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Research Article

An Inverse Problem for Parabolic Partial Differential Equations with Nonlinear Conductivity Term

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We consider an inverse problem for partial differential equation with nonlinear conductivity term in one-dimensional space within a finite interval. In the considered problem, a temperature history is unknown in a boundary of domain. The homotopy perturbation technique is used. Moreover, we have presented a numerical example.

1. Introduction

Inverse heat conduction problems (IHCPs) rely on temperature heat flux measurements for estimating unknown quantities in the analysis of physical problems in thermal engineering. As an example, inverse problems dealing with heat conduction have been generally associated with estimating an unknown boundary heat flux by using temperature measurements taken below the boundary surfaces. Therefore, while in the classical direct heat conduction problem the cause (boundary heat flux) is given and the effect (temperature field in the body) is determined, the inverse problem involves the estimation of the cause from the knowledge of the effect. An advantage of IHCP is that it enables a much closer collaboration between experimental and theoretical researchers in order to obtain the maximum of information regarding the physical problem under study.

Difficulties encountered in the solution of IHCPs should be recognized. IHCPs are mathematically classified as ill-posed in a general sense because their solutions may become unstable, as a result of the errors inherent to the measurements used in the analysis. Inverse problems were initially taken as not of physical interest due to their ill-posedness.

In recent years, some new methods for estimating surface heat flux in IHCPs developed in the theory and practice. Consequently, some methods for approximate solution of these problems have developed. To obtain stable results, special numerical techniques should be used. Examples include iterative gradient methods, optimization algorithms, regularization methods, function specification methods, space-marching method, conjugate gradient method, Levenberg-Marquardt method, iterative techniques, variational iteration, finite elements, finite volumes, boundary elements methods, and so on. In some works, the numerical methods have been used for IHCPs [1–14]. For instance, in [14] the variational iteration method is used to find the exact solution of a control parameter in parabolic equations. Unfortunately, most of these methods are useful in the linear IHCPs merely.

The homotopy perturbation method (HPM) was first proposed by the Chinese mathematician He [15–17]. The essential idea of this method is to introduce a homotopy parameter, say p, which takes values from 0 to 1. When $p = 0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p is gradually increased to 1, the system goes through a sequence of deformations, the solution for each of which is close to that at the previous stage of deformation. Eventually, at $p = 1$, the system takes the original form of the equation and the final stage of deformation gives the desired solution. One of the most remarkable features of the HPM is that usually just few perturbation terms are sufficient for obtaining a reasonably accurate solution. Considerable research works have been conducted recently in applying this method to a class of linear and nonlinear equations [14–23]. The interested reader can see [21–23] for last development of HPM. This homotopy perturbation method will become a much more interesting method to solving nonlinear differential equations in science and engineering.

In this paper, we consider a one-dimensional nonlinear inverse heat conduction problem with nonlinear diffusivity term that temperature history is unknown in a boundary. Then, using finite difference method and discrete time variable, the partial differential equation converts to a system of nonlinear ordinary differential equations. Consequently, by applying the homotopy perturbation technique and estimate solution, the temperature distribution in domain at discrete times will be found. Finally, a numerical experiment is given.

2. Homotopy Perturbation Technique

We begin with the following definition which is presented in [24].

Definition 1. Let X and Y be topologic spaces. If f and g are continuous maps of the space X into space Y, it is said that f is homotopic to g, if there is a continuous map $F : X \times I = [0,1] \to Y$ such that

$$F(x,0) = f(x), \quad F(x,1) = g(x) \quad \text{for each } x \in X. \tag{1}$$

The map F is called a homotopy between f and g.

