t),\varsigma_2(t)) \quad \forall \varsigma_1,\varsigma_2 \in \mathscr{C}_{\lambda^2}. \tag{2.9}$$

In addition to the familiar Young, Hölder, and Minkowski inequalities, the inequality of the form $(\sum_{i=1}^n a_i)^m \le m^{n-1}\sum_{i=1}^n a_i^m$, where a_i is a nonnegative constant $(i = 1,\dots,n)$ and $m,n \in \mathbb{N}$, will be used to establish various estimates.

We conclude this section with some comments regarding probability measures. The probability measure P induced by a U-valued random variable X, denoted P_X, is defined by $P \circ X^{-1} : \mathfrak{B}(U) \to [0,1]$. A sequence $\{P_n\} \subset \wp(U)$ is said to be weakly convergent to

P if $\int_\Omega f\,dP_n \to \int_\Omega f\,dP$, for every bounded, continuous function $f : U \to \mathbb{R}$; in such case, we write $P_n \xrightarrow{w} P$. Next, a family $\{P_n\}$ is tight if for each $\varepsilon > 0$, there exists a compact set K_ε such that $P_n(K_\varepsilon) \geq 1 - \varepsilon$ for all n. Bergström [25] established the equivalence of tightness and relative compactness of a family of probability measures. Consequently, the Arzelá-Ascoli theorem can be used to establish tightness.

Definition 2.3. Let $P \in \wp(U)$ and $0 \leq t_1 < t_2 < \cdots < t_k \leq T$. Define $\pi_{t_1,\ldots,t_k} : C([0,T];U) \to U^k$ by $\pi_{t_1,\ldots,t_k}(X) = (X(t_1),\ldots,X(t_k))$. The probability measures induced by $\pi_{t_1,\ldots,t_k}$ are the finite dimensional joint distributions of P.

Proposition 2.4 [28, page 37]. *If a sequence $\{X_n\}$ of U-valued random variables converges weakly to a U-valued random variable X in the mean-square sense, then the sequence of finite dimensional joint distributions corresponding to $\{P_{X_n}\}$ converges weakly to the finite dimensional joint distribution of P_X.*

The next theorem, in conjunction with Proposition 2.4, is the main tool used to prove one of the convergence results in this paper.

Theorem 2.5. *Let $\{P_n\} \subset \wp(U)$. If the sequence of finite dimensional joint distributions corresponding to $\{P_n\}$ converges weakly to the finite dimensional joint distribution of P and $\{P_n\}$ is relatively compact, then $P_n \xrightarrow{w} P$.*

3. Statement of results

We consider mild solutions of (1.1) in the following sense.

Definition 3.1. A stochastic process $x \in C([0,T];U)$ is a mild solution of (1.1) if

(i) $x(t) = S(t)x_0 + \int_0^t S(t-s)f(s,x(s),\mu(s))ds + \int_0^t S(t-s)g(s)dB^H(s)$ for all $0 \leq t \leq T$,

(ii) $\mu(t)$ is the probability distribution of $x(t)$ for all $0 \leq t \leq T$.

The following conditions on (1.1) are assumed throughout the manuscript unless otherwise specified.

(A1) $A : D(A) \subset U \to U$ is the infinitesimal generator of a strongly continuous semigroup $\{S(t) : t \geq 0\}$ on U such that $\|S(t)\|_{\mathfrak{B}\mathfrak{L}(U)} \leq M\exp(\alpha t)$ for all $0 \leq t \leq T$, for some $M \geq 1$, $\alpha > 0$;

(A2) $f : [0,T] \times U \times \wp_{\lambda^2}(U) \to U$ satisfies the following:

(i) there exists a positive constant M_f such that

$$\|f(t,z_1,\varsigma_1) - f(t,z_2,\varsigma_2)\|_U^2 \leq M_f\left[\|z_1 - z_2\|_U^2 + \rho^2(\varsigma_1,\varsigma_2)\right], \tag{3.1}$$

globally on $[0,T] \times U \times \wp_{\lambda^2}(U)$;

(ii) there exists a positive constant $\overline{M_f}$ such that

$$\|f(t,z,\varsigma)\|_U^2 \leq \overline{M_f}[\|z\|_U^2 + \|\varsigma\|_{\lambda^2}^2] \tag{3.2}$$

globally on $[0,T] \times U \times \wp_{\lambda^2}(U)$ (cf. Equation (2.7));

(A3) $g : [0,T] \to \mathfrak{L}(V,U)$ is a bounded, measurable mapping;

(A4) $\{B^H(t) : t \geq 0\}$ is a U-valued fBm;

(A5) $x_0 \in L^2(\Omega;U)$ is $(\mathcal{F}_0, \mathfrak{B}(U))$-measurable, where $\{\mathcal{F}_t : 0 \le t \le T\}$ is the family of σ-algebras $\mathcal{F}_t$ generated by $\{B^H(s) : 0 \le s \le t\}$.

(Henceforth, we write $M_S = \max_{0\le t\le T} \|S(t)\|_{\mathfrak{B}\mathfrak{L}(U)}$, which is finite by (A1).)

The following more general version of [12, Lemma 6], stated without proof, is critical in establishing several estimates.

Lemma 3.2. *Assume that $g : [0,T] \to \mathfrak{L}(V,U)$ satisfies (A3). Then, for all $0 \le t \le T$,*

$$E\left\|\int_0^t S(t-s)g(s)dB^H(s)\right\|_U^2 \le C_t \sum_{j=1}^{\infty} \nu_j, \tag{3.3}$$

where C_t is a positive constant depending on t, M_S, and the growth bound on g, and $\{\nu_j : j \in \mathbb{N}\}$ is defined as in the discussion leading to Definition 2.1.

Consider the solution map $\Phi : C([0,T];U) \to C([0,T];U)$ defined by

$$\Phi(x)(t) = S(t)x_0 + \int_0^t S(t-s)f(s,x(s),\mu(s))ds + \int_0^t S(t-s)g(s)dB^H(s), \quad 0 \le t \le T. \tag{3.4}$$

The first integral on the right-hand side of (3.4) is taken in the Bochner sense while the second is defined in Section 2. The operator Φ satisfies the following properties.

Lemma 3.3. *Assume that (A1)–(A5) hold. Then, Φ is a well defined, L^2-continuous mapping.*

The main existence-uniqueness result is as follows.

Theorem 3.4. *If (A1)–(A5) hold, then (1.1) has a unique mild solution x on $[0,T]$ with corresponding probability law $\mu \in \mathcal{C}_{\lambda^2}$, provided that $TM_f M_S < 1$.*

Mild solutions of (1.1) depend continuously on the initial data and probability distribution of the state process in the following sense.

Proposition 3.5. *Assume that (A1)–(A5) hold, and let x and y be the mild solutions of (1.1) (as guaranteed to exist by Theorem 3.4) corresponding to initial data x_0 and y_0 with respective probability distributions μ_x and μ_y. Then, there exists a positive constant M^* such that*

$$E\|x(t) - y(t)\|_U^2 \le M^*\left[\|x_0 - y_0\|_{L^2(\Omega;U)}^2 + D_T^2(\mu_x,\mu_y)\right]. \tag{3.5}$$

We now formulate various convergence and approximation results. For the first such result, let $n \ge 1$ and consider the Yosida approximation of (1.1) given by

$$\begin{aligned} dx_n(t) &= Ax_n(t)dt + nR(n;A)f(t,x_n(t),\mu_n(t))dt + nR(n;A)g(t)dB^H(t), \quad 0 \le t \le T, \\ x_n(t) &= nR(n;A)x_0, \end{aligned} \tag{3.6}$$

where $\mu_n(t)$ is the probability law of $x_n(t)$, and $R(n;A) = (I - nA)^{-1}$ is the resolvent operator of A. Assuming that (A1)–(A5) hold, one can invoke Theorem 3.4 to deduce that

(3.6) has a unique mild solution $x_n \in C([0,T];U)$ with probability law $\mu_n \in \mathscr{C}_{\lambda^2}$. The following convergence result holds.

Theorem 3.6. *Let x denote the unique mild solution of (1.1) on $[0,T]$ as guaranteed by Theorem 3.4. Then, the sequence of solutions of (3.6) converges to x in $C([0,T];U)$ as $n \to \infty$.*

The following corollary is needed to establish the weak convergence of probability measures.

Corollary 3.7. *The sequence of probability laws μ_n corresponding to the mild solutions x_n of (3.6) converges in $\mathscr{C}_{\lambda^2}$ to the probability law μ corresponding to the mild solution x of (1.1) as $n \to \infty$.*

Remark 3.8. We observe for later purposes that Corollary 3.7 implies that

$$\sup_{n\in\mathbb{N}} \sup_{0\le s\le T} \|\mu_n(s)\|_{\lambda^2}^2 < \infty. \tag{3.7}$$

In view of Theorem 3.6 and Corollary 3.7, the following continuity-type result can be established as in [19]. The details are omitted.

