

## Research Article

# Formulas for the Amplitude of the van der Pol Limit Cycle through the Homotopy Analysis Method

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The limit cycle of the van der Pol oscillator,  $\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$ , is studied in the plane  $(x, \dot{x})$  by applying the homotopy analysis method. A recursive set of formulas that approximate the amplitude and form of this limit cycle for the whole range of the parameter  $\epsilon$  is obtained. These formulas generate the amplitude with an error less than 0.1%. To our knowledge, this is the first time where an analytical approximation of the amplitude of the van der Pol limit cycle, with validity from the weakly up to the strongly nonlinear regime, is given.

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## 1. Introduction

A dynamical system whose time evolution  $x(t)$  is determined by the differential equation,

$$\ddot{x}(t) + \epsilon(x^2 - 1)\dot{x}(t) + x(t) = 0, \quad t \geq 0, \quad (1)$$

with  $\epsilon$  a real parameter, and the dot denoting the time derivative is called the van der Pol oscillator [1]. For  $\epsilon > 0$ , and due to the nonlinear term  $\epsilon(x^2 - 1)\dot{x}$ , the system accumulates energy in the region  $|x| < 1$  and dissipates this energy in the region  $|x| > 1$ . This constraint implies the existence of a stable periodic motion (*limit cycle* [2]) when  $\epsilon > 0$ . If the nonlinearity is increased, the dynamics in the time domain runs from near-harmonic oscillations when  $\epsilon \rightarrow 0$  to relaxation oscillations when  $\epsilon \rightarrow \infty$ , making it a good model for many practical situations [3, 4]. The closed curve representing this oscillation in the plane  $(x, \dot{x})$  is quasi circular when  $\epsilon \rightarrow 0$  and a sharp figure when  $\epsilon \rightarrow \infty$ . For  $\epsilon < 0$ , the dynamics is dissipative in the region  $|x| < 1$  and amplificative for  $|x| > 1$ . Under these conditions, the periodic motion is still possible but unstable. In this case, the limit cycle can be derived from that one with  $\epsilon > 0$  taking into account the symmetry  $(\epsilon, x(t)) \rightarrow (-\epsilon, -x(-t))$ .

Therefore, it is enough to study the case  $\epsilon > 0$  to obtain also the behavior of the system for  $\epsilon < 0$ .

Different standard methods (perturbative, nonperturbative, geometrical) [1–7] have been used to study the limit cycle of the van der Pol equation, in the weakly ( $\epsilon \rightarrow 0$ ) and in the strongly ( $\epsilon \rightarrow \infty$ ) nonlinear regimes. However, investigations giving analytical information of this object in the intermediate regime of  $\epsilon$  are lacking in literature. In this paper, it is our aim to fill in this gap by applying to (1) the homotopy analysis method (HAM) introduced by Liao [8, 9] in the nineties. This method has been shown to be very useful to solve different nonlinear problems [10–20]. In particular, it has been applied in [21] to Liénard equation,  $\ddot{x} + \epsilon f(x)\dot{x} + x = 0$ , which is the generalization of the van der Pol system when  $f(x)$  is an arbitrary function. As the interest in that work [21] was the amplitude and the frequency of the periodic motions, the calculations were performed with the time variable being explicit. As here we are only interested in the amplitude and form of the limit cycles, the time dependence of the solutions can be omitted, and we can work directly in the phase space  $(x, \dot{x})$ . Our results for the van der Pol limit cycle are presented in Section 2. In Section 3 it is explained how these results have been obtained from our specific application of the HAM to the van der Pol equation. Some conclusions are given in the last Section.