To illustrate homotopy perturbation method, we consider the following nonlinear equation:

$$A(u) - f(r) = 0, \quad (r \in \Omega), \tag{2}$$

with the boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad (r \in \Gamma), \tag{3}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of domain Ω. The operator A can be generally divided in to two parts F and N, where F and N are linear and nonlinear parts of A, respectively. However, (2) converts to the following form:

$$L(u) + N(u) - f(r) = 0. \tag{4}$$

In [15], the author constructs a homotopy $V : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(v_0)] + p[A(v) - f(r)] = 0, \tag{5}$$

or

$$H(v,p) = L(v) - L(v_0) + pL(v_0) + p[N(v) - f(r)] = 0, \tag{6}$$

where $r \in \mathbb{R}$ and $p \in [0,1]$. In this situation, the parameter p is called homotopy parameter and v_0 is an initial approximation of (2) which satisfies boundary conditions. When $p = 0$ or $p = 1$, we have

$$H(v,0) = L(v) - L(v_0) = 0, \quad H(v,1) = A(v) - f(r) = 0. \tag{7}$$

On the other hand, if $p \in (0,1)$, then the homotopy $H(v,p)$ changes from $L(v) - L(v_0)$ to $A(v) - f(r)$.

Noticing that $0 \le p \le 1$ can be considered as a small parameter, applying the perturbation technique, we may assume that the solution of (5) or (6) can be expressed as a series in p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots . \tag{8}$$

When $p \to 1$, (5) or (6) corresponds to (4) and becomes the approximate solution of (4), that is, we have

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{9}$$

Series (9) is convergent for most cases and the rate of convergence depends on $A(v)$ (for more details see [16, 17]).

3. Solution of Nonlinear Parabolic Problem by Homotopy Perturbation Technique

Let $\Phi(x,t)$ be given function in $\Omega \equiv [0,l] \times [0,T]$, $f(t)$, $g(t)$ and $s(x)$ known functions on $[0,T]$ and $[0,l]$, respectively.

Now, consider the following nonlinear differential equation:

$$u_t - \{(a(t)u + b(t))u_x\}_x = \Phi(x,t), \quad (x,t) \in \Omega_0 \equiv (0,l) \times (0,T), \tag{10}$$

with initial condition:

$$u(x,0) = s(x), \quad x \in [0,l], \tag{11}$$

and boundary conditions:

$$u(0,t) = f(t), \quad t \in [0,T], \quad u_x(0,t) = g(t), \quad t \in [0,T], \tag{12}$$

where a, b, f, g, and s are known function such that $b(t)$ is far from zero in $[0,T]$.

Suppose that $u_0(x) = u(x,0) = s(x)$, $n\Delta t = T$, $k = 1/\Delta t$, $t_j = j\Delta t$, $u_j(x) = u(x,t_j)$, and $\Phi_j(x) = \Phi(x,t_j)$ for $j \in J_n = \{1,2,\ldots,n\}$. By using the backward finite difference scheme for term u_t in (10) in the form

$$u_t(x) \simeq k(u_j(x) - u_{j-1}(x)), \quad j \in J_n, \tag{13}$$

and substituting to (10), we find a system of second-order differential equations with respect to x. Then, we obtain

$$k(u_j(x) - u_{j-1}(x)) - \frac{d}{dx}\left\{(a(t_j)u_j(x) + b(t_j))\frac{d}{dx}u_j(x)\right\} = \Phi_j(x), \quad 1 \le j \le n, \tag{14}$$

or

$$\frac{d^2}{dx^2}u_j(x) - \left\{\frac{k}{b(t_j)}(u_j(x) - u_{j-1}(x)) - \frac{a(t_j)}{b(t_j)}\left(\frac{d}{dx}\left(u_j(x)\frac{d}{dx}u_j(x)\right)\right)\right\} = \frac{-1}{b(t_j)}\Phi_j(x). \tag{15}$$

Now, we can write (2) as follows:

$$Au = L_x u - Nu = \Psi(x,t), \tag{16}$$

where $\Psi(x,t) = (-1/b(t_j))\,\Phi_j(x)$, and $L_x = d^2/dx^2$ and $Nu = (k/b(t_j))(u_j(x) - u_{j-1}(x)) + (a(t_j)/b(t_j))((d/dx)(u_j(x)(d/dx)u_j(x)))$ are the linear and nonlinear parts of operator A, respectively.