Proposition 3.9. *Assume that $E\|x_0\|_U^4 < \infty$. Then, for any function $F : [0,T] \times U \to \mathbb{R}$ satisfying the following:*

(i) *for each $N \in \mathbb{N}$, there exists a positive continuous function $k_N(t)$ such that*

$$|F(t,x) - F(t,y)| \le k_N(t)\|x-y\|_U \quad \forall 0 \le t \le T,\ \|x\|_U \le N,\ \|y\|_U \le N; \tag{3.8}$$

(ii) *there exists a positive continuous function $l(t)$ such that*

$$|F(t,x)| \le l(t)\lambda^2(x) \quad \forall 0 \le t \le T,\ x \in U, \tag{3.9}$$

it is the case that $\int_0^T \int_U F(t,x)d(\mu_n(t) - \mu(t))dt \to 0$ as $n \to \infty$.

We now consider the weak convergence of the probability measures induced by the mild solutions of (3.6). Let P_x denote the probability measure generated by the mild solution x of (1.1) and P_{x_n} the probability measure generated by x_n as in (3.6). We have the following.

Theorem 3.10. *If (A1)–(A5) hold and $x_0 \in L^4(\Omega,U)$, then $P_{x_n} \xrightarrow{w} P_x$ as $n \to \infty$.*

Next, we present a generalization of [12, Theorem 2] which allows for measure dependence in the nonlinearity. Specifically, let $m \in \mathbb{N}$ and $t \in [0,T]$ be given, and partition the interval $[0,t]$ using the points $\{t_j^m = (t/m)(j) : j = 0,1,\dots,m\}$. For each $j \in \{1,\dots,m\}$, consider the following recursively defined sequence:

$$\begin{gathered} x_j^m(s) = S(s-t_j^m)x_j^m(t_j^m) + \int_{t_j^m}^{s} S(s-\tau) f(\tau, x_j^m(\tau), \mu_j^m(\tau))d\tau, \quad s \in [t_j^m, t_{j+1}^m), \\ x_0^m(0) = x_0, \end{gathered} \tag{3.10}$$

where

$$x_j^m(t_j^m) = S(t_j^m - t_{j-1}^m)x_{j-1}^m(t_j^m) + \int_{t_{j-1}^m}^{t_j^m} S(t_j^m - \tau)g(\tau)dB^H(\tau). \tag{3.11}$$

Arguing as in Theorem 3.4, one can show that (3.10) has a unique mild solution $x_j^m \in C([t_j^m, t_{j+1}^m]; U)$ with probability distribution $\{\mu_j^m(\tau) : \tau \in [t_j^m, t_{j+1}^m]\}$. As such, it is meaningful to use (3.10) to define the sequence of processes $\{y_m(s) : 0 \le s \le t\}$ by

$$\begin{aligned} y_m(s) = S(s)x_0 &+ \int_0^s S(s-\tau)f(\tau, y_m(\tau), \mu_m(\tau))d\tau \\ &+ \int_0^{t_{m-1}^m} S(s - t_{m-1}^m)g(\tau)dB^H(\tau), \quad 0 \le s \le t, \end{aligned} \tag{3.12}$$

where $\mu_m(s) = \mu_m^j(s)$, $s \in [t_j^m, t_{j+1}^m]$, $j = 0, 1, \ldots, m$.

Lemma 3.11. *For each $0 \le t \le T$, there exists a positive constant C_t (independent of m) such that $\sup\{E\|y_m(s)\|_U^2 : 0 \le s \le t,\ m \in \mathbb{N}\} \le C_t$. Moreover, $\sup_{0\le t\le T} C_t < \infty$.*

Using this lemma, together with a standard Gronwall-type argument, yields the following approximation result.

Theorem 3.12. *Let $\{x(t) : 0 \le t \le T\}$ be the (unique) mild solution process of (1.1) with probability law $\{\mu(t) : 0 \le t \le T\}$. Then, for each $0 \le t \le T$, $\lim_{m\to\infty} E\|y_m(t) - x(t)\|_U^2 = 0$.*

Next, we formulate a result in which a deterministic initial-value problem is approximated by a sequence of stochastic equations of a particular form of (1.1) arising frequently in applications. Specifically, consider the deterministic initial-value problem

$$\begin{aligned} z'(t) &= Az(t) + F(t, z(t)), \quad 0 \le t \le T, \\ z(0) &= z_0, \end{aligned} \tag{3.13}$$

and for each $\varepsilon > 0$, consider the stochastic initial-value problem

$$\begin{aligned} dx_\varepsilon(t) &= \left(A_\varepsilon x_\varepsilon(t) + \int_U F_{1\varepsilon}(t,z)\mu_\varepsilon(t)(dz) + F_{2\varepsilon}(t, x_\varepsilon(t))\right)dt + g_\varepsilon(t)dB^H(t), \quad 0 \le t \le T, \\ x_\varepsilon(0) &= z_0, \\ \mu_\varepsilon(t) &= \text{probability distribution of } x_\varepsilon(t), \end{aligned} \tag{3.14}$$

in U. Here, $z_0 \in D(A_\varepsilon) = D(A)$ and $F_{i\varepsilon} : [0,T] \times U \to U$ $(i = 1, 2)$ are given mappings. Regarding (3.13), we assume that A satisfies (A1) and that the following hold that.

(A6) $F : [0,T] \times U \to U$ satisfies

(i) there exists a positive constant M_F such that $\|F(t,z_1) - F(t,z_2)\|_U \le M_F\|z_1 - z_2\|_U$ globally on $[0,T] \times U$,

(ii) there exists a positive constant $\overline{M_F}$ such that $\|F(t,z)\|_U \le \overline{M_F}\|z\|_U$ globally on $[0,T]\times U$, (A1) and (A6) guarantee the existence of a unique mild solution z of (3.13) on $[0,T]$ given by

$$z(t) = S(t)z_0 + \int_0^t S(t-s)F(s,z(s))\,ds, \quad 0 \le t \le T. \tag{3.15}$$

As for (3.14), we impose the following conditions on the data for each $\varepsilon > 0$:

(A7) $A_\varepsilon : D(A_\varepsilon) = D(A) \to D(A_\varepsilon)$ generates a strongly continuous semigroup $\{S_\varepsilon(t) : t \ge 0\}$ satisfying $S_\varepsilon(t) \to S(t)$ strongly as $\varepsilon \to 0^+$, uniformly in $t \in [0,T]$, and $\sup_{0\le t\le T}\|S_\varepsilon(t)\|_{\mathfrak{B}\mathfrak{L}(U)} \le M_S$ (the same growth bound as for the semigroup generated by A);

(A8) $F_{2\varepsilon} : [0,T]\times U \to U$ is Lipschitz in the second variable (with the same Lipschitz constant M_F used for F in (A6)), and $F_{2\varepsilon}(t,u) \to F(t,u)$ as $\varepsilon \to 0^+$ for all $u \in U$, uniformly in $t \in [0,T]$;

(A9) $F_{1\varepsilon} : [0,T]\times U \to U$ is a continuous mapping such that $\int_U F_{1\varepsilon}(t,z)\mu_\varepsilon(t)(dz) \to 0$ uniformly in t as $\varepsilon \to 0^+$;

(A10) $g_\varepsilon : [0,T] \to \mathfrak{L}(V,U)$ is a bounded, measurable function such that $g_\varepsilon(t) \to 0$ as $\varepsilon \to 0^+$, uniformly in $t \in [0,T]$.

Under these assumptions, the following result holds.

Theorem 3.13. *Let z and x_ε be the mild solutions of (3.13) and (3.14) on $[0,T]$, respectively. Then, there exist a positive constant ς and a positive function $\psi(\varepsilon)$ (which decreases to 0 as $\varepsilon \to 0^+$) such that $E\|x_\varepsilon(t) - z(t)\|_U^2 \le \psi(\varepsilon)\exp(\varsigma t)$ for all $t \in [0,T]$.*

4. Proofs

Proof of Lemma 3.3. Let $\mu \in \mathscr{C}_{\lambda^2}$ be fixed and consider the solution map Φ defined in (3.4).