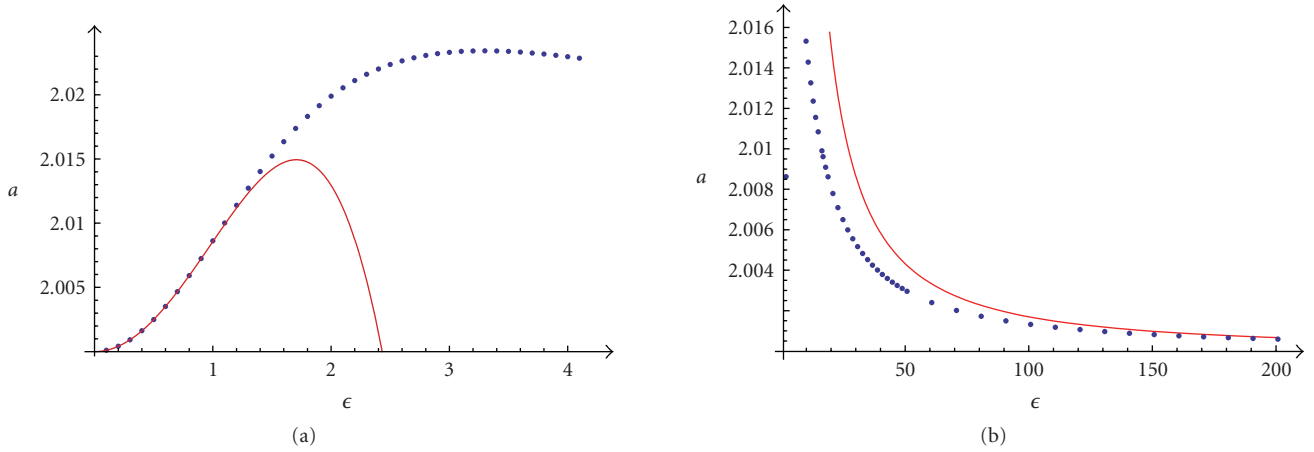


FIGURE 1: Comparison of the “experimental” amplitude  $a_E(\epsilon)$  (dotted curve) with (a) amplitude  $a(\epsilon)_{\epsilon \rightarrow 0}$  given by formula (3) (solid line) and (b) amplitude  $a(\epsilon)_{\epsilon \rightarrow \infty}$  given by formula (4) (solid line).

TABLE 1: The value  $a_E$  represents the amplitude  $a$  of the van der Pol limit cycle obtained with five decimal digits by integrating directly (1) with a fourth order Runge-Kutta method for the indicated values of  $\epsilon$ .

$\epsilon$	0.1	0.5	1.0	1.5	2.0	2.5
$a_E$	2.00010	2.00249	2.00862	2.01522	2.01989	2.02235
$\epsilon$	3.0	3.5	4.0	5.0	10	50
$a_E$	2.02330	2.02337	2.02296	2.02151	2.01428	2.00295

## 2. The Amplitude of the van der Pol Limit Cycle

Nowadays, the amplitude  $a$  of the van der Pol limit cycle can be easily computed with a classical Runge-Kutta method. The results of this computational calculation  $a_E$  are shown in Table 1. Let us observe that  $a_E(\epsilon) > 2$  for  $\epsilon > 0$ , with the asymptotic values:  $a_E(\epsilon \rightarrow 0) = a_E(\epsilon \rightarrow \infty) = 2$ . An upper bound rigorously established in [22] for the amplitude  $a$  is 2.3233. However, as it was also signaled in that work, the maximum of  $a_E$  is 2.02342, and it is obtained for  $\epsilon = 3.3$ . In view of this result, Odani [22] conjectured that *the amplitude of the limit cycle of the van der Pol equation is estimated by  $2 < a(\epsilon) < 2.0235$  for every  $\epsilon > 0$ .*

A closed formula for the amplitude  $a$  as function of  $\epsilon$  is unknown. By inspecting Table 1, one could propose as a good solution the constant amplitude

$$\bar{a} = 2 \quad (2)$$

for the whole range of the parameter  $\epsilon$ . Knowing that the experimental upper bound for  $a$  is 2.02342, that is,  $2 < a(\epsilon) < 2.02342$  for every  $\epsilon > 0$ , the error made with this approximation,  $(a(\epsilon) - \bar{a})/a(\epsilon)$ , would be about 1%.

If more precision is needed, the different analytical expansions in  $\epsilon$  that have been found for the amplitude in the weakly and in the strongly nonlinear regimes can be considered. Evidently, the error becomes very large in the regions where these approximations are not valid. In [21, 23, 24], a recursive perturbation approximation is used

to find the formula for the amplitude when  $\epsilon \rightarrow 0$ . This is, up to order  $\mathcal{O}(\epsilon^8)$ ,

$$a(\epsilon)_{\epsilon \rightarrow 0} = 2 + \frac{1}{96}\epsilon^2 - \frac{1033}{552960}\epsilon^4 + \frac{1019689}{55738368000}\epsilon^6. \quad (3)$$

This expansion agrees for small  $\epsilon$  with the computational calculation  $a_E$  presented in Table 1. For  $0 < \epsilon < 2$ , the error is less than 1%, for  $\epsilon \approx 4$ , the error is bigger than 10%, and for  $\epsilon \approx 6$ , the formula has lost its validity, and the error is bigger than 50%. In [25], the asymptotic dependence of the amplitude on  $\epsilon$  is given for sufficiently large  $\epsilon > 0$ ,

$$a(\epsilon)_{\epsilon \rightarrow \infty} = 2 + 0.7793\epsilon^{-4/3}. \quad (4)$$

Compared with  $a_E$ , this formula generates the amplitude with an error bigger than 15% for  $\epsilon < 2$ . The formula starts to have validity for  $\epsilon \approx 10$  with an error around 1%, that passes to be less than 0.1% when  $\epsilon > 50$ . In Figures 1(a) and 1(b), formulas (3) and (4) are, respectively, plotted versus  $\epsilon$  in their regions of validity.