For simplicity, define $\mathbf{u}(x) = (u_1(x), u_2(x), \ldots, u_n(x))^T$ and $D(\mathbf{u}(x), \mathbf{v}(x)) = (u_1(x)(d/dx)v_1(x), u_2(x)(d/dx)v_2 \times (x), \ldots, u_n(x)(d/dx)v_n(x))^T$. Now, by putting

$$\mathbf{m} = \left(\frac{-k}{b(t_1)} s(x), 0, \ldots, 0\right),$$

$$\mathbf{M}_1 = \begin{pmatrix} \frac{k}{b(t_1)} & & & 0 \\ \frac{-k}{b(t_2)} & \frac{k}{b(t_2)} & & \\ & \ddots & \ddots & \\ 0 & & \frac{-k}{b(t_n)} & \frac{k}{b(t_n)} \end{pmatrix}, \tag{17}$$

$$\mathbf{M_2} = \operatorname{diag}\left(\frac{a(t_1)}{b(t_1)}, \frac{a(t_2)}{b(t_2)}, \ldots, \frac{a(t_n)}{b(t_n)}\right),$$

and $D(\mathbf{u}(x), \mathbf{u}(x)) = (u_1(x)(d/dx)u_1(x), \ldots, u_n(x)(d/dx)u_n(x))^T$, where $u_j(x)$ are the values of $\mathbf{u}$ at $t = t_j$ for $1 \le j \le n$, and the notation $\operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ refers to a diagonal matrix in the form

$$\begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n \end{pmatrix}. \tag{18}$$

We can express (16) to the matrix form

$$N\mathbf{u} = -\mathbf{M}_2 \frac{d}{dx}\left(D(\mathbf{u}, \mathbf{u})\right) + \mathbf{M}_1 \mathbf{u} + \mathbf{m}. \tag{19}$$

By twice integration of (10) with respect to x and applying the initial conditions, we obtain

$$\begin{aligned} \mathbf{u}(x) - x\mathbf{g}(t) - \mathbf{f}(t) - \int_0^x \int_0^x N\mathbf{u}(x)\,dx\,dx &= \int_0^x \int_0^x \Psi(x)\,dx\,dx \\ &= -\int_0^x \int_0^x \Phi_b(x)\,dx\,dx, \end{aligned} \tag{20}$$

where $\mathbf{g}(t) = (g(t_1), \ldots, g(t_n))^T$, $\mathbf{f}(t) = (f(t_1), \ldots, f(t_n))^T$, $\Psi(x) = \operatorname{diag}(\Psi_1(x), \ldots, \Psi_n(x))$, and $\Phi_b(x) = \operatorname{diag}(\Phi_1(x)/b(t_1), \ldots, \Phi_n(x)/b(t_n))$.

By homotopy perturbation method, we may choose a convex homotopy such that [18, 20]

$$\mathbf{H}(v(x), p) = \mathbf{v}(x) - \mathbf{h}(x) - p\int_0^x \int_0^x N\mathbf{v}(x)\,dx\,dx = 0, \tag{21}$$

$$\mathbf{F}(\mathbf{u}(x)) = \mathbf{u}(x) - \mathbf{h}(x) = 0, \tag{22}$$

where

$$\mathbf{h}(x) = x\mathbf{g}(t) + \mathbf{f}(t) - \int_0^x \int_0^x \Psi(x)\,dx\,dx, \tag{23}$$

$\mathbf{v}(x) = (v(t_1), \ldots, v(t_n))^T$, and $\mathbf{h}(x) = (h(t_1), \ldots, h(t_n))^T$. By using (16), we can write

$$\mathbf{v}(x) = \mathbf{h}(x) + p\int_0^x \int_0^x N\mathbf{v}(x)\,dx\,dx. \tag{24}$$