Using the discussion in Section 2 and the properties of x, one can see that for any $x \in C([0,T];U)$, $\Phi(x)(t)$ is a well defined stochastic process for each $0 \le t \le T$. In order to verify the L^2-continuity of Φ on $[0,T]$, let $z \in C([0,T];U)$ and consider $0 \le t^* \le T$ and $|h|$ sufficiently small. Observe that

$$\begin{aligned}
&E\|\Phi(z)(t^*+h) - \Phi(z)(t^*)\|_U^2 \\
&\quad \le 4\Big[E\|[S(t^*+h-s) - S(t^*-s)]x_0\|_U^2 \\
&\qquad + E\Big\|\int_0^{t^*+h} S(t^*+h-s)f(s,x(s),\mu(s))\,ds - \int_0^{t^*} S(t^*-s)f(s,x(s),\mu(s))\,ds\Big\|_U^2 \\
&\qquad + E\Big\|\int_0^{t^*+h} S(t^*+h-s)g(s)\,dB^H(s) - \int_0^{t^*} S(t^*-s)g(s)\,dB^H(s)\Big\|_U^2\Big] \\
&\quad = 4\sum_{i=1}^{3}[I_i(t^*+h) - I_i(t^*)].
\end{aligned} \tag{4.1}$$

Since the semigroup property enables us to write

$$I_1(t^*+h)-I_1(t^*)=E\|((S(t^*+h)-S(t^*))x_0)\|_U^2=E\|(S(h)(S(t^*)x_0)-S(t^*)x_0)\|_U^2, \tag{4.2}$$

the strong continuity of $S(t)$ implies that the right-hand side of (4.2) goes to 0 as $|h|\to 0$. Next, using the Hölder inequality with (A2) yields

$$\begin{aligned} &E\left\|\int_{t^*}^{t^*+h} S(t^*+h-s)f(s,x(s),\mu(s))\,ds\right\|_U^2 \\ &\quad\le 4(\overline{M}_f)^2M_S^2h^2\left[1+\|x\|^2_{C([0,T];U)}+\sup_{t^*\le s\le t^*+h}\|\mu(s)\|^2_{\lambda^2}\right] \end{aligned} \tag{4.3}$$

which clearly goes to 0 as $|h|\to 0$. Also,

$$\begin{aligned} &E\left\|\int_0^{t^*}[S(h)-I]S(t^*-s)f(s,x(s),\mu(s))\,ds\right\|_U^2 \\ &\quad\le T\int_0^{t^*}\left\|[S(h)-I]S(t^*-s)\right\|^2_{\mathfrak{BL}(U)}E\|f(s,x(s),\mu(s))\|_U^2\,ds. \end{aligned} \tag{4.4}$$

Subsequently, using (A2)(ii) together with the strong continuity of $S(t)$, we can invoke the dominated convergence theorem to conclude that the right-hand side of (4.4) goes to 0 as $|h|\to 0$. Consequently, since $I_2(t^*+h)-I_2(t^*)$ is dominated by a sum of constant multiples of the right-hand sides of (4.3) and (4.4), we conclude that $I_2(t^*+h)-I_2(t^*)\to 0$ as $|h|\to 0$. It remains to show that $I_3(t^*+h)-I_3(t^*)\to 0$ as $|h|\to 0$. Observe that

$$\begin{aligned} &I_3(t^*+h)-I_3(t^*) \\ &=E\left\|\int_0^{t^*+h}S(t^*+h-s)g(s)dB^H(s)-\int_0^{t^*}S(t^*-s)g(s)dB^H(s)\right\|_U^2 \\ &=E\left\|\sum_{j=1}^{\infty}\int_0^{t^*+h}S(t^*+h-s)g(s)e_j\,dB_j^H(s)-\sum_{j=1}^{\infty}\int_0^{t^*}S(t^*-s)g(s)e_j\,dB_j^H(s)\right\|_U^2 \\ &=E\left\|\sum_{j=1}^{\infty}\int_{t^*}^{t^*+h}S(t^*+h-s)g(s)e_j\,dB_j^H(s)\right. \\ &\qquad\left.+\sum_{j=1}^{\infty}\int_0^{t^*}[S(t^*+h-s)-S(t^*-s)]g(s)e_j\,dB_j^H(s)\right\|_U^2 \\ &\le 2E\left[\left\|\sum_{j=1}^{\infty}\int_{t^*}^{t^*+h}S(t^*+h-s)g(s)e_j\,dB_j^H(s)\right\|_U^2\right. \\ &\qquad\left.+\left\|\sum_{j=1}^{\infty}\int_0^{t^*}[S(t^*+h-s)-S(t^*-s)]g(s)e_j\,dB_j^H(s)\right\|_U^2\right]. \end{aligned} \tag{4.5}$$

For the moment, assume that g is a simple function as defined in (2.2). Observe that for $m \in \mathbb{N}$, arguing as in [12, Lemma 6] yields

$$\begin{aligned}
&E\left\|\sum_{j=1}^{m}\int_0^{t^*}[S(t^*+h-s)-S(t^*-s)]g(s)e_j\,dB_j^H(s)\right\|_U^2\\
&\quad= E\left\|\sum_{j=1}^{m}\sum_{k=0}^{n-1}\overline{S}_h g_k e_j\left(B_j^H(t_{k+1})-B_j^H(t_k)\right)\right\|_U^2\\
&\quad= E\left\langle\sum_{j=1}^{m}\sum_{k=0}^{n-1}\overline{S}_h g_k e_j\left(B_j^H(t_{k+1})-B_j^H(t_k)\right),\sum_{j=1}^{m}\sum_{k=0}^{n-1}\overline{S}_h g_k e_j\left(B_j^H(t_{k+1})-B_j^H(t_k)\right)\right\rangle_U\\
&\quad\le \sum_{j=1}^{m}\sum_{k=0}^{n-1}\|\overline{S}_h\|^2_{\mathfrak{BL}(U)}K^2E\left(B_j^H(t_{k+1})-B_j^H(t_k),B_j^H(t_{k+1})-B_j^H(t_k)\right)_{\mathbb{R}}\\
&\quad\le \sup_{0\le s\le t^*}\|\overline{S}_h\|^2_{\mathfrak{BL}(U)}K^2E\left(B_j^H\right)^2\\
&\quad= \sup_{0\le s\le t^*}\|\overline{S}_h\|^2_{\mathfrak{BL}(U)}K^2\left(t^*\right)^{2H}\sum_{j=1}^{m}\nu_j,
\end{aligned}\tag{4.6}$$

where $\overline{S}_h = S(t^*-t_k+h)-S(t^*-t_k)$. Hence,

$$\begin{aligned}
&E\left\|\sum_{j=1}^{\infty}\int_0^{t^*}[S(t^*+h-s)-S(t^*-s)]g(s)e_j\,dB_j^H(s)\right\|_U^2\\
&\quad\le \lim_{m\to\infty}E\left\|\sum_{j=1}^{m}\int_0^{t^*}[S(t^*+h-s)-S(t^*-s)]g(s)e_j\,dB_j^H(s)\right\|_U^2\\
&\quad\le \lim_{m\to\infty}\sup_{0\le s\le t^*}\|\overline{S}_h\|^2_{\mathfrak{BL}(U)}K^2\left(t^*\right)^{2H}\sum_{j=1}^{m}\nu_j\\
&\quad= \sup_{0\le s\le t^*}\|\overline{S}_h\|^2_{\mathfrak{BL}(U)}K^2\left(t^*\right)^{2H}\sum_{j=1}^{\infty}\nu_j
\end{aligned}\tag{4.7}$$

and the right-hand side of (4.7) goes to 0 as $|h|\to 0$. Next, observe that

$$\begin{aligned}
&E\left\|\sum_{j=1}^{\infty}\int_{t^*}^{t^*+h}S(t^*+h-s)g(s)e_j\,dB_j^H(s)\right\|_U^2\\
&\quad= E\left\|\sum_{j=1}^{\infty}\int_0^{h}S(u)g(t^*+h-u)e_j\,dB_j^H(t^*+h-u)\right\|_U^2.
\end{aligned}\tag{4.8}$$

Using the property $E(B_j^H(s)-B_j^H(t))^2 = |t-s|^{2H}\nu_j$ with $s = t^*+h$ and $t = t^*$, we can argue as above to conclude that the right-hand side of (4.8) goes to 0 as $|h|\to 0$. Consequently, $I_3(t^*+h)-I_3(t^*)\to 0$ as $|h|\to 0$ when g is a simple function. Since the set of all

such simple functions is dense in $\mathfrak{L}(V,U)$, a standard density argument can be used to extend this conclusion to a general bounded, measurable function g. This establishes the L^2-continuity of Φ.

Finally, we assert that $\Phi(C([0,T];U)) \subset C([0,T];U)$. Indeed, the necessary estimates can be established as above, and when used in conjunction with Lemma 3.2, one can readily verify that $\sup_{0\le t\le T} E\|\Phi(x)(t)\|_U^2 < \infty$ for any $x \in C([0,T];U)$. Thus, we conclude that Φ is well defined, and the proof of Lemma 3.3 is complete. □

Proof of Theorem 3.4. Let $\mu \in \mathscr{C}_{\lambda^2}$ be fixed and consider the operator as Φ defined in (3.4). We know that Φ is well defined and L^2-continuous from Lemma 3.3. We now prove that Φ has a unique fixed point in $C([0,T;U])$. Indeed, for any $x, y \in C([0,T;U])$, (3.4) implies that

$$\begin{aligned} E\|(\Phi x)(t) - (\Phi y)(t)\|_U^2 &\le E\left\|\int_0^t S(t-s)[f(s,x(s),\mu(s)) - f(s,y(s),\mu(s))]ds\right\|_U^2 \\ &\le (TM_fM_S)^2\|x-y\|_{C([0,T];U)}^2, \quad 0 \le t \le T. \end{aligned} \tag{4.9}$$

Consequently, for a given $\mu \in \mathscr{C}_{\lambda^2}$ and $T > 0$, Φ has a unique fixed point $x_\mu \in C([0,T];U)$, provided that $TM_fM_S < 1$. In such case, we conclude that x_μ is a mild solution of (1.1).

