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Research Letter

On the Spheroidal Semiseparation for Stokes Flow

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Many heat and mass transport problems involve particle-fluid systems, where the assumption of Stokes flow provides a very good approximation for representing small particles embedded within a viscous, incompressible fluid characterizing the steady, creeping flow. The present work is concerned with some interesting practical aspects of the theoretical analysis of Stokes flow in spheroidal domains. The stream function ψ, for axisymmetric Stokes flow, satisfies the well-known equation $E^4\psi = 0$. Despite the fact that in spherical coordinates this equation admits separable solutions, this property is not preserved when one seeks solutions in the spheroidal geometry. Nevertheless, defining some kind of semiseparability, the complete solution for ψ in spheroidal coordinates has been obtained in the form of products combining Gegenbauer functions of different degrees. Thus, the general solution is represented in a full-series expansion in terms of eigenfunctions, which are elements of the space ker E^2 (separable solutions), and in terms of generalized eigenfunctions, which are elements of the space ker E^4 (semiseparable solutions). In this work we revisit this aspect by introducing a different and simpler way of representing the aforementioned generalized eigenfunctions. Consequently, additional semiseparable solutions are provided in terms of the Gegenbauer functions, whereas the completeness is preserved and the full-series expansion is rewritten in terms of these functions.

1. INTRODUCTION

Particle-fluid systems are encountered in many important technological applications and the study of Stokes flow [1] through a swarm of particles is of great theoretical and practical interest. The small size of the suspended particles allows us to assume the so-called Stokes flow approximation, which concerns the steady and creeping flow of an incompressible, viscous fluid. On the other hand, spheroidal geometry [2] is employed as a good approximation, since the particular analysis enjoys rotational symmetry and the flow is considered to be axisymmetric [3].

Two-dimensional Stokes flow is characterized by the existence of a stream function ψ [1], necessary to obtain the velocity and the total pressure fields, and it belongs to the kernel of the fourth-order differential operator E^4. In fact, rotational flows are described by stream functions that belong to the kernel of E^4, while $E^2\psi = 0$ provides us with irrotational fields. The importance of the Stokes stream function in low-Reynolds-number hydrodynamics is due to the fact that a single scalar function expresses the basic flow fields and at the same time carries all the physical interpretation [1].

The well-known equation of motion for the Stokes hypothesis is separable in Cartesian, cylindrical, and spherical coordinates. In addition, in spheroidal coordinates the equation $E^2\psi = 0$ admits separable solutions in the form of products of Gegenbauer functions of the first and of the second kind [4]. Unfortunately, this property of separability is not preserved when one is searching for solutions of the equation of motion $E^4\psi = 0$, a fact that has impeded considerably the development of several analyses or numerical implementations in order to solve physical problems. Dassios et al. [5] recently resolved this difficulty with the introduction of the method of semiseparation of variables, which is based on an appropriate finite-dimensional spectral decomposition of the operator E^4. In particular, the complete solution for Stokes flow in spheroidal coordinates can be obtained through the theory of generalized eigenfunctions and this stream function enjoys the representation of a full-series expansion in terms of semiseparable eigenmodes [5].

The solution for the Kuwabara model [5, 6], as well as the solution of other particle-in-cell physical problems [7, 8], has been obtained as a demonstration of this method.

However, indeterminacies appear when the Happel-type [8] or the Kuwabara-type [5] spheroidal models are solved in terms of the function ψ. These indeterminacies were overcome through the imposition of an additional geometrical condition that secured the correct reduction to the sphere case. Nowadays, many efforts within the analytical framework are being made using the representation theory [3, 9] in order to avoid such kind of problems.

In this work, we attempt to find, through a new approach of the difficulty of nonseparability, the proper tools to eliminate the indeterminacies, which appear in basic problems of creeping flow. Our purpose is focusing on the construction of a new set of generalized eigenfunctions that belong to the kernel of the differential operator E^4, rewritten in terms of the Gegenbauer functions of the first and of the second kind (see the appendix). Of course, the general solution is then presented as a series expansion in terms of these new eigenmodes. This procedure is based on the complete decomposition of axisymmetric Stokes flow [10], where the Stokes stream function that solves the equation $E^4\psi = 0$ is written as the linear combination of two other functions that belong to the $\ker E^2$. These generalized eigenfunctions are complete and the novelty of our work is established via an application.