We propose here two formulas,  $a_{R1}(\epsilon)$  and  $a_{R2}(\epsilon)$ , that approximate the amplitude of the van der Pol limit cycle for every  $\epsilon > 0$ : the first one with an error less than 0.1% and the second one with an error less than 0.05%. These formulas have been obtained by applying the HAM to the van der Pol equation. The details of the derivation of these formulas are given in Section 3.

The first formula is

$$a_{R1}(\epsilon) = 2 + \frac{1.737\epsilon^2}{(8\pi + 9\epsilon)(4 + \epsilon^2)}, \quad (5)$$

that derives from expression (38). The error obtained,  $|a(\epsilon) - a_{R1}(\epsilon)|/a(\epsilon)$ , with this formula is less than 0.1% for every  $\epsilon > 0$ . Figure 2(a) shows  $a_{R1}$  in comparison with the “experimental” amplitude  $a_E$ . The maximum of  $a_{R1}$  is 2.02317, and it is taken for  $\epsilon = 3.3$ .

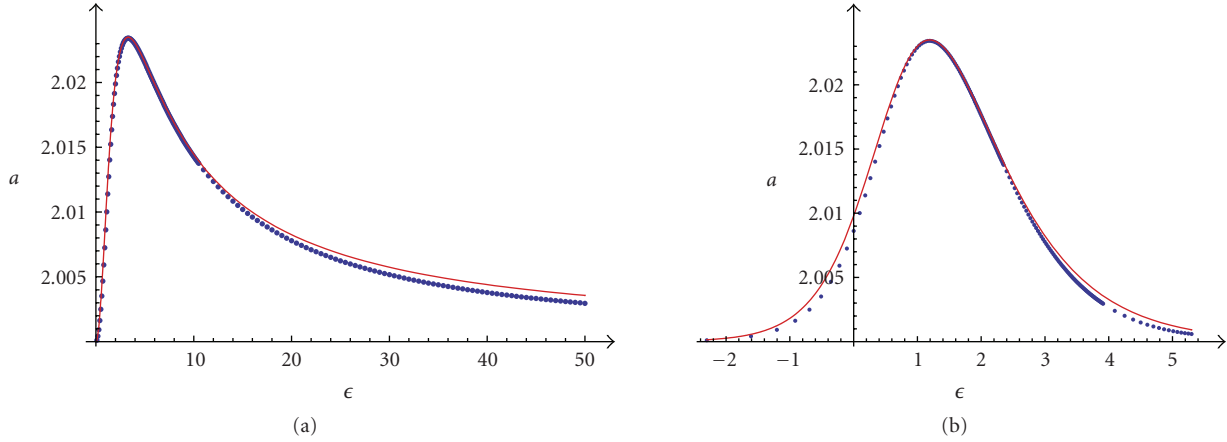


FIGURE 2: (a) Amplitude  $a_{R1}(\epsilon)$  given by formula (5) (solid curve) and experimental amplitude  $a_E(\epsilon)$ . (b) Amplitude  $a_{R2}(\epsilon)$  given by formula (6) (solid curve) and experimental amplitude  $a_E(\epsilon)$  (dotted curve), and  $\epsilon$  given in logarithmic scale.

The second formula is

$$a_{R2}(\epsilon) = 2 + \frac{0.74958\epsilon^2}{(8\pi + 9\epsilon)(4 + \epsilon^2)} + \frac{\epsilon^2(75.3562 + 43.0023\epsilon + 28.1589\epsilon^2 + 8.3479\epsilon^3)}{(8\pi + 9\epsilon)^2(4 + \epsilon^2)^2}, \quad (6)$$

that derives from expression (41). The error obtained,  $|a(\epsilon) - a_{R2}(\epsilon)|/a(\epsilon)$ , with this formula is less than 0.05% for every  $\epsilon > 0$ . The maximum of  $a_{R2}$  is 2.02346, and it is taken for  $\epsilon = 3.29482$ . The plot of  $a_{R2}$  in comparison with the amplitude  $a_E$  obtained computationally can be seen in Figure 2(b). Let us observe that it can be guessed from Figure 2(b) that  $a_{R2}(\epsilon) > a_E(\epsilon)$  for every  $\epsilon > 0$ .

The expansion of expression (6) for small  $\epsilon$  gives

$$a_{R2}(\epsilon)_{\epsilon \rightarrow 0} = 2 + 0.01491\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (7)$$

For large  $\epsilon$ , we obtain

$$a_{R2}(\epsilon)_{\epsilon \rightarrow \infty} = 2 + 0.18648\epsilon^{-1} + \mathcal{O}(\epsilon^{-2}). \quad (8)$$

Let us note the different scaling of this last expression (with behavior  $\epsilon^{-1}$ ) with respect to expansion (4) (with behavior  $\epsilon^{-1.33}$ ). Taking into account that these approximations start to be valid when  $\epsilon > 100$ , it can be easily seen that this difference is negligible, in fact less than 0.01%. Nevertheless, we have tried to modify formula (6) to obtain the correct scaling  $\epsilon^{-1.33}$  of expression (4) but the increase of the error in other regions of the parameter  $\epsilon$  does not recommend this possibility.