To complete the proof, we must show that μ is, in fact, the probability law of x_μ. To this end, let $\mathscr{L}(x_\mu) = \{\mathscr{L}(x_\mu(t)) : t \in [0,T]\}$ represent the probability law of x_μ and define the map $\Psi : \mathscr{C}_{\lambda^2} \to \mathscr{C}_{\lambda^2}$ by $\Psi(\mu) = L(x_\mu)$. It is not difficult to see that $\mathscr{L}(x_\mu(t)) \in \wp_{\lambda^2}(U)$ for all $t \in [0,T]$ since $x_\mu \in C([0,T];U)$. Concerning the continuity of the map $t \mapsto \mathscr{L}(x_\mu(t))$, we first comment that an argument similar to the one used to establish Lemma 3.3 can be used to show that for sufficiently small $|h| > 0$,

$$\lim_{h\to 0} E\|x_\mu(t+h) - x_\mu(t)\|_U^2 = 0 \quad \forall 0 \le t \le T. \tag{4.10}$$

Consequently, since for all $t \in [0,T]$ and $\varphi \in \mathscr{C}_{\lambda^2}$, it is the case that

$$\begin{aligned} \left|\int_U \varphi(x)(\mathscr{L}(x_\mu(t+h)) - \mathscr{L}(x_\mu(t)))(dx)\right| &= |E[\varphi(x_\mu(t+h)) - \varphi(x_\mu(t))]| \\ &\le \|\varphi\|_{\lambda^2} E\|x_\mu(t+h) - x_\mu(t)\|_U, \end{aligned} \tag{4.11}$$

and hence

$$\begin{aligned} &\rho(\mathscr{L}(x_\mu(t+h)), \mathscr{L}(x_\mu(t))) \\ &\quad = \sup_{\|\varphi\|_{\lambda^2}\le 1} \int_U \varphi(x)(\mathscr{L}(x_\mu(t+h)) - \mathscr{L}(x_\mu(t)))(dx) \longrightarrow 0 \quad \text{as } |h| \longrightarrow 0 \end{aligned} \tag{4.12}$$

for any $0 \le t \le T$. Thus, $t \mapsto \mathscr{L}(x_\mu(t))$ is a continuous map, so that $\mathscr{L}(x_\mu) \in \mathscr{C}_{\lambda^2}$, thereby showing that Ψ is well defined. In order to show that Ψ has a unique fixed point in $\mathscr{C}_{\lambda^2}$,

let $\mu, \nu \in \mathscr{C}_{\lambda^2}$ and let x_μ, x_ν be the corresponding mild solutions of (1.1). A standard computation yields

$$\begin{aligned} &E\|x_\mu(t) - x_\nu(t)\|_U^2 \\ &\quad \le T(M_f M_S)^2 D_T^2(\mu,\nu) + (TM_f M_S)^2 \int_0^t E\|x_\mu(s) - x_\nu(s)\|_U^2\, ds, \quad 0 \le t \le T. \end{aligned} \tag{4.13}$$

Applying Gronwall's lemma and then taking the supremum over $[0,T]$ yields

$$\|x_\mu - x_\nu\|_{C([0,T];U)}^2 \le \varsigma_T D_T^2(\mu,\nu), \tag{4.14}$$

where $\varsigma_T = (TM_f M_S)^2 \exp((TM_f M_S)^2)$. Now, choose T small enough to ensure that $\varsigma_T < 1$; denote such a T by T_0. Then, since

$$\rho(\mathscr{L}(x_\mu(t)), \mathscr{L}(x_\nu(t))) \le E\|x_\mu(t) - x_\nu(t)\|_U \quad \forall 0 \le t \le T \tag{4.15}$$

(which follows directly from (4.11) and (4.12)), we have

$$\begin{aligned} \|\Psi(\mu) - \Psi(\nu)\|_{\lambda^2}^2 = D_{T_0}^2(\Psi(\mu),\Psi(\nu)) &\le \sup_{t\in[0,T_0]} E\|x_\mu(t) - x_\nu(t)\|_U^2 \\ &= \|x_\mu - x_\nu\|_{C([0,T_0];U)}^2 < \varsigma_{T_0} D_{T_0}^2(\mu,\nu), \end{aligned} \tag{4.16}$$

so that Ψ is a strict contraction on $\mathscr{C}_{\lambda^2}([0,T_0];(\wp_{\lambda^2}(U),\rho))$. Thus, (1.1) has a unique mild solution on $[0,T_0]$ with probability distribution $\mu \in \mathscr{C}_{\lambda^2}([0,T_0];(\wp_{\lambda^2}(U),\rho))$. The solution can then be extended, by continuity, to the entire interval $[0,T]$ in finitely many steps, thereby completing the proof of the theorem. □

Proof of Proposition 3.5. Computations similar to those used leading to the contractivity of the solution map in Theorem 3.4 can be used, along with Gronwall's lemma, to establish this result. The details are omitted. □

Proof of Theorem 3.6. Observe that

$$\begin{aligned} &E\|x_n(t) - x(t)\|_U^2 \\ &\quad \le 8\Bigg[E\|(nR(n;A) - I)S(t)x_0\|_U^2 \\ &\qquad + M_S^2 T \int_0^t E\|nR(n;A) f(s,x_n(s),\mu_n(s)) - f(s,x(s),\mu(s))\|_U^2\, ds \\ &\qquad + E\left\|\int_0^t S(t-s)[nR(n;A) - I]g(s)\,dB^H(s)\right\|_U^2\Bigg] \\ &\quad = 8\left[E\|(nR(n;A) - I)S(t)x_0\|_U^2 + \sum_{i=4}^{5} I_i(t)\right], \quad 0 \le t \le T. \end{aligned} \tag{4.17}$$

Standard computations imply that

$$I_4(t) \le 2M_S^2 T \int_0^t [E\|(nR(n;A)-I)f(s,x_n(s),\mu_n(s))\|_U^2 + 2M_f^2(E\|x_n(s)-x(s)\|_U^2 + \rho^2(\mu_n(s),\mu(s)))]ds, \quad 0 \le t \le T. \tag{4.18}$$

Further, the triangle inequality and (A2), together, imply

$$\begin{aligned}\int_0^t [E\|(nR(n;A)-I)f(s,x_n(s),\mu_n(s))\|_U^2\,ds \\ \le 2\int_0^t E\|nR(n;A)-I\|_{\mathfrak{B}\mathfrak{L}(U)}^2 \Big[2M_f^2(E\|x_n(s)-x(s)\|_U^2 + \rho^2(\mu_n(s),\mu(s))) \\ + E\|f(s,x(s),\mu(s))\|_U^2\Big]ds, \quad 0 \le t \le T.\end{aligned} \tag{4.19}$$

The boundedness of $E\|f(s,x(s),\mu(s)\|_U^2$ independent of n, together with the strong convergence of $nR(n;A)-I$ to 0, enables us to infer that the right-hand side of (4.19) goes to 0 as $n \to \infty$. Next, using Lemma 3.2 yields

$$I_5(t) \le M_S^2 E\left\|\int_0^t [nR(n;A)-I]g(s)dB^H(s)\right\|_U^2 \le \left(\sup_{0\le t\le T} C_t \sum_{j=1}^{\infty} \nu_j\right)\|nR(n;A)-I\|_{\mathfrak{B}\mathfrak{L}(U)}^2. \tag{4.20}$$

Using (4.18)–(4.20) in (4.17) gives rise to an inequality of the form

$$E\|x_n(t)-x(t)\|_U^2 \le \overline{\beta}_1 E\|x_0\|_U^2 + \overline{\beta}_2 \int_0^t E\|x_n(s)-x(s)\|_U^2\,ds, \quad 0 \le t \le T, \tag{4.21}$$

where $\overline{\beta}_i$ $(i = 1,2)$ are constant multiples of the quantity $\|nR(n;A)-I\|_{\mathfrak{B}\mathfrak{L}(U)}^2$. Consequently, applying Gronwall's lemma and then taking the supremum over $0 \le t \le T$ yields

$$\|x_n - x\|_{C([0,T];U)}^2 \le \overline{\beta}_1[1+E\|x_0\|_U^2]\exp(\overline{\beta}_2 T) \quad \forall n \ge 1. \tag{4.22}$$

Since the right-hand side of (4.22) goes to 0 as $n \to \infty$, the conclusion of the theorem follows. □

Proof of Corollary 3.7. This follows from the fact that

$$\begin{aligned}D_T^2(\mu_n,\mu) = \sup_{0\le t\le T} \rho^2(\mu_n(t),\mu(t)) \le \sup_{0\le t\le T} E\|x_n(t)-x(t)\|_U^2 \\ \le \|x_n - x\|_{C([0,T];U)}^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{4.23}$$

□

Proof of Theorem 3.10. Throughout the proof, C_i denotes a suitable positive constant independent of n. We will show that $\{P_{x_n}\}_{n=1}^{\infty}$ is relatively compact using the Arzela-Ascoli theorem.