Finally, we restrict our attention to prolate spheroids, since the results for the oblate spheroid can be obtained through a well-known transformation [2].

2. MATHEMATICAL FORMULATION

The governing equations that characterize the creeping, incompressible, and viscous motion within smooth and bounded domains $\Omega(\mathbb{R}^3)$ are provided in terms of the biharmonic velocity field $\mathbf{v}$ and the harmonic total pressure field P, that is,

$$\begin{aligned} \mu\,\Delta\mathbf{v}(\mathbf{r}) &= \nabla P(\mathbf{r}), \quad \mathbf{r}\in\Omega(\mathbb{R}^3), \\ \nabla\cdot\mathbf{v}(\mathbf{r}) &= 0, \quad \mathbf{r}\in\Omega(\mathbb{R}^3), \end{aligned} \tag{1}$$

with dynamic viscosity μ and mass density ρ. Given $\mathbf{v}$, the vorticity of the flow $\boldsymbol{\omega}$ is expressed via $\boldsymbol{\omega} = \nabla\times\mathbf{v}$. Additionally, if the flow is symmetric with respect to an axis, Stokes flow can thus be described through the stream function ψ, whereas the latter satisfies the equation of motion ($E^4 = E^2\circ E^2$)

$$E^4\psi(\mathbf{r}) = 0, \quad \mathbf{r}\in\Omega(\mathbb{R}^2). \tag{2}$$

Given a fixed positive number $c > 0$, which we consider to be the semifocal distance of our system, we define the transformed prolate spheroidal coordinates for $1\le\tau<+\infty$, $-1\le\zeta\le 1$, and $0\le\varphi<2\pi$:

$$x_1 = c\tau\zeta, \qquad \begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix} = c\sqrt{\tau^2-1}\sqrt{1-\zeta^2}\begin{Bmatrix} \cos\varphi \\ \sin\varphi \end{Bmatrix}, \tag{3}$$

while the outward unit normal vector yields

$$\hat{\mathbf{n}} = \frac{\left(\sqrt{\tau^2-1}\zeta, \tau\sqrt{1-\zeta^2}\cos\varphi,\ \tau\sqrt{1-\zeta^2}\sin\varphi\right)}{\sqrt{\tau^2-\zeta^2}} \tag{4}$$

and the differential operator E^2 assumes the form

$$E^2 = \frac{1}{c^2(\tau^2-\zeta^2)}\left[(\tau^2-1)\frac{\partial^2}{\partial\tau^2} + (1-\zeta^2)\frac{\partial^2}{\partial\zeta^2}\right]. \tag{5}$$

The vorticity field is proved to be expressed as

$$\boldsymbol{\omega}(\tau,\zeta) = \frac{\hat{\boldsymbol{\varphi}}E^2\psi(\tau,\zeta)}{c\sqrt{\tau^2-1}\sqrt{1-\zeta^2}}, \quad \tau>1,\ |\zeta|\le 1, \tag{6}$$

showing that irrotational fields are described by a stream function ψ which belongs to the kernel of E^2. Hence, every axisymmetric Stokes flow problem is being solved once ψ is known via the theory of semiseparability of (2) introduced in [5].

Focusing our effort onto the production of new generalized eigenfunctions in order to solve basic mathematical problems in fluid dynamics, we mainly refer to the decomposition theorem [10] for a stream function that belongs to the kernel of E^4. According to this theorem, which is identical to the Almansi theorem for biharmonic functions [3], every stream function that solves the equation of motion (2) can be written, in a nonunique way, as the sum of two terms, one which belongs to the kernel of E^2 and one which is the product of r^2 with a stream function of $\ker E^2$. Mathematically speaking, it is proved [10] that if $\psi_1,\psi_2\in\ker E^2$ and $\psi = \psi_1 + r^2\psi_2$ then $\psi\in\ker E^4$. Conversely, given $\psi\in\ker E^4$, there exist $\psi_1,\psi_2\in\ker E^2$ such that $\psi = \psi_1 + r^2\psi_2$. This decomposition becomes unique for flows regular on the axis of symmetry.