In summary, let us remark the exceptional fit of the “experimental” points  $a_E$  by the amplitudes  $a_{R1}$  and  $a_{R2}$  generated with formulas (5) and (6), respectively, (see Figures 2(a) and 2(b)). Moreover, by inspection of Figure 2(b), let us finish this section by posing the following.

*Conjecture.* The amplitude  $a_{R2}(\epsilon)$  obtained with formula (6) is an upper bound for the amplitude  $a(\epsilon)$  of the van der Pol limit cycle, that is,

$$a_{R2}(\epsilon) > a(\epsilon) > 2 \quad \text{for every } \epsilon > 0. \quad (9)$$

### 3. The HAM and the van der Pol Equation

The generalization of the van der Pol equation,  $\ddot{u} + \epsilon(u^2 - 1)\dot{u} + u = 0$ , is called the Liénard equation. In coordinates  $(u, v)$ , it reads

$$\ddot{u}(t) + \epsilon f(u)\dot{u}(t) + u(t) = 0, \quad t \geq 0, \quad (10)$$

where we consider that  $f(u)$  is an even function. The HAM has been applied to this equation in [21] working explicitly in the time domain. We proceed now to apply the HAM to this system by following a different strategy, namely, by omitting the time variable of (10).

*3.1. HAM Applied to Liénard Equation.* If we define the variable  $v = \dot{u}$ , and suppose that  $v$  is a function of  $u$ , the acceleration  $\ddot{u}$  can be rewritten as  $v'v$ , where the prime denotes the derivative respect to  $u$ . The result is a new equation with the time variable eliminated:

$$vv' + \epsilon f(u)v + u = 0. \quad (11)$$

When  $f(u)$  is even [26, 27], the limit cycles of this equation are closed trajectories of amplitude  $a$  in the  $(u, v)$  plane, with  $-a \leq u \leq a$  and  $v(a) = v(-a) = 0$ .

After the rescaling  $(u, v) = (ax, ay)$  to the new variables  $(x, y)$ , the limit cycles are transformed in solutions of the equation

$$yy' + \epsilon yf(ax) + x = 0, \quad -1 \leq x \leq 1, \quad (12)$$

$$y(-1) = y(1) = 0,$$

where the prime is now denoting the derivative with respect to  $x$ .

From [26, 27], we know that an approximate solution of the positive branch solution of (12), for any value of  $\epsilon > 0$ , is

$$\tilde{y}(x) = \sqrt{1 - x^2} + \frac{\epsilon}{\tilde{a}} \begin{cases} 0 & \text{if } -1 \leq x < \frac{x_0}{\tilde{a}}, \\ F(\tilde{a}) - F(\tilde{a}x) & \text{if } \frac{x_0}{\tilde{a}} \leq x \leq 1, \end{cases} \quad (13)$$

where  $F'(x) = f(x)$ ,  $x_0$  is a negative root of  $f(x)$ :  $f(x_0) = 0$  with  $x_0 < 0$ , and  $\tilde{a}$  is a positive root of  $F(x_0) - F(x)$ :  $F(x_0) = F(\tilde{a})$  with  $\tilde{a} > 0$ . The number of possible stable limit cycles in the strongly nonlinear regime (for large  $\epsilon$ ) is obtained from the number of positive roots of  $F(x_0) - F(x)$  when  $x_0$  runs over the set of all the negative roots of  $f(x_0) = 0$ . We take the positive roots  $\tilde{a}$  selected by the algorithm explained in [26, 27] as the corresponding approximate amplitudes. Let us observe that the initial approximate limit cycle  $\tilde{y}(x)$  is justified because it recovers the exact circular form when  $\epsilon \rightarrow 0$ , and also, when  $\epsilon \rightarrow \infty$ , it becomes exactly the two-piecewise limit cycle proposed and plotted in [26]. Moreover, it must be noticed that  $\tilde{y}(x)$  makes sense only when  $\epsilon > 0$ , and then the results obtained from this initial guess only will have validity for  $\epsilon > 0$ .

After eliminating the time variable, the nonlinear differential equation  $\gamma y' + \epsilon y f(ax) + x = 0$  is of order one. Then, to construct the homotopy [8, 9], we build a linear differential equation of order one for the whole range  $[0, 1]$  of what is called the homotopy parameter  $p$ .