First, we can show that $\{x_n\}$ is uniformly bounded in $C([0,T];U)$, that is, $\sup_{n\in\mathbb{N}} \sup_{0\le t\le T} E\|x_n(t)\|_U^2 < \infty$. The mild solution x_n of (3.6) is given by the variation

of parameters formula

$$\begin{aligned} x_n(t) &= S(t)nR(n;A)x_0 + \int_0^t S(t-s)nR(n;A)f\big(s,x_n(s),\mu_n(s)\big)\,ds \\ &\quad + \int_0^t S(t-s)nR(n;A)g(s)\,dB^H(s) \\ &= S(t)nR(n;A)x_0 + \sum_{i=6}^{7} I_i(t). \end{aligned} \tag{4.24}$$

Let $t \in [0,T]$. We consider each of the three terms on the right-hand side of (4.24) separately. First, since $nR(n;A)$ is contractive for each n, it follows that

$$E\big\|S(t)nR(n;A)x_0\big\|_U^2 \le M_S^2 E\|x_0\|_U^2. \tag{4.25}$$

Routine arguments involving (A2) and Remark 3.8 yield

$$E\|I_6(t)\|_U^2 \le 2TM_S^2\overline{M}_f^2\left[TC_1 + \int_0^t E\|x_n(s)\|_U^2\,ds\right]. \tag{4.26}$$

Arguing as in Lemma 3.3, using the contractivity of $nR(n;A)$, yields

$$E\left\|\int_0^t S(t-s)nR(n;A)g(s)\,dB^H(s)\right\|_U^2 = E\left\|\sum_{j=1}^{\infty}\int_0^t S(t-s)nR(n;A)g(s)e_j\,dB_j^H(s)\right\|_U^2 \le C_2. \tag{4.27}$$

Combining (4.25)–(4.27), we obtain that

$$E\|x_n(t)\|_U^2 \le C_3 + C_4\int_0^t E\|x_n(s)\|_U^2\,ds, \quad 0 \le t \le T. \tag{4.28}$$

Applying Gronwall's lemma now yields the uniform boundedness of $\{x_n\}$ in $C([0,T];U)$.

Next, we establish the equicontinuity of $\{x_n\}$. We will show that for every $n \in \mathbb{N}$ and for fixed $0 \le s \le t \le T$, $E\|x_n(t) - x_n(s)\|_U^4 \to 0$ (independently of n) as $t - s \to 0$. Indeed, for $0 \le s \le t \le T$, using the semigroup properties yields

$$E\big\|\big(S(t)-S(s)\big)nR(n;A)x_0\big\|_U^4 \le E\left(\int_s^t \|S(\tau)AnR(n;A)x_0\|_U\,d\tau\right)^4 \le M_S^4M_A^4E\|x_0\|_U^4(t-s)^4. \tag{4.29}$$

Also,

$$
\begin{aligned}
&E\|I_6(t)-I_6(s)\|_U^4 \\
&\quad\le E\Bigg(\int_0^s \|[S(t-\tau)-S(s-\tau)]nR(n;A)f(\tau,x_n(\tau),\mu_n(\tau))\|_U\,d\tau \\
&\qquad+\int_s^t \|S(t-\tau)nR(n;A)f(\tau,x_n(\tau),\mu_n(\tau))\|_U\,d\tau\Bigg)^4 \\
&\quad\le 4E\Bigg[\Bigg(\int_0^s \|[S(t-\tau)-S(s-\tau)]nR(n;A)f(\tau,x_n(\tau),\mu_n(\tau))\|_U\,d\tau\Bigg)^4 \\
&\qquad+\Bigg(\int_s^t \|S(t-\tau)nR(n;A)f(\tau,x_n(\tau),\mu_n(\tau))\|_U\,d\tau\Bigg)^4\Bigg] \\
&\quad\le 4E\Bigg[T^{1/4}\int_0^s \|[S(t-\tau)-S(s-\tau)]nR(n;A)f(\tau,x_n(\tau),\mu_n(\tau))\|_U^4\,d\tau \\
&\qquad+\Bigg(\int_s^t \|S(t-\tau)nR(n;A)f(\tau,x_n(\tau),\mu_n(\tau))\|_U\,d\tau\Bigg)^4\Bigg] \\
&\quad\le 4\Bigg[T^{1/4}\int_0^s E\Bigg(\int_{s-\tau}^{t-\tau} \|S(u)AnR(n;A)f(\tau,x_n(\tau),\mu_n(\tau))\|_U du\Bigg)^4 d\tau \\
&\qquad+E\Bigg(\int_s^t \|S(t-\tau)nR(n;A)f(\tau,x_n(\tau),\mu_n(\tau))\|_U d\tau\Bigg)^4\Bigg] \\
&\quad\le 4\Bigg[T^{1/4}(t-s)^4\int_0^s\int_{s-\tau}^{t-\tau} E\|S(u)AnR(n;A)f(\tau,x_n(\tau),\mu_n(\tau))\|_U^4 du\,d\tau \\
&\qquad+E\Bigg(\int_s^t \|S(t-\tau)nR(n;A)f(\tau,x_n(\tau),\mu_n(\tau))\|_U d\tau\Bigg)^4\Bigg] \\
&\quad\le 16(t-s)^5 M_S^4 M_A \overline{M_f}^4 (T^{5/4}+1)\left(1+\|x\|_{C([0,T];U)}^4+\left(\sup_{0\le t\le T}\|\mu_n(t)\|_{\lambda^2}\right)^4\right) \\
&\quad\le C_5(t-s)^5,
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
&E\|I_7(t)-I_7(s)\|_U^4 \\
&\quad\le E\Bigg\|\sum_{j=1}^{\infty}\Bigg[\int_0^t S(t-\tau)nR(n;A)g(\tau)e_j\,dB_j^H(\tau) \\
&\qquad\qquad-\int_0^s S(s-\tau)nR(n;A)g(\tau)e_j\,dB_j^H(\tau)\Bigg]\Bigg\|_U^4 \\
&\quad\le E\Bigg\|\sum_{j=1}^{\infty}\int_0^s [S(t-\tau)-S(s-\tau)]nR(n;A)g(\tau)e_j\,dB_j^H(\tau)\Bigg\|_U^4 \\
&\qquad+E\Bigg\|\sum_{j=1}^{\infty}\int_s^t S(t-\tau)nR(n;A)g(\tau)e_j\,dB_j^H(\tau)\Bigg\|_U^4.
\end{aligned}
\tag{4.31}
$$

One can argue as in Lemma 3.3 to show that the first term on the right-hand side of (4.31) goes to 0 as $(t-s)\to 0$. Likewise, an application of Lemma 3.2 shows that the second term also goes to 0 as $(t-s)\to 0$. Thus, the estimates (4.29)–(4.31) then yield the equicontinuity of $\{x_n\}$. Therefore, we conclude that the family $\{P_{x_n}\}_{n=1}^{\infty}$ is relatively compact by Arzela-Ascoli, and therefore tight (cf. Section 2). Hence, by Proposition 2.4, the finite dimensional joint distributions of P_{x_n} converge weakly to P and so, by Theorem 2.5, $P_{x_n}\xrightarrow{w}P_x$, as $n\to\infty$. □

Proof of Lemma 3.11. A standard Gronwall argument involving (A2)(ii) and Lemma 3.2 can be used to establish this result. □

Proof of Theorem 3.12. Since

$$\begin{aligned} y_m(t)-x(t) &= \int_0^t S(t-\tau)\left[f(\tau,y_m(\tau),\mu_m(\tau))-f(\tau,x(\tau),\mu(\tau))\right]d\tau \\ &\quad+\int_{t_{m-1}^m}^t S(t-t_{m-1}^m)g(\tau)dB^H(\tau) \end{aligned} \tag{4.32}$$

for all $0\le t\le T$, using (A2)(i), Lemma 3.2, and the observations in the proof of Corollary 3.7 yields

$$\begin{aligned} &E\|y_m(t)-x(t)\|_U^2 \\ &\quad\le 2M_S^2M_f^2T\int_0^t\left[E\|y_m(\tau)-x(\tau)\|_U^2+\rho^2(\mu_m(\tau),\mu(\tau))\right]d\tau+M_S^2C_t^2(t-t_{m-1}^m)\sum_{j=1}^{\infty}\nu_j \\ &\quad\le 4M_S^2M_f^2T\int_0^t E\|y_m(\tau)-x(\tau)\|_U^2d\tau+M_S^2C_t^2(t-t_{m-1}^m)\sum_{j=1}^{\infty}\nu_j. \end{aligned} \tag{4.33}$$