Our purpose herein is to derive new generalized eigenmodes based on the previous important theorem in spheroidal geometry and hence, we are led to a new complete representation of generalized eigenfunctions.

3. GENERALIZED EIGENFUNCTIONS

Introducing the eigenfunctions $\Theta_n^{(i)}$ of kind $i = 1,2,3,4$ and of order $n = 0,1,2,\ldots$ in terms of the Gegenbauer functions of the first G_n and of the second H_n kind via the formulae ($\tau>1$ and $|\zeta|\le 1$)

$$\begin{aligned} \Theta_n^{(1)}(\tau,\zeta) &= G_n(\tau)G_n(\zeta), \\ \Theta_n^{(2)}(\tau,\zeta) &= G_n(\tau)H_n(\zeta), \\ \Theta_n^{(3)}(\tau,\zeta) &= H_n(\tau)G_n(\zeta), \\ \Theta_n^{(4)}(\tau,\zeta) &= H_n(\tau)H_n(\zeta), \end{aligned} \tag{7}$$

the following complete representations of the kernel space of E^2 are obtained for ψ_1 and ψ_2, that is,

$$\psi_{1/2}(\tau,\zeta) = \sum_{n=0}^{\infty}\sum_{i=1}^{4} A_{n,1/2}^{i}\Theta_n^{(i)}(\tau,\zeta), \quad \tau>1,\ |\zeta|\le 1, \tag{8}$$

where $A^i_{n,1}$ and $A^i_{n,2}$ are constants. Then, writing

$$r^2 = c^2(\tau^2+\zeta^2-1), \quad \tau>1,\ |\zeta|\le 1,\ c>0, \tag{9}$$

and substituting (8), (9) into the previous decomposition theorem $\psi = \psi_1 + r^2\psi_2$, we reach the following representation for $\tau>1$, $|\zeta|\le 1$, by

$$\begin{aligned}\psi(\tau,\zeta) = \sum_{n=0}^{\infty}\sum_{i=1}^{4}\{&(A^i_{n,1}-c^2A^i_{n,2})\Theta^{(i)}_n(\tau,\zeta)\\ &+(c^2A^i_{n,2})[(\tau^2+\zeta^2)\Theta^{(i)}_n(\tau,\zeta)]\}.\end{aligned} \tag{10}$$

Expression (10) stands for the complete solution of (2), while the new generalized eigenfunctions $\Pi^{(i)}_n$ appearing in (10) are defined for $\tau>1$, $|\zeta|\le 1$, as

$$\Pi^{(i)}_n(\tau,\zeta) = (\tau^2+\zeta^2)\Theta^{(i)}_n(\tau,\zeta), \quad n\ge 0,\ i=1,2,3,4, \tag{11}$$