By combining (10) and (22), we obtain following results:

$$\begin{aligned} \mathbf{v}(x) = {} & x\mathbf{g}(t) + \mathbf{f}(t) - \int_0^x \int_0^x \Phi_b(x)\,dx\,dx \\ & + p\int_0^x \int_0^x \Big\{ \mathbf{M}_1(\mathbf{v}(x) - \mathbf{u}_s(x)) \\ & - \mathbf{M}_2 \frac{d}{dx} D(\mathbf{u}(x), \mathbf{u}(x)) \Big\}\,dx\,dx, \end{aligned} \tag{25}$$

where $\mathbf{u}_s(x) = (s(x), u_1(x), \ldots, u_{n-1}x))^T$, or

$$\begin{aligned} \mathbf{v}_0 &= \mathbf{h}(x) = x\mathbf{g}(t) + \mathbf{f}(t) - \int_0^x \int_0^x \Phi_b(x)\,dx\,dx, \\ \mathbf{v}_1 &= \int_0^x \int_0^x \Big\{ \mathbf{M}_1(\mathbf{v}_0 - \mathbf{u}_s(x)) - \mathbf{M}_2 \frac{d}{dx} D(\mathbf{v}_0, \mathbf{v}_0) \Big\}\,dx\,dx, \\ \mathbf{v}_2 &= \int_0^x \int_0^x \Big\{ \operatorname{diag}\left(\frac{k}{b(t_1)}, \ldots, \frac{k}{b(t_n)}\right) \mathbf{v}_1 \\ &\quad - \mathbf{M}_2 \frac{d}{dx}\left(D(\mathbf{v}_0, \mathbf{v}_1) + D(\mathbf{v}_1, \mathbf{v}_0)\right) \Big\}\,dx\,dx, \end{aligned} \tag{26}$$

where the above relations are obtained of equating the terms with identical powers of p in (25). Obviously, if $p \to 1$, then the approximate solution is

$$\mathbf{u}(x) \simeq \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2. \tag{27}$$

In Section 5, we give a numerical sample. By using of homotopy perturbation technique, an approximate solution for nonlinear diffusion problem is obtained [15, 21].

4. Numerical Results

Let

$$\begin{aligned} & u_t - \frac{\partial}{\partial x}\left\{ \left(\frac{1}{6} e^{-t} u + (t+5) e^{-t} \right) \frac{\partial u}{\partial x} \right\} \\ & \quad = -\frac{7}{3} t - 9, \quad (x,t) \in [0,1] \times [0,1], \\ & u(x,0) = x^2, \quad 0 \le x \le 1, \\ & u(0,t) = t, \quad 0 \le t \le 1, \\ & u_x(0,t) = 0, \quad 0 \le t \le 1. \end{aligned} \tag{28}$$

Table 1

(a) Exact and approximate solution of $u_j(x)$ at $t_j = 0.25$.

x	Exact solution	Approximate solution	Relative error
0.1	0.2628402542	0.2628399122	1.30×10^{-6}
0.2	0.3013610167	0.3013563938	1.53×10^{-5}
0.3	0.3655622875	0.3655396925	6.18×10^{-5}
0.4	0.4554440667	0.4553735995	1.54×10^{-4}
0.5	0.5710063542	0.5708355086	2.29×10^{-4}
0.6	0.7122491501	0.7118965132	4.95×10^{-4}
0.7	0.8791724543	0.8785215163	7.40×10^{-4}
0.8	1.071776267	1.070669376	1.03×10^{-3}
0.9	1.290060588	1.288293071	1.37×10^{-3}
1	1.534025417	1.531339891	1.75×10^{-3}

(b) Exact and approximate solution of $u_j(x)$ at $t_j = 0.5$.

x	Exact solution	Approximate solution	Relative error
0.1	0.5164872127	0.5164865028	1.37×10^{-6}
0.2	0.5659488508	0.5659409374	1.39×10^{-5}
0.3	0.6483849144	0.6483481508	5.67×10^{-5}
0.4	0.7637954034	0.7636830954	1.47×10^{-4}
0.5	0.9121803178	0.9119111460	2.95×10^{-4}
0.6	1.093539658	1.092988531	5.03×10^{-4}
0.7	1.307873423	1.306862885	7.72×10^{-4}
0.8	1.555181613	1.553473905	1.09×10^{-3}
0.9	1.835464230	1.832754112	1.47×10^{-3}
1	2.148721271	2.144629711	1.90×10^{-3}

(c) Exact and approximate solution of $u_j(x)$ at $t_j = 0.75$.