So, an application of Gronwall's lemma yields

$$E\|y_m(t)-x(t)\|_U^2\le\left(M_S^2C_t^2\sum_{j=1}^{\infty}\nu_j\right)(t-t_{m-1}^m)\exp\left(4M_S^2M_f^2t\right),\quad 0\le t\le T,\ m\in\mathbb{N}. \tag{4.34}$$

Observe that as $m\to\infty$ the right-hand side of (4.34) goes to 0 since $t-t_{m-1}^m\to 0$ as $m\to\infty$. This completes the proof. □

Proof of Theorem 3.13. We estimate each term of the representation formula for $E\|x_\varepsilon(t)-z(t)\|_U^2$ separately. First, (A7) guarantees the existence of a positive constant K_1 and a positive function $\alpha_1(\varepsilon)$ (which decreases to 0 as $\varepsilon\to 0$) such that for sufficiently small $\varepsilon>0$,

$$E\|S_\varepsilon(t)z_0-S(t)z_0\|_U^2\le K_1\alpha_1(\varepsilon),\quad 0\le t\le T. \tag{4.35}$$

Next, we estimate $E\|\int_0^t(S_\varepsilon(t-s)F_{2\varepsilon}(s,x_\varepsilon(s))-S(t-s)F(s,z(s)))ds\|_U^2$. To this end, note that the continuity of F, together with (A6), enables us to infer the existence of $K_2>0$

and $\alpha_2(\varepsilon)$ (as above) such that for sufficiently small $\varepsilon > 0$,

$$\int_0^t E\|[S_\varepsilon(t-s) - S(t-s)]F(s,z(s))\|_U^2\, ds \le K_2\alpha_2(\varepsilon) \tag{4.36}$$

for all $0 \le t \le T$. Also, observe that Young's inequality and (A8), together, imply

$$\begin{aligned}
&\int_0^t E\|S_\varepsilon(t-s)[F_{2\varepsilon}(s,x_\varepsilon(s)) - F(s,z(s))]\|_U^2\, ds \\
&\quad \le M_S^2 \int_0^t E\|F_{2\varepsilon}(s,x_\varepsilon(s)) - F_{2\varepsilon}(s,z(s)) + F_{2\varepsilon}(s,z(s)) - F(s,z(s))\|_U^2\, ds \\
&\quad \le 4M_S^2 \int_0^t \left[M_F^2 E\|x_\varepsilon(s) - z(s)\|_U^2 + E\|F_{2\varepsilon}(s,z(s)) - F(s,z(s))\|_U^2\right] ds.
\end{aligned} \tag{4.37}$$

Note that (A8) guarantees the existence of $K_3 > 0$ and $\alpha_3(\varepsilon)$ (as above) such that for sufficiently small $\varepsilon > 0$, $E\|F_{2\varepsilon}(s,z(s)) - F(s,z(s))\|_U^2 \le K_3\alpha_3(\varepsilon)$ for all $0 \le t \le T$. So, we can continue the inequality (4.37) to conclude that

$$\int_0^t E\|S_\varepsilon(t-s)[F_{2\varepsilon}(s,x_\varepsilon(s)) - F(s,z(s))]\|_U^2\, ds \le 4M_F^2 \int_0^t E\|x_\varepsilon(s) - z(s)\|_U^2\, ds + 4TK_3\alpha_3(\varepsilon). \tag{4.38}$$

Using (4.37) and (4.38), together with the Hölder, Minkowski, and Young inequalities, yields

$$\begin{aligned}
&E\left\|\int_0^t (S_\varepsilon(t-s)F_{2\varepsilon}(s,x_\varepsilon(s)) - S(t-s)F(s,z(s)))\, ds\right\|_U^2 \\
&\quad \le 4T^{1/2}\left[K_2\alpha_2(\varepsilon) + 4TK_3\alpha_3(\varepsilon) + 4M_S^2M_F^2 \int_0^t E\|x_\varepsilon(s) - z(s)\|_U^2\, ds\right].
\end{aligned} \tag{4.39}$$

Next, (A9) guarantees the existence of $K_4 > 0$ and $\alpha_4(\varepsilon)$ (as above) such that for sufficiently small $\varepsilon > 0$,

$$E\left\|\int_U F_{1\varepsilon}(s,z)\mu_\varepsilon(s)(dz)\right\|_U^2 \le K_4\alpha_4(\varepsilon) \quad \forall 0 \le s \le T. \tag{4.40}$$

As such, we have

$$\begin{aligned}
&E\left\|\int_0^t S_\varepsilon(t-s)\int_U F_{1\varepsilon}(s,z)\mu_\varepsilon(s)(dz)ds\right\|_U^2 ds \\
&\quad \le M_S^2 T \int_0^t E\left\|\int_U F_{1\varepsilon}(s,z)\mu_\varepsilon(s)(dz)\right\|_U^2 \le M_S^2T^2K_4\alpha_4(\varepsilon)
\end{aligned} \tag{4.41}$$

for all $0 \le t \le T$.

It remains to estimate $E\|\int_0^t S_\varepsilon(t-s)g_\varepsilon(s)dB^H(s)\|_U^2$. Note that (A10) implies the existence of $K_5 > 0$ and $\alpha_5(\varepsilon)$ (as above) such that for sufficiently small $\varepsilon > 0$,

$$E\|g_\varepsilon(t)\|_{\mathfrak{L}(V;U)}^2 \le K_5\alpha_5(\varepsilon), \quad 0 \le t \le T. \tag{4.42}$$

First, assume that g_ε is a simple function. Observe that for $m \in \mathbb{N}$, we proceed in a manner similar to the one used to establish (4.6) to obtain that

$$\begin{aligned}
&E\left\|\sum_{j=1}^{m}\int_0^t S_\varepsilon(t-s)g_\varepsilon(s)e_j\,dB_j^H(s)\right\|_U^2 \\
&\quad= E\left\|\sum_{j=1}^{m}\sum_{k=0}^{n-1} S_\varepsilon(t-t_k)g_k e_j\left(B_j^H(t_{k+1})-B_j^H(t_k)\right)\right\|_U^2 \\
&\quad= E\left\langle \sum_{j=1}^{m}\sum_{k=0}^{n-1} S_\varepsilon(t-t_k)g_k e_j\left(B_j^H(t_{k+1})-B_j^H(t_k)\right),\right. \\
&\qquad\left.\sum_{j=1}^{m}\sum_{k=0}^{n-1} S_\varepsilon(t-t_k)g_k e_j\left(B_j^H(t_{k+1})-B_j^H(t_k)\right)\right\rangle_U \\
&\quad\le \sum_{j=1}^{m}\sum_{k=0}^{n-1} \|S_\varepsilon(t-t_k)\|_{\mathfrak{BL}(U)}^2 K_5\alpha_5(\varepsilon)E\left(B_j^H(t_{k+1})-B_j^H(t_k),B_j^H(t_{k+1})-B_j^H(t_k)\right)_{\mathbb{R}} \\
&\quad\le \sup_{0\le s\le t}\|S_\varepsilon(t-t_k)\|_{\mathfrak{BL}(U)}^2 K_5\alpha_5(\varepsilon)E\left(B_j^H\right)^2 \\
&\quad= \sup_{0\le s\le t}\|S_\varepsilon(t-t_k)\|_{\mathfrak{BL}(U)}^2 K_5\alpha_5(\varepsilon)t^{2H}\sum_{j=1}^{m}\nu_j.
\end{aligned} \tag{4.43}$$

Hence,

$$\begin{aligned}
&E\left\|\int_0^t S_\varepsilon(t-s)g_\varepsilon(s)dB^H(s)\right\|_U^2 \\
&\quad= E\left\|\sum_{j=1}^{\infty}\int_0^t S_\varepsilon(t-s)g_\varepsilon(s)e_j\,dB_j^H(s)\right\|_U^2 \\
&\quad\le \lim_{m\to\infty} E\left\|\sum_{j=1}^{m}\int_0^t S_\varepsilon(t-s)g_\varepsilon(s)e_j\,dB_j^H(s)\right\|_U^2 \\
&\quad\le \lim_{m\to\infty}\left(\sup_{0\le s\le t}\|S_\varepsilon(t-t_k)\|_{\mathfrak{BL}(U)}^2 K_5\alpha_5(\varepsilon)t^{2H}\sum_{j=1}^{m}\nu_j\right) \\
&\quad\le \left(M_S^2 t^{2H}\sum_{j=1}^{\infty}\nu_j\right)K_5\alpha_5(\varepsilon), \quad 0\le t\le T.
\end{aligned} \tag{4.44}$$

Since the set of all such simple functions is dense in $\mathfrak{L}(V,U)$, a standard density argument can be used to establish estimate (4.44) for a general bounded, measurable function g_ε.