where using relations (A.4)–(A.5) they take the form

$$\begin{aligned}
\Pi^{(1)}_0(\tau,\zeta) &= 2(G_0(\tau)-G_2(\tau))G_0(\zeta)-2G_0(\tau)G_2(\zeta),\\
\Pi^{(2)}_0(\tau,\zeta) &= (G_0(\tau)-2G_2(\tau))H_0(\zeta)\\
&\quad + G_0(\tau)G_1(\zeta)+2G_0(\tau)G_3(\zeta),\\
\Pi^{(3)}_0(\tau,\zeta) &= (G_1(\tau)+2G_3(\tau)+H_0(\tau))G_0(\zeta)-2H_0(\tau)G_2(\zeta),\\
\Pi^{(4)}_0(\tau,\zeta) &= (G_1(\tau)+2G_3(\tau))H_0(\zeta)\\
&\quad + H_0(\tau)G_1(\zeta)+2H_0(\tau)G_3(\zeta),\\
\Pi^{(1)}_1(\tau,\zeta) &= 2(G_1(\tau)+G_3(\tau))G_1(\zeta)+2G_1(\tau)G_3(\zeta),\\
\Pi^{(2)}_1(\tau,\zeta) &= -G_1(\tau)G_0(\zeta)+(G_1(\tau)+2G_3(\tau))H_1(\zeta)\\
&\quad + 2G_1(\tau)G_2(\zeta),\\
\Pi^{(3)}_1(\tau,\zeta) &= (2G_2(\tau)-G_0(\tau)+H_1(\tau))G_1(\zeta)+2H_1(\tau)G_3(\zeta),\\
\Pi^{(4)}_1(\tau,\zeta) &= -H_1(\tau)G_0(\zeta)+(2G_2(\tau)-G_0(\tau))H_1(\zeta)\\
&\quad + 2H_1(\tau)G_2(\zeta),\\
\Pi^{(1)}_2(\tau,\zeta) &= \frac{2}{5}(G_2(\tau)+2G_4(\tau))G_2(\zeta)+\frac{4}{5}G_2(\tau)G_4(\zeta),\\
\Pi^{(2)}_2(\tau,\zeta) &= -\frac{1}{3}G_2(\tau)G_1(\zeta)+\frac{2}{5}(G_2(\tau)+2G_4(\tau))H_2(\zeta)\\
&\quad + \frac{4}{5}G_2(\tau)H_4(\zeta),\\
\Pi^{(3)}_2(\tau,\zeta) &= \left(\frac{2}{5}H_2(\tau)+\frac{4}{5}H_4(\tau)-\frac{1}{3}G_1(\tau)\right)G_2(\zeta)\\
&\quad + \frac{4}{5}H_2(\tau)G_4(\zeta),\\
\Pi^{(4)}_2(\tau,\zeta) &= \left(\frac{2}{5}H_2(\tau)+\frac{4}{5}H_4(\tau)-\frac{1}{3}G_1(\tau)\right)H_2(\zeta)\\
&\quad - \frac{1}{3}H_2(\tau)G_1(\zeta)+\frac{4}{5}H_2(\tau)H_4(\zeta),\\
\Pi^{(1)}_3(\tau,\zeta) &= \frac{2}{7}(3G_3(\tau)+2G_5(\tau))G_3(\zeta)+\frac{4}{7}G_3(\tau)G_5(\zeta),\\
\Pi^{(2)}_3(\tau,\zeta) &= \frac{2}{7}(3G_3(\tau)+2G_5(\tau))H_3(\zeta)\\
&\quad + \frac{1}{15}G_3(\tau)G_0(\zeta)+\frac{4}{7}G_3(\tau)H_5(\zeta),\\
\Pi^{(3)}_3(\tau,\zeta) &= \left(\frac{6}{7}H_3(\tau)+\frac{4}{7}H_5(\tau)+\frac{1}{15}G_0(\tau)\right)G_3(\zeta)\\
&\quad + \frac{4}{7}H_3(\tau)G_5(\zeta),\\
\Pi^{(4)}_3(\tau,\zeta) &= \left(\frac{6}{7}H_3(\tau)+\frac{4}{7}H_5(\tau)+\frac{1}{15}G_0(\tau)\right)H_3(\zeta)\\
&\quad + \frac{1}{15}H_3(\tau)G_0(\zeta)+\frac{4}{7}H_3(\tau)H_5(\zeta),
\end{aligned} \tag{12}$$

