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Research Article

Relativistic Quasilinear Description of Three-Dimensional Diffusion

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Quasilinear theory is developed by using canonical variables for a relativistic plasma. It is self-consistent, including momentum, pitch angle, and spatial diffusions. By assuming the wave field as a superposition of known toroidal and poloidal Fourier modes, the quasilinear diffusion coefficients are written in a form which can be directly evaluated using the output of a spectral full-wave solver of Maxwell equations in toroidal plasmas. The formalism is special for tokamas and, therefore, simple and suitable for simulations of cyclotron heating, current drive, and radio-frequency wave-induced radial transport in ITER.

1. Introduction

Interaction of radio-frequency (RF) wave with plasma in magnetic confinement devices has been a very important discipline of plasma physics. To approach more realistic description of wave-plasma interaction in a time scale longer than the kinetic time scales bounce-average is needed. The long-time evolution of the kinetic distribution can be treated with Fokker-Planck equation. The behavior of the plasma and the most interesting macroscopic effects are obtained by balancing the diffusion term with a collision term.

For the relativistic particles the action and angle variables initiated by Kaufman [1] are introduced. The technique of the area-conserved transformation proposed by Lichtenberg and Lieberman [2] is employed. A new invariant which actually is an implicit Hamiltonian is formed by using bounce average and from which the bounce frequency and precession frequency can be calculated. Using new action and angle variables the relativistic quasi-linear equation is derived, including spatial diffusion. It is different from Brizard and Chan [3] where magnetic field does not have the toroidal component. An elegant form of the quasi-linear kinetic equation including a term describing RF wave-induced radial diffusion has been derived by Eriksson and Helander [4]. This paper is an extension of their work with relativistic effects. In most of the above derivations for the quasi-linear operator it is assumed that the wave field in the vicinity of the resonances can be represented in an Eikonal form. Here, the field is supposed as a superposition of given toroidal and poloidal Fourier modes, instead. This is the representation used to solve the Maxwell equation in tokamak plasmas with a spectral full-wave code, for example, TORIC [5]. For the circulating particles, under the conditions of small Larmor radius and first harmonic resonance, the diffusion coefficient is compatible with the numerical code developed by Cardinali et al. [6].

The rest of the paper is organized as follows. In next section the exact guiding center variables are derived with Hamiltonian transformation. The center variables are derived with Hamiltonian transformation. The bounce-averaged quasi-linear equation is carried out in Section 3. A brief summary is presented in the last section.

2. Exact Guiding Center Variables

In tokamak configuration, the relativistic Hamiltonian of a charged particle can be expressed as

$$H = \sqrt{\left[\left(P_R - \frac{e}{c}A_R\right)^2 + \left(P_Z - \frac{e}{c}A_Z\right)^2 + \mathfrak{A}/R^2\right]c^2 + m_0^2c^4} + e\Phi, \tag{1}$$

where $\mathfrak{A}$ denotes $(P_\phi - (e/c)RA_\phi)^2$, A_R, A_Z, and A_ϕ are the vector potential components of the magnetic field, and Φ is the electrical potential and assumed to be a function of Ψ.

Here, Ψ is the poloidal flux of the magnetic field. m_0 and e are the rest mass and charge. P_R, P_ϕ, and P_Z are the canonical momenta conjugate to the cylindrical coordinates R, ϕ and Z, respectively.

The magnetic field can be expressed as

$$B = \nabla\phi \times \nabla\Psi + I\nabla\phi, \tag{2}$$

where I is related to the poloidal current. Then, in tokamaks, we have

$$A_R = 0, \qquad A_Z = -I\ln\frac{R}{R_0}, \qquad A_\phi = -\frac{\Psi}{R}. \tag{3}$$

We introduce a generating function [7] for changing to the guiding center variables:

$$F_1 = -\frac{m_0\Omega_0 R_0^2}{2}\exp\left(\frac{X}{m_0\Omega_0 R_0}\right)\left(\ln\frac{R}{R_0} - \frac{X}{m_0\Omega_0 R_0}\right)^2 tg\alpha - ZX, \tag{4}$$

where

$$X = m_0\Omega_0 R_0 \ln\frac{R_C}{R_0}, \tag{5}$$

and Ω_0 is the gyrofrequency in the toroidal field, ρ is the Larmor radius, α is the gyrophase, and subscripts 0 and c refer to the values at the magnetic axis and the guiding center position, respectively. X and α are the new coordinates conjugate to the momenta

$$P_X = Z + \rho\sin\alpha + \frac{\rho^2}{4R_C}\sin 2\alpha, \qquad P_\alpha = \frac{1}{2}m_0\Omega_C\rho^2. \tag{6}$$

The other two canonical variables P_ϕ and ϕ do not change in the new coordinates. The old coordinates are connected with the new ones through four equations:

$$P_R = m_0\Omega_c\rho e^{(\rho/R_c)\cos\alpha}\sin\alpha, \tag{7}$$

$$P_Z = -X = -m_0\Omega_0 R_0\ln\frac{R_C}{R_0}, \tag{8}$$

$$R = R_C\exp\left(-\frac{\rho\cos\alpha}{R_C}\right), \tag{9}$$

$$Z = P_X - \rho\sin\alpha - \frac{\rho^2}{4R_C}\sin 2\alpha, \tag{10}$$

where P_X is actually the Z coordinate of the guiding center, Z_c. Such fact that a momentum is turned to be a coordinate often occurs during area-conserved canonical transformation [2].