We define an auxiliary linear operator  $\mathcal{L}$  by

$$\mathcal{L}[\phi(x; p)] = \frac{\partial}{\partial x} \phi(x; p), \tag{14}$$

with the property

$$\mathcal{L}[C_1] = 0, \tag{15}$$

where  $C_1$  is a constant function (the kernel of operator  $\mathcal{L}$ ), and  $p$  is a (homotopy) parameter explained below.

From (12) we define a nonlinear operator

$$\begin{aligned} \mathcal{N}[\phi(x; p), A(p)] &= \phi(x; p) \frac{\partial \phi(x; p)}{\partial x} \\ &+ \epsilon \phi(x; p) f(A(p)x) + x, \end{aligned} \tag{16}$$

and then construct the homotopy

$$\begin{aligned} \mathcal{H}[\phi(x; p), A(p)] &= (1 - p) \mathcal{L}[\phi(x; p) - \tilde{y}(x)] \\ &+ hp \mathcal{N}[\phi(x; p), A(p)], \end{aligned} \tag{17}$$

where  $h$  is a nonzero auxiliary parameter. Setting  $\mathcal{H}[\phi(x; p), A(p)] = 0$ , we have the zero-order deformation equation

$$(1 - p) \mathcal{L}[\phi(x; p) - \tilde{y}(x)] + hp \mathcal{N}[\phi(x; p), A(p)] = 0, \tag{18}$$

subject to the boundary conditions

$$\phi(-1; p) = 0, \quad \phi(1; p) = 0, \tag{19}$$

where  $p \in [0, 1]$  is an embedding parameter. When the parameter  $p$  increases from 0 to 1, the solution  $\phi(x; p)$  varies from  $y_0(x) = \tilde{y}(x)$  to  $y(x)$ , and  $A(p)$  varies from  $a_0$  to  $a$ . Assume that  $\phi(x; p)$  and  $A(p)$  are analytic in  $p \in [0, 1]$  and can be expanded in the Maclaurin series of  $p$  as follows:

$$\phi(x; p) = \sum_{n=0}^{+\infty} y_n(x) p^n, \quad A(p) = \sum_{n=0}^{+\infty} a_n p^n, \tag{20}$$

where

$$y_n(x) = \frac{1}{n!} \left. \frac{\partial^n \phi(x; p)}{\partial p^n} \right|_{p=0}, \quad a_n = \frac{1}{n!} \left. \frac{\partial^n A(p)}{\partial p^n} \right|_{p=0}. \tag{21}$$

Notice that series (20) contain the auxiliary parameter  $h$ , which has influence on their convergence regions. Assume that  $h$  is properly chosen such that all of these Maclaurin series are convergent at  $p = 1$ . Hence at  $p = 1$  we have

$$y(x) = y_0(x) + \sum_{n=1}^{+\infty} y_n(x), \quad a = a_0 + \sum_{n=1}^{+\infty} a_n. \tag{22}$$

At the  $N$ th-order approximation, we have the analytic solution of (12), namely,

$$y(x) \approx Y_N(x) := \sum_{n=0}^N y_n(x), \quad a \approx A_N := \sum_{n=0}^N a_n. \tag{23}$$

The parameter  $h$  is free and can be chosen arbitrarily; in particular, it can be a function of  $\epsilon$ . Nevertheless, the function  $\tilde{y}(x)$  is the exact limit cycle solution of the Liénard equation (12) in the limits  $\epsilon \rightarrow 0$  and  $\epsilon \rightarrow \infty$ . This means that the general solution  $y(x, h)$  of (18) should tend to  $\tilde{y}(x)$  for any  $p$  in those limits of  $\epsilon$ , losing its dependence on  $h$ . Then, a reasonable property for  $h$  would be to vanish in those limits of  $\epsilon$ . Hence, the solution of (18) would be the exact solution  $\tilde{y}(x)$  for any value of the parameter  $p$  in the limits  $\epsilon \rightarrow 0$  and  $\epsilon \rightarrow \infty$ . A simple function  $h(\epsilon)$  satisfying these properties is

$$h(\epsilon) = \frac{b\epsilon}{c + \epsilon^2}, \quad \text{with } b, c \in \mathbb{R}. \tag{24}$$

Differentiating (18) and (19)  $n$  times with respect to  $p$ , then setting  $p = 0$ , and finally dividing by  $n!$ , we obtain the  $n$ th-order deformation equation

$$\mathcal{L}[y_n(x) - \chi_n y_{n-1}(x)] + h R_n(x) = 0, \quad (n = 1, 2, 3, \dots), \tag{25}$$

subject to the boundary conditions

$$y_n(-1) = 0, \quad y_n(1) = 0, \tag{26}$$

where  $R_n(x)$  is defined by

$$\begin{aligned} R_n(x) &= \frac{1}{(n-1)!} \left. \frac{\partial^{n-1} \mathcal{N}[\phi(x; p), A(p)]}{\partial p^{n-1}} \right|_{p=0}, \\ \chi_n &= \begin{cases} 0, & n \leq 1, \\ 1, & n > 1. \end{cases} \end{aligned} \tag{27}$$

At each iteration, we have two unknowns:  $C_1$ , in (15), and  $a_n$ . These unknowns are obtained by considering the boundary conditions (26) as follows.