for $n=0,1,2,3$, and for $n\ge 4$ their general form is

$$\begin{aligned}
\Pi^{(1)}_n(\tau,\zeta) &= 2\gamma_nG_n(\tau)G_n(\zeta)\\
&\quad + \alpha_n(G_{n-2}(\tau)G_n(\zeta)+G_n(\tau)G_{n-2}(\zeta))\\
&\quad + \beta_n(G_{n+2}(\tau)G_n(\zeta)+G_n(\tau)G_{n+2}(\zeta)),\\
\Pi^{(2)}_n(\tau,\zeta) &= 2\gamma_nG_n(\tau)H_n(\zeta)\\
&\quad + \alpha_n(G_{n-2}(\tau)H_n(\zeta)+G_n(\tau)H_{n-2}(\zeta))\\
&\quad + \beta_n(G_{n+2}(\tau)H_n(\zeta)+G_n(\tau)H_{n+2}(\zeta)),\\
\Pi^{(3)}_n(\tau,\zeta) &= 2\gamma_nH_n(\tau)G_n(\zeta)\\
&\quad + \alpha_n(H_{n-2}(\tau)G_n(\zeta)+H_n(\tau)G_{n-2}(\zeta))\\
&\quad + \beta_n(H_{n+2}(\tau)G_n(\zeta)+H_n(\tau)G_{n+2}(\zeta)),\\
\Pi^{(4)}_n(\tau,\zeta) &= 2\gamma_nH_n(\tau)H_n(\zeta)\\
&\quad + \alpha_n(H_{n-2}(\tau)H_n(\zeta)+H_n(\tau)H_{n-2}(\zeta))\\
&\quad + \beta_n(H_{n+2}(\tau)H_n(\zeta)+H_n(\tau)H_{n+2}(\zeta)),
\end{aligned} \tag{13}$$

where the constants α_n, β_n, and γ_n are given by the expressions (A.6).

The corresponding results for an oblate spheroid are obtained through the simple transformation [2]

$$\tau \longrightarrow i\lambda, \qquad c \longrightarrow -i\bar{c}, \tag{14}$$

where $0\le\lambda\equiv\sinh\eta<+\infty$ and $\bar{c}>0$ are the new characteristic variables.

As an application of our generalized formulae, we demonstrate the Stokes flow problem, preserving axial symmetry, of a constant uniform streaming flow $-U\hat{\mathbf{x}}_1$ passing a solid prolate spheroid in the negative direction parallel to its axis of revolution. Suppose that the stationary spheroid with axes a (x_1-direction of symmetry) and b is defined as $\tau=\tau_s=a/c$, where the semifocal distance is $c=\sqrt{a^2-b^2}$. In terms of the stream function ψ, the uniform velocity implies

$$\psi(\tau,\zeta)\sim 2^{-1}Uc^2(\tau^2-1)(1-\zeta^2), \quad \tau\longrightarrow\infty, \tag{15}$$

while the nonslip boundary conditions on the surface of the spheroid assume the forms

$$\psi(\tau,\zeta)=0,\qquad \partial\psi(\tau,\zeta)/\partial\tau=0,\quad \tau=\tau_s, \tag{16}$$

for every $|\zeta|\le 1$. Combining the asymptotic condition (15) with the orthogonality (A.3) of the Gegenbauer functions, we see that the solution of our exterior problem requires certain terms of the expansion (10), those that include only $G_2(\zeta)=2^{-1}(1-\zeta^2)$ in view of formulae (12)–(13) and relations (A.1)–(A.2). Hence, if we apply boundary conditions (16) to the reduced form of (10), we arrive to the solution

$$\psi(\tau,\zeta)=2^{-1}Uc^2(\tau^2-1)(1-\zeta^2)\times\left\{1-\frac{[(\tau_s^2+1)/(\tau_s^2-1)]\coth^{-1}\tau-[\tau/(\tau^2-1)]}{[(\tau_s^2+1)/(\tau_s^2-1)]\coth^{-1}\tau_s-[\tau_s/(\tau_s^2-1)]}\right\}, \tag{17}$$

which is the stream function for the flow past a spheroid given in [1]. Here, we utilized the definition of the associated Legendre functions [4], (A.1), and the relationship $\coth^{-1}\tau=2^{-1}\ln[(\tau+1)/(\tau-1)]$ for $\tau>1$.

APPENDIX

The Gegenbauer functions G_n and H_n of the first and of the second kind, respectively, are defined in terms of the Legendre functions P_n, Q_n [4] as follows:

$$G_n(x)=\frac{P_{n-2}(x)-P_n(x)}{(2n-1)},\qquad H_n(x)=\frac{Q_{n-2}(x)-Q_n(x)}{(2n-1)}, \tag{A.1}$$

for every $n\ge 2$ and $x\in\mathbb{R}$, while

$$G_0(x)=-H_1(x)=1,\qquad G_1(x)=H_0(x)=-x. \tag{A.2}$$

Furthermore, G_n satisfies the orthogonality relation

$$\int_{-1}^{+1}\frac{G_n(x)G_{n'}(x)}{(1-x^2)}\,dx=\frac{2}{n(n-1)(2n-1)}\delta_{nn'},\quad |x|<1, \tag{A.3}$$

for $n,n'\ge 2$, where $\delta_{nn'}$ denotes the Kronecker delta.