x	Exact solution	Approximate solution	Relative error
0.1	0.7711700002	0.7711686828	1.70×10^{-6}
0.2	0.8346800007	0.8346667665	1.58×10^{-5}
0.3	0.9405300015	0.9404704769	6.32×10^{-5}
0.4	1.088720003	1.088540598	1.64×10^{-4}
0.5	1.279250004	1.278823053	3.33×10^{-4}
0.6	1.512120006	1.511249735	5.75×10^{-4}
0.7	1.787330008	1.785739520	8.89×10^{-4}
0.8	2.104880011	2.102199488	1.27×10^{-3}
0.9	2.464770014	2.460526294	1.72×10^{-3}
1	2.867000017	2.860607703	2.22×10^{-3}

(d) Exact and approximate solution of $u_j(x)$ at $t_j = 1$.

x	Exact solution	Approximate solution	Relative error
0.1	1.027182818	1.027180602	2.15×10^{-6}
0.2	1.108731273	1.108709841	1.93×10^{-5}
0.3	1.244645364	1.244550232	7.64×10^{-5}
0.4	1.434925092	1.434640074	1.98×10^{-4}
0.5	1.679570457	1.678894579	4.02×10^{-4}
0.6	1.978581458	1.977207417	6.94×10^{-4}
0.7	2.331958096	2.329452661	1.07×10^{-3}
0.8	2.739700370	2.735487036	1.53×10^{-3}
0.9	3.201808281	3.195152502	2.07×10^{-3}
1	3.718281828	3.708279035	2.69×10^{-3}

If we want to use our last notation, we have

$$\Phi(x,t) = -\frac{7}{3}t - 9,$$
$$a(t) = \frac{1}{6}e^{-t}, \quad (29)$$
$$b(t) = (t+5)e^{-t}.$$

Obviously, the above assumptions satisfy consideration of conditions. The exact solution is $u(x,t) = x^2e^t + t$. In this sample, we obtain the solution in $x = 0.2, 0.4, 0.8, 1$ at $t = 0.25, 0.5, 1$. Assume that $\Delta t = 0.25$, then we construct a homotopy as the same form as we describe in previous sections. Consequently, the solution will be constructed by

$$\mathbf{v}_{0_j}(x) = h(x,t_j) = t_j - \frac{1}{(t_j+5)e^{-t_j}}\left(-\frac{7}{3}t - 9\right)\frac{x^2}{2},$$
$$\mathbf{v}_{1_j}(x) = \int_0^x\int_0^x\left\{\frac{4e^{t_j}}{(t_j+5)}\left(\mathbf{v}_{0_j}(x) - u(x,t_{j-1})\right) - \frac{1/6}{(t_j+5)}\frac{d}{dx}\left(\mathbf{v}_{0_j}(x)\frac{d}{dx}\mathbf{v}_{0_j}(x)\right)\right\}dx\,dx,$$
$$\mathbf{v}_{2_j}(x) = \int_0^x\int_0^x\left\{\frac{4e^{t_j}}{(t_j+5)}\mathbf{v}_{1_j}(x) - \frac{1/6}{(t_j+5)}\frac{d}{dx} \times\left(\mathbf{v}_{0_j}(x)\frac{d}{dx}\mathbf{v}_{1_j}(x)+\mathbf{v}_{1_j}(x)\frac{d}{dx}\mathbf{v}_{0_j}(x)\right)\right\}dx\,dx, \quad (30)$$

for $j = 1, 2, 3, 4$.

Exact solution, approximate solution, and relative error for the above problem are given in Table 1 at $t = t_j = j\Delta t$, $j = 1, 2, 3, 4$.

5. Conclusion

In the obtained results of problem, we see that the approximate solutions for small increment Δt have relative error at least of order $O(E-3)$. This technique applied for some inverse problems and results of this approach are acceptable for small mesh size Δt. In [25–27], the stability and convergency of HPM for heat transfer, KdV equation, and couple of systems of reaction-diffusion equations are discussed. The authors' aim is to find the stability conditions of boundaries and initial data such that the solution is stable.

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