At zero order, we obtain

$$y_0(x) = \tilde{y}(x). \tag{28}$$

At 1st-order, we obtain

$$y_1(x) = \frac{h}{2}[1 - x^2 - y_0^2(x)] - h\epsilon \int_{-1}^x y_0(t)f(a_0t) dt, \quad (29)$$

where we have imposed the condition  $y_1(-1) = 0$  choosing the integration constant,  $C_1$ , appropriately. At this moment  $a_0$  is free but we can fix the value of  $a_0$  by imposing  $y_1(1) = 0$ :

$$\int_{-1}^1 y_0(t)f(a_0t) dt = 0. \quad (30)$$

This is a nonlinear equation for  $a_0$ . When  $f(x)$  is an even polynomial of degree  $2n$ , it is a polynomial equation of degree  $n$  in the variable  $a_0^2$ . Therefore, at this order, the number of limit cycles is the number of positive roots of this equation, at most  $n$  (see [27] for a longer discussion of this point).

Then, for every one of the  $a_0$  solutions of the above equation, that is, for every one of the limit cycles of the system, we proceed order by order in  $p$  to obtain higher order corrections to the amplitude  $a$  and to the shape of the limit cycle  $y(x)$ .

For example, at 2th-order, we obtain

$$y_2(x) = y_1(x) - hy_0(x)y_1(x) - h\epsilon \int_{-1}^x [y_1(t)f(a_0t) + ty_0(t)f'(a_0t)a_1] dt, \quad (31)$$

$$a_1 = \frac{-[\int_{-1}^1 y_1(x)f(a_0x) dx]}{[\int_{-1}^1 xy_0(x)f'(a_0x) dx]}. \quad (32)$$

Therefore, a first approximation to the shape of the limit cycle is  $y_0(x) + y_1(x)$ , and a first-order approximation to the amplitude of the limit cycle is  $a = a_0 + a_1$ .

At 3th-order, we have

$$y_3(x) = y_2(x) - hy_0(x)y_2(x) - \frac{h}{2}y_1^2(x) - h\epsilon \int_{-1}^x \left[ y_2(t)f(a_0t) + a_1ty_1(t)f'(a_0t) + \frac{1}{2}a_1^2t^2y_0(t)f''(a_0t) + ty_0(t)f'(a_0t)a_2 \right] dt, \quad (33)$$

$$a_2 = -\left[ \int_{-1}^1 \left[ y_2(x)f(a_0x) + a_1xy_1(x)f'(a_0x) + \frac{1}{2}a_1^2x^2y_0(x)f''(a_0x) \right] dx \right] / \left[ \int_{-1}^1 xy_0(x)f'(a_0x) dx \right]. \quad (34)$$

Therefore, a second-order approximation to the shape of the limit cycle is  $y_0(x) + y_1(x) + y_2(x)$ , and a second-order approximation to the amplitude of the limit cycle is  $a = a_0 + a_1 + a_2$ .

Moreover, at every order in  $p$  the appropriate function  $h(\epsilon)$  is indicated by the experiment, and in some sense,  $h(\epsilon)$  should be seen as somewhat provisional if a better guess could be inferred.

**3.2. Results for the van der Pol Equation.** For the van der Pol system, we have  $f(x) = x^2 - 1$ ,  $F(x) = x^3/3 - x$ ,  $x_0 = -1$ ,  $\tilde{a} = 2$  and

$$\tilde{y}(x) = \sqrt{1 - x^2} + \frac{\epsilon}{2} \begin{cases} 0 & \text{if } -1 \leq x < -\frac{1}{2}, \\ F(2) - F(2x) & \text{if } -\frac{1}{2} \leq x \leq 1. \end{cases} \quad (35)$$

The expressions for  $y_0(x)$ ,  $y_1(x)$ , and  $y_2(x)$  can be calculated as explained in Section 3.1. From (30) we get

$$\int_{-1}^1 y_0(t)f(a_0t) dt = \frac{9\epsilon + 8\pi}{64}(a_0^2 - 4) = 0 \implies a_0 = 2. \quad (36)$$

From (32) we obtain

$$a_1(h) = -\frac{27\sqrt{3}\epsilon h}{35(8\pi + 9\epsilon)}. \quad (37)$$

Therefore, a first approximation to the amplitude of the limit cycle is

$$a(h) \simeq 2 - \frac{27\sqrt{3}\epsilon h}{35(8\pi + 9\epsilon)}. \quad (38)$$

As it was advanced at the beginning of this section, note that this expression is only valid for positive  $\epsilon$ . Choosing  $b = -1.3$  and  $c = 4$  in (24),

$$h(\epsilon) = -\frac{1.3\epsilon}{4 + \epsilon^2}, \quad (39)$$

the formula  $a_{R1}(\epsilon)$  given in (5) is obtained. The values for  $b$  and  $c$  are selected by guess and check.