Now, using (4.35)–(4.44), we conclude that there exist positive constants $\overline{K_i}$ $(i = 1,\dots,5)$ such that

$$E\|x_\varepsilon(t) - z(t)\|_U^2 \le \sum_{i=1}^{5} \overline{K_i}\alpha_i(\varepsilon) + 4M_S^2 M_F^2 \int_0^t E\|x_\varepsilon(s) - z(s)\|_U^2\, ds, \quad 0 \le t \le T, \tag{4.45}$$

so that an application of Gronwall's lemma implies

$$E\|x_\varepsilon(t) - z(t)\|_U^2 \le \Psi(\varepsilon)\exp(\varsigma t), \quad 0 \le t \le T, \tag{4.46}$$

where $\varsigma = 4M_S^2 M_F^2$ and $\Psi(\varepsilon) = \sum_{i=1}^{5} \overline{K_i}\alpha_i(\varepsilon)$. This completes the proof. □

5. Examples

Example 5.1. Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\mathcal{D}$. Consider the following initial boundary value problem:

$$\begin{aligned}
\partial x(t,z) = {} & \Big(\Delta_z x(t,z) + F_1(t,z,x(t,z)) \\
& + \int_{L^2(\mathcal{D})} F_2(t,z,y)\mu(t,z)(dy)\Big)\partial t + g(t,z)dB^H(t), \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. on } (0,T)\times\mathcal{D}, \\
x(t,z) = {} & 0, \quad \text{a.e. on } (0,T)\times\partial\mathcal{D}, \\
x(0,z) = {} & x_0(z), \quad \text{a.e. on } \mathcal{D},
\end{aligned} \tag{5.1}$$

where $x : [0,T]\times D \to \mathbb{R}$, $F_1 : [0,T]\times\mathcal{D}\times\mathbb{R} \to \mathbb{R}$, $F_2 : [0,T]\times\mathcal{D}\times L^2(\mathcal{D}) \to L^2(\mathcal{D})$, $\mu(t,\cdot) \in \wp_{\lambda^2}(L^2(\mathcal{D}))$ is the probability law of $x(t,\cdot)$, $g : [0,T]\times\mathcal{D} \to \mathfrak{L}(\mathbb{R}, L^2(\mathcal{D}))$, and $\{B^H(t) : 0 \le t \le T\}$ is a real fBm. We impose the following conditions:

(A11) F_1 satisfies the Caratheodory conditions (i.e., measurable in (t,z) and continuous in the third variable) such that
  (i) $|F_1(t,y,z)| \le \overline{M}_{F_1}[1+|z|]$ for all $0 \le t \le T$, $y \in \mathcal{D}$, $z \in \mathbb{R}$, and some $\overline{M}_{F_1} > 0$,
  (ii) $|F_1(t,y,z_1) - F_1(t,y,z_2)| \le M_{F_1}|z_1 - z_2|$ for all $0 \le t \le T$, $y \in \mathcal{D}$, $z_1, z_2 \in \mathbb{R}$, and some $M_{F_1} > 0$;

(A12) F_2 satisfies the Caratheodory conditions and
  (i) $\|F_2(t,y,z)\|_{L^2(\mathcal{D})} \le \overline{M}_{F_2}[1 + \|z\|_{L^2(\mathcal{D})}]$ for all $0 \le t \le T$, $y \in \mathcal{D}$, $z \in L^2(\mathcal{D})$, and some $\overline{M}_{F_2} > 0$,
  (ii) $F_2(t,y,\cdot) : L^2(\mathcal{D}) \to L^2(\mathcal{D})$ is in $\mathcal{C}$ for each $0 \le t \le T$, $y \in \mathcal{D}$;

(A13) $g : [0,T]\times\mathfrak{D} \to \mathfrak{L}(\mathbb{R}, L^2(\mathcal{D}))$ is a bounded, measurable function.

We have the following theorem.

Theorem 5.2. *Assume that (A11)–(A13) hold. If $M_S M_f T < 1$, then (5.1) has a unique mild solution $x \in C([0,T]; L^2(\mathcal{D}))$ with probability law $\{\mu(t,\cdot) : 0 \le t \le T\}$.*

Proof. Let $U = L^2(\mathfrak{D})$ and $V = \mathbb{R}^N$. Define the operator

$$Ax(t,\cdot) = \Delta_z x(t,\cdot), \quad x \in H^2(\mathfrak{D}) \cap H_0^1(\mathfrak{D}). \tag{5.2}$$

It is known that A generates a strongly continuous semigroup $\{S(t)\}$ on $L^2(\mathfrak{D})$ (see [11]), so that (A1) is satisfied. Define $f : [0,T] \times U \times \wp_{\lambda^2}(U) \to U$, $g : [0,T] \times U \to \mathfrak{L}(V,U)$, and $x_0 \in L^2(\Omega;U)$, respectively, by

$$f(t,x(t),\mu(t))(z) = F_1(t,z,x(t,z)) + \int_{L^2(\mathfrak{D})} F_2(t,z,y)\mu(t,z)(dy), \tag{5.3}$$

$$\begin{aligned} g(t)(z) &= g(t,z) \quad \forall 0 \le t \le T,\ z \in D, \\ x_0(0)(z) &= x_0(z) \quad \forall z \in D. \end{aligned} \tag{5.4}$$

Using these identifications, (5.1) can be written in the abstract form (1.1). Clearly, (A3)–(A5) are satisfied. We now show that f (as defined in (5.3)) satisfies (A2). To this end, observe that from (A11)(i),

$$\begin{aligned} \|F_1(t,\cdot,x(t,\cdot))\|_{L^2(\mathfrak{D})} &\le \overline{M}_{F_1}\left[\int_D [1+|x(t,z)|]^2 dz\right]^{1/2} \\ &\le 2\overline{M}_{F_1}\left[m(\mathfrak{D}) + \|x(t,\cdot)\|^2_{L^2(\mathfrak{D})}\right]^{1/2} \\ &\le 2\overline{M}_{F_1}\left[\sqrt{m(\mathfrak{D})} + \|x\|_{C([0,T];L^2(\mathfrak{D}))}\right] \\ &\le M^*_{F_1}\left[1 + \|x\|_{C([0,T];L^2(\mathfrak{D}))}\right], \end{aligned} \tag{5.5}$$

for all $0 \le t \le T$ and $x \in C([0,T];L^2(\mathfrak{D}))$, where

$$M^*_{F_1} = \begin{cases} 2\overline{M}_{F_1}\sqrt{m(\mathfrak{D})} & \text{if } m(\mathfrak{D}) > 1, \\ 2\overline{M}_{F_1} & \text{if } m(\mathfrak{D}) \le 1. \end{cases} \tag{5.6}$$

(Here, m denotes Lebesgue measure in $\mathbb{R}^N$.) Also, from (A11)(ii), we obtain that

$$\begin{aligned} \|F_1(t,\cdot,x(t,\cdot)) - F_1(t,\cdot,y(t,\cdot))\|_{L^2(\mathfrak{D})} &\le M_{F_1}\left[\int_D |x(t,z) - y(t,z)|^2 dz\right]^{1/2} \\ &= M_{F_1}\|x-y\|_{C([0,T];L^2(\mathfrak{D}))}. \end{aligned} \tag{5.7}$$

Next, using (A12)(i) together with the Hölder inequality, we observe that

$$\begin{aligned}
&\left\|\int_{L^2(\mathfrak{D})} F_2(t,\cdot,y)\mu(t,\cdot)(dy)\right\|_{L^2(\mathfrak{D})} \\
&\quad = \left[\int_{\mathfrak{D}}\left[\int_{L^2(\mathfrak{D})} F_2(t,z,y)\mu(t,z)(dy)\right]^2 dz\right]^{1/2} \\
&\quad \le \left[\int_{\mathfrak{D}}\int_{L^2(\mathfrak{D})} \|F_2(t,z,y)\|^2_{L^2(\mathfrak{D})}\mu(t,z)(dy)dz\right]^{1/2} \\
&\quad \le \overline{M}_{F_2}\left[\int_{D}\left(\int_{L^2(\mathfrak{D})} (1+\|y\|_{L^2(\mathfrak{D})})^2\mu(t,z)(dy)\right)dz\right]^{1/2} \\
&\quad \le \overline{M}_{F_2}\sqrt{m(\mathfrak{D})}\sqrt{\|\mu(t)\|_{\lambda^2}} \\
&\quad \le \overline{M}_{F_2}\sqrt{m(\mathfrak{D})}(1+\|\mu(t)\|_{\lambda^2}) \quad \forall 0\le t\le T,\ \mu\in\wp_{\lambda^2}(U).
\end{aligned} \tag{5.8}$$