Some important recurrence relations are

$$\begin{aligned}
x^2G_0(x)&=G_0(x)-2G_2(x),\\
x^2G_1(x)&=G_1(x)+2G_3(x),\\
x^2G_2(x)&=\frac{1}{5}G_2(x)+\frac{4}{5}G_4(x),\\
x^2G_3(x)&=\frac{3}{7}G_3(x)+\frac{4}{7}G_5(x),\\
x^2G_n(x)&=\alpha_nG_{n-2}(x)+\gamma_nG_n(x)+\beta_nG_{n+2}(x),
\end{aligned} \tag{A.4}$$

for $n\ge 4$ and $x\in\mathbb{R}$, while

$$\begin{aligned}
x^2H_0(x)&=x^2G_1(x),\\
x^2H_1(x)&=-x^2G_0(x),\\
x^2H_2(x)&=-\frac{1}{3}G_1(x)+\frac{1}{5}H_2(x)+\frac{4}{5}H_4(x),\\
x^2H_3(x)&=\frac{1}{15}G_0(x)+\frac{3}{7}H_3(x)+\frac{4}{7}H_5(x),\\
x^2H_n(x)&=\alpha_nH_{n-2}(x)+\gamma_nH_n(x)+\beta_nH_{n+2}(x),
\end{aligned} \tag{A.5}$$

for $n\ge 4$ and $x\in\mathbb{R}$, where

$$\alpha_n=\frac{(n-3)(n-2)}{(2n-3)(2n-1)},\qquad \beta_n=\frac{(n+1)(n+2)}{(2n-1)(2n+1)},$$
$$\gamma_n=(2n^2-2n-3)\,/\,(2n+1)(2n-3),\quad n\ge 4. \tag{A.6}$$

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REFERENCES

[1] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics*, Prentice Hall, Englewood Cliffs, NJ, USA, 1965.

[2] P. Moon and E. Spencer, *Field Theory Handbook*, Springer, Berlin, Germany, 2nd edition, 1971.

[3] X. Xu and M. Wang, "General complete solutions of the equations of spatial and axisymmetric Stokes flow," *The Quarterly Journal of Mechanics & Applied Mathematics*, vol. 44, no. 4, pp. 537–548, 1991.

[4] E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Chelsea, New York, NY, USA, 1965.

[5] G. Dassios, M. Hadjinicolaou, and A. C. Peyatakes, "Generalized eigenfunctions and complete semiseparable solutions for Stokes flow in spheroidal coordinates," *Quarterly of Applied Mathematics*, vol. 52, no. 1, pp. 157–191, 1994.

[6] S. Kuwabara, "The forces experienced by randomly distributed parallel circular cylinders or spheres in a viscous flow at small Reynolds number," *Journal of the Physical Society of Japan*, vol. 14, pp. 527–532, 1959.

[7] J. Happel, "Viscous flow in multiparticle systems: slow motion of fluids relative to beds of spherical particles," *AIChE Journal*, vol. 4, pp. 197–201, 1958.

[8] G. Dassios, M. Hadjinicolaou, F. A. Coutelieris, and A. C. Payatakes, "Stokes flow in spheroidal particle-in-cell models with rappel and kuwabara boundary conditions," *International Journal of Engineering Science*, vol. 33, no. 10, pp. 1465–1490, 1995.

[9] G. Dassios, A. C. Payatakes, and P. Vafeas, "Interrelation between Papkovich-Neuber and Stokes general solutions of the stokes equations in spheroidal geometry," *The Quarterly Journal of Mechanics & Applied Mathematics*, vol. 57, no. 2, pp. 181–203, 2004.

[10] A. Charalambopoulos and G. Dassios, "Complete decomposition of axisymmetric Stokes flow," *International Journal of Engineering Science*, vol. 40, no. 10, pp. 1099–1111, 2002.