From (34) we get

$$a_2(h) = (3\epsilon h[19197\sqrt{3}\epsilon^2 h - 5040(8\sqrt{3} - 21h)\pi + 2\epsilon(-22680\sqrt{3} + h(54837 + 7700\sqrt{3}\pi))]) / 19600(9\epsilon + 8\pi)^2. \quad (40)$$

Therefore, a second approximation to the amplitude of the limit cycle is

$$a(h) \simeq 2 - \frac{27\sqrt{3}\epsilon h}{35(8\pi + 9\epsilon)} + (3\epsilon h[19197\sqrt{3}\epsilon^2 h - 5040(8\sqrt{3} - 21h)\pi + 2\epsilon(-22680\sqrt{3} + h(54837 + 7700\sqrt{3}\pi))]) / 19600(9\epsilon + 8\pi)^2. \quad (41)$$

Choosing  $b = -0.561$  and  $c = 4$  in (24),

$$h(\epsilon) = -\frac{0.561\epsilon}{4 + \epsilon^2}, \quad (42)$$

the formula  $a_{R2}(\epsilon)$  shown in (6) is finally obtained.

As an example, and to conclude this section, we plot in Figure 3 the form of the van der Pol limit cycle when  $y_0(x)$ ,  $y_0(x) + y_1(x)$ , and  $y_0(x) + y_1(x) + y_2(x)$  are used as approximated limit cycles. It is not banal to recall here that the reconstruction of the shape of limit cycles is not a less difficult problem than that of the calculation of their amplitudes.



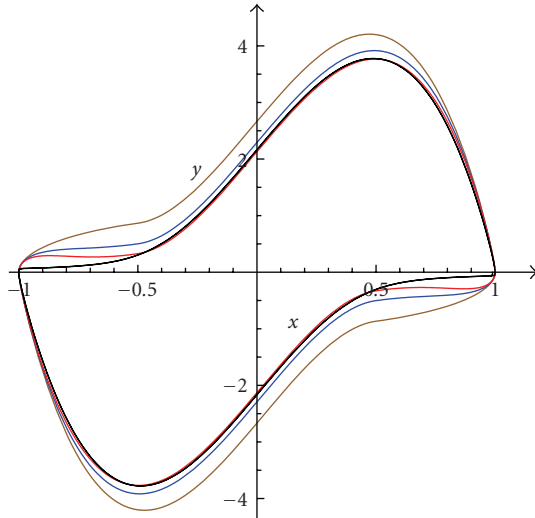


FIGURE 3: Approximated shape of the van der Pol limit cycle for  $\epsilon = 5$  and  $h(\epsilon)$  given in (42). Black line: the experimental limit cycle. Brown, blue, and red lines: the curves  $y_0(x)$ ,  $y_0 + y_1(x)$  and  $y_0 + y_1(x) + y_2(x)$ , respectively.

#### 4. Conclusions

In this work, the conjecture (9) has been posed. This is a consequence of the different formulas here presented for approximating the amplitude  $a$  of the van der Pol limit cycle. In addition to the well-known constant approximation  $\bar{a} = 2$  that generates an error less than 1%, we establish a family of recursive formulas that are valid for the whole range of the parameter  $\epsilon$ . Two of them,  $a_{R1}(\epsilon)$  and  $a_{R2}(\epsilon)$ , have been explicitly given. The first one,  $a_{R1}$ , produces an error less than 0.1%, and the second one,  $a_{R2}$ , reduces the error to less than 0.05%. Moreover,  $a_{R2}$  is conjectured to be an upper bound of  $a$ .

As far as we know, this is the first time where an analytical approximation of the amplitude of the van der Pol limit cycle, with validity from the weakly up to the strongly nonlinear regime, is proposed.