Also, invoking (A11)(ii) enables us to see that for all $\mu,\nu\in\wp_{\lambda^2}(U)$,

$$\begin{aligned}
&\left\|\int_{L^2(\mathfrak{D})} F_2(t,\cdot,y)\mu(t,\cdot)(dy)-\int_{L^2(\mathfrak{D})} F_2(t,\cdot,y)\nu(t,\cdot)(dy)\right\|_{L^2(\mathfrak{D})} \\
&\quad = \left\|\int_{L^2(\mathfrak{D})} F_2(t,\cdot,y)(\mu(t,\cdot)-\nu(t,\cdot))(dy)\right\|_{L^2(\mathfrak{D})} \\
&\quad \le \|\rho(\mu(t),\nu(t))\|_{L^2(\mathfrak{D})} \\
&\quad \le \sqrt{m(\mathfrak{D})}\rho(\mu(t),\nu(t)) \quad \forall 0\le t\le T.
\end{aligned} \tag{5.9}$$

Combining (5.5) and (5.8), we see that f satisfies (A2)(i) with $\overline{M}_{f_1} = 2\cdot\max\{\overline{M}_{F_2}\sqrt{m(\mathfrak{D})}, M^*_{F_1}\}$, and combining (5.7) and (5.9), we see that f satisfies (A2)(ii) with $M_{f_1} = \max\{M_{F_1},\sqrt{m(\mathfrak{D})}\}$. Thus, we can invoke Theorem 3.4 to conclude that (5.1) has a unique mild solution $x\in C([0,T];L^2(\mathfrak{D})))$ with probability law $\{\mu(t,\cdot):0\le t\le T\}$. □

Example 5.3. We now consider a modified version of Example 5.1 which constitutes a generalization of the model considered in [12]. Precisely, let $\mathfrak{D}=\mathbb{R}$ and consider the initial boundary value problem given by

$$\begin{aligned}
\partial x(t,z) &= \big(-(I-\Delta_z)^{\gamma/2}(-\Delta_z)^{\alpha/2}x(t,z)+F_1(t,z,x(t,z)) \\
&\qquad + \int_{L^2(\mathfrak{D})} F_2(t,z,y)\mu(t,z)(dy)\big)\partial t + g(t,z)dB^H(t), \quad \text{a.e. on } (0,T)\times\mathfrak{D} \\
x(t,z) &= 0, \quad \text{a.e. on } (0,T)\times\partial\mathfrak{D}, \\
x(0,z) &= x_0(z), \quad \text{a.e. on } \mathfrak{D},
\end{aligned} \tag{5.10}$$

The operator $(-\Delta_z)^{\alpha/2}$ is defined by

$$\begin{aligned}(-\Delta_z)^{\alpha/2}h(z) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{izy}|y|^{\alpha}\widehat{h}(y)dy,\\ D\Big((-\Delta_z)^{\alpha/2}\Big) &= \Big\{h\in L^p_w(\mathbb{R}) : h,\ |y|^{\alpha}\widehat{h}(y)\in L^1(\mathbb{R})\cap L^2(\mathbb{R}),\\ &\qquad \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{izy}|y|^{\alpha}\widehat{h}(y)dy\in L^p_w(\mathbb{R})\Big\},\end{aligned} \tag{5.11}$$

where $\widehat{h}$ denotes the Fourier transform of h, and for $p>1$, the Banach space $L^p_w(\mathbb{R})$ is given by

$$L^p_w(\mathbb{R}) = \Big\{h : h \text{ is measurable and } \|h\|_{L^p_w(\mathbb{R})} = \int_{\mathbb{R}} |h(z)|^p w(z)dz < \infty\Big\}, \tag{5.12}$$

where $w(z)=(1+z^2)^{-\xi/2}$ for $\xi>1$. Also, the operator $(I-\Delta_z)^{\gamma/2}$ is defined by

$$\begin{aligned}(I-\Delta_z)^{\gamma/2}h(z) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{izy}(1+y^2)^{\gamma/2}\widehat{h}(y)dy,\\ D\Big((I-\Delta_z)^{\gamma/2}\Big) &= \Big\{h\in L^p_w(\mathbb{R}) : h,\ (1+y^2)^{\gamma/2}\widehat{h}(y)\in L^1(\mathbb{R})\cap L^2(\mathbb{R}),\\ &\qquad \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{izy}(1+y^2)^{\gamma/2}\widehat{h}(y)dy\in L^p_w(\mathbb{R})\Big\}.\end{aligned} \tag{5.13}$$

As shown in [12, Proposition 1], the operator $-(I-\Delta_z)^{\gamma/2}(-\Delta_z)^{\alpha/2}$ generates a strongly continuous semigroup on $L^p_w(\mathbb{R})$, assuming that $\alpha+\gamma>(\lambda-1)/p$ and $\lambda<p/q$, (where $q\in(1,2]$). As such, by taking $U=L^p_w(\mathbb{R})$ and $V=\mathbb{R}$, and defining the operator $A=-(I-\Delta_z)^{\gamma/2}(-\Delta_z)^{\alpha/2}$, we can argue as in Example 5.1 to show that (5.10) has a unique mild solution $x\in C([0,T];L^p_w(\mathbb{R}))$ with probability law $\{\mu(t,\cdot):0\le t\le T\}$.

Example 5.4. Consider the following initial-boundary value problem of Sobolev type:

$$\begin{aligned}&\partial\big(x(t,z)-x_{zz}(t,z)\big)-x_{zz}(t,z)\partial t\\ &\quad=\Big(F_1(t,z,x(t,z))+\int_{L^2(0,\pi)}F_2(t,z,y)\mu(t,z)(dy)\Big)\partial t+g(t,z)dB^H(t),\\ &\qquad\qquad 0\le z\le\pi,\ 0\le t\le T,\end{aligned} \tag{5.14}$$

$$\begin{aligned}&x(t,0)=x(t,\pi)=0,\quad 0\le t\le T,\\ &x(0,z)=x_0(z),\quad 0\le z\le\pi,\end{aligned} \tag{5.15}$$

where $x:[0,T]\times[0,\pi]\to\mathbb{R}$, $F_1:[0,T]\times[0,\pi]\times\mathbb{R}\to\mathbb{R}$ and $F_2:[0,T]\times[0,\pi]\times L^2(0,\pi)\to L^2(0,\pi)$ satisfy (A11) and (A12), $\mu(t,\cdot)\in\wp_{\lambda^2}(L^2(0,\pi))$ is the probability law of $x(t,\cdot)$, $g:[0,T]\times[0,\pi]\to\mathfrak{L}(\mathbb{R}^N,L^2(0,\pi))$ is a bounded measurable mapping, and $\{\beta^H(t):0\le t\le T\}$ is an fBm. We have the following theorem.

Theorem 5.5. *Under these assumptions, (5.15) has a unique mild solution* $x \in C([0,T]; L^2(0,\pi))$, *provided that* $M_S M_f T < 1$.

Proof. Let $U = L^2(0,\pi)$, $V = \mathbb{R}$, and define the operators $A : D(A) \subset U \to U$ and $B : D(B) \subset U \to U$, respectively, by

$$Ax(t,\cdot) = -x_{zz}(t,\cdot), \qquad Bx(t,\cdot) = x(t,\cdot) - x_{zz}(t,\cdot), \tag{5.16}$$

with domains

$$\begin{aligned} D(A) = D(B) = \{x \in L^2(0,\pi) : x,\ x_z \text{ are absolutely continuous,} \\ x_{zz} \in L^2(0,\pi),\ x(0) = x(\pi) = 0\}. \end{aligned} \tag{5.17}$$

Define f and g as in Example 5.1 (with $L^2(0,\pi)$ in place of $L^2(\mathcal{D})$). Then, (5.15) can be written in the abstract form

$$\begin{aligned} dv(t) + AB^{-1}v(t)dt &= f\left(t, B^{-1}v(t), \mu(t)\right)dt + g(t)dB^H(t), \quad 0 \le t \le T, \\ v(0) &= Bx_0, \end{aligned} \tag{5.18}$$

where $v(t) = Bx(t)$. It is known that B is a bijective operator possessing a continuous inverse and that $-AB^{-1}$ is a bounded linear operator on $L^2(0,\pi)$ which generates a strongly continuous semigroup on $L^2(0,\pi)$ satisfying (A1) with $M_S = \alpha = 1$ (see [11]). Further, f satisfies (A2) as in Example 5.1. Consequently, we can invoke Theorem 3.4 (assuming that $M_S M_f T < 1$) to conclude that (5.18) has a unique mild solution $v \in C([0,T]; L^2(0,\pi))$. Consequently, $x = B^{-1}v$ is the corresponding mild solution of (5.15). □

6. Remark

This example provides a generalization of the work in [20, 30–32] to a more general setting. Such initial-boundary value problems arise naturally in the mathematical modeling of various physical phenomena (see [8, 23, 24]).

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Eduardo Hernandez: Departamento de Mathemática, I.C.M.C., Universidade de Sao Paulo, Caixa Postal 668, 13560-970 Sao Carlos SP, Brazil
Email address: lalohm@icmc.sc.usp.br

David N. Keck: Department of Mathematics, Ohio University, Athens, OH 45701, USA
Email address: dkeck@math.ohiou.edu

Mark A. McKibben: Mathematics and Computer Science Department, Goucher College, Baltimore, MD 21204, USA
Email address: mmckibben@goucher.edu