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#### References

- [1] B. van der Pol, "On relaxation oscillations," *Philosophical Magazine*, vol. 2, pp. 978–992, 1926.
- [2] A. A. Andronov, A. A. Vitt, and S. E. Khaikin, *Theory of Oscillators*, Dover, New York, NY, USA, 1st edition, 1989.
- [3] B. van der Pol and M. van der Mark, "The beating of the heart considered as relaxation oscillation and an electric model of the heart," *L'Onde électrique*, vol. 7, pp. 365–392, 1928.
- [4] R. López-Ruiz and Y. Pomeau, "Transition between two oscillation modes," *Physical Review E*, vol. 55, no. 4, pp. R3820–R3823, 1997.
- [5] K. Odani, "The limit cycle of the van der Pol equation is not algebraic," *Journal of Differential Equations*, vol. 115, no. 1, pp. 146–152, 1995.
- [6] A. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations*, Oxford University Press, Oxford, UK, 1st edition, 1977.
- [7] Y. Ye, *Theory of Limit Cycles*, vol. 66 of *Translations of Mathematical Monographs*, American Mathematical Society, Boston, Mass, USA, 1986.
- [8] S. J. Liao, *The proposed homotopy analysis technique for the solutions of non-linear problems*, Ph.D. thesis, Shanghai Jiao Tong University, Shanghai, China, 1992.
- [9] S. J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman & Hall/CRC Press, Boca Raton, Fla, USA, 2003.
- [10] S.-J. Liao, "An analytic approximate approach for free oscillations of self-excited systems," *International Journal of Non-Linear Mechanics*, vol. 39, no. 2, pp. 271–280, 2004.
- [11] W. Wu and S.-J. Liao, "Solving solitary waves with discontinuity by means of the homotopy analysis method," *Chaos, Solitons & Fractals*, vol. 26, no. 1, pp. 177–185, 2005.
- [12] T. Hayat and M. Khan, "Homotopy solutions for a generalized second-grade fluid past a porous plate," *Nonlinear Dynamics*, vol. 42, no. 4, pp. 395–405, 2005.
- [13] T. Hayat, M. Khan, and M. Ayub, "On non-linear flows with slip boundary condition," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 56, no. 6, pp. 1012–1029, 2005.
- [14] T. Hayat, M. Khan, M. Sajid, and M. Ayub, "Steady flow of an Oldroyd 8-constant fluid between coaxial cylinders in a porous medium," *Journal of Porous Media*, vol. 9, no. 8, pp. 709–722, 2006.
- [15] T. Hayat, M. Khan, and M. Ayub, "The effect of the slip condition on flows of an Oldroyd 6-constant fluid," *Journal of Computational and Applied Mathematics*, vol. 202, no. 2, pp. 402–413, 2007.
- [16] M. Khan, Z. Abbas, and T. Hayat, "Analytic solution for flow of Sisko fluid through a porous medium," *Transport in Porous Media*, vol. 71, no. 1, pp. 23–37, 2008.
- [17] S. Abbasbandy, "The application of homotopy analysis method to solve a generalized Hirota-Satsuma coupled KdV equation," *Physics Letters A*, vol. 361, no. 6, pp. 478–483, 2007.
- [18] S. Abbasbandy, "Homotopy analysis method for heat radiation equations," *International Communications in Heat and Mass Transfer*, vol. 34, no. 3, pp. 380–387, 2007.
- [19] S. Abbasbandy, Y. Tan, and S. J. Liao, "Newton-homotopy analysis method for nonlinear equations," *Applied Mathematics and Computation*, vol. 188, no. 2, pp. 1794–1800, 2007.
- [20] Y. Tan and S. Abbasbandy, "Homotopy analysis method for quadratic Riccati differential equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 3, pp. 539–546, 2008.
- [21] S. Abbasbandy, J. L. López, and R. López-Ruiz, "The homotopy analysis method and the Liénard equation," <http://arxiv.org/abs/0805.3916>.
- [22] K. Odani, "On the limit cycle of the Liénard equation," *Archivum Mathematicum*, vol. 36, no. 1, pp. 25–31, 2000.
- [23] J. L. López and R. López-Ruiz, "Approximating the amplitude and form of limit cycles in the weakly nonlinear regime of Liénard systems," *Chaos, Solitons & Fractals*, vol. 34, no. 4, pp. 1307–1317, 2007.
- [24] J. L. López and R. López-Ruiz, "The limit cycles of Liénard equations in the weakly nonlinear regime," to appear in *Far East Journal of Dynamical Systems* (2009); also available at <http://arxiv.org/abs/nlin/0605025>.

- [25] M. L. Cartwright, "Van der Pol's equation for relaxation oscillation," in *Contributions to the Theory of Non-Linear Oscillations II*, S. Lefschetz, Ed., vol. 29 of *Annals of Mathematics Studies*, pp. 3–18, Princeton University Press, Princeton, NJ, USA, 1952.
- [26] J. L. López and R. López-Ruiz, "The limit cycles of Liénard equations in the strongly nonlinear regime," *Chaos, Solitons & Fractals*, vol. 11, no. 5, pp. 747–756, 2000.
- [27] R. López-Ruiz and J. L. López, "Bifurcation curves of limit cycles in some Liénard systems," *International Journal of Bifurcation and Chaos*, vol. 10, no. 5, pp. 971–980, 2000.