The Jacobian in the area-conserved transformation is unity [2], that is,

$$d\tau = dP_\alpha dP_X dP_\phi d\alpha dX d\phi. \tag{11}$$

Substituting (7) and (8) into (1), the exact Hamiltonian for the relativistic particles is

$$H = \sqrt{\left\{2m_0\Omega_c P_\alpha\mathfrak{B} + \frac{1}{R^2}\left[P_\phi + e\Psi\right]^2\right\}c^2 + m_0^2c^4} + e\Phi, \tag{12}$$

where $\mathfrak{B}$ denotes $[(R_c/R)^2\sin^2\alpha + \cos^2\alpha]$. It is suitable for particle simulation. The equations of motion and Vlasov's equation can be derived from the Hamiltonian.

3. QuasiLinear Equation

If the Larmor radius is smaller than the scale length of the system, a small parameter $\delta = \rho/b$ may be introduced where b is the scale length.

For the gyrokinetics the Hamiltonian in (12) could be averaged and reduced as

$$H = \sqrt{\left(2m_0\Omega_c P_\alpha + m_0^2u_\phi^2\right)c^2 + m_0^2c^4} + e\Phi \tag{13}$$

to the first order of δ, where $u_\phi = (P_\phi + e\Psi)/m_0R$ is the toroidal velocity.

We form a new invariant [2]:

$$\Pi = \frac{1}{2\pi}\oint P_x dX. \tag{14}$$

The action invariant is the toroidal flux enclosed by a particle orbit which actually is an implicit Hamiltonian from which the bounce frequency and precession frequency can be calculated.

For trapped particles in a large aspect ratio configuration, that is, $\varepsilon = r/R_0 \ll 1$, we get

$$\Pi_t = \frac{8qR_0m_0(\varepsilon\Omega_0P_\alpha/m_0)^{0.5}}{\pi}\left[E(k_1) - \left(1 - {k_1}^2\right)K(k_1)\right]. \tag{15}$$

The bounce frequency and the precession frequency are obtained from (15) as in [8, 9]:

$$\omega_{bt} = \frac{\pi(\varepsilon\Omega_0P_\alpha/m_0)^{0.5}}{2\gamma qR_0K(k_1)},$$
$$\omega_{\zeta t} = \frac{2\Omega_0P_\alpha}{\gamma\Omega_p m_0R_0^2}\left[\frac{E(k_1)}{K(k_1)} - \frac{1}{2}\right] + \frac{4\Omega_0P_\alpha\hat{s}}{\gamma\Omega_p m_0R_0^2}\left[\frac{E(k_1)}{K(k_1)} - \left(1 - {k_1}^2\right)\right], \tag{16}$$

where Ω_p is the gyrofrequency in poloidal magnetic field, $k_1^2 = u_{\phi 0}^2/(4\varepsilon\Omega_0P_\alpha)$, and $\hat{s}$ is the magnetic shear. $E(k_1)$ and $K(k_1)$ are complete elliptic function of the first and second kinds.

For the circulating particles,

$$\Pi_c = \frac{\Omega_0 m_0 r^2}{2} + \frac{2qR_0 m_0 u_{\phi 0}\sigma}{\pi} E(k),$$
$$\omega_{bc} = \frac{\pi u_{\phi 0}\sigma}{2\gamma q R_0 K(k)},$$
$$\omega_{\zeta c} = q\omega_{bc}\sigma + \frac{u_{\phi 0}^2}{2\gamma\Omega_p r R_0}\left[\frac{E(k)}{K(k)} - \left(1 - \frac{k^2}{2}\right)\right] + \frac{u_{\phi 0}^2 \hat{s}}{\gamma\Omega_p R_0^2}\frac{E(k)}{K(k)}, \quad (17)$$

where σ represents direction of the circulating particle and $k^2 = k_1^{-2}$. The bounce-averaged gyrofrequency of the trapped particles is

$$\Omega = \frac{\Omega_0}{\gamma} * \left[1 - 2\varepsilon\left(\frac{E(k_1)}{K(k_1)} - \frac{1}{2}\right)\right], \quad (18)$$

while for the circulating particles,

$$\Omega = \frac{\Omega_0}{\gamma} * \left\{1 - \frac{2\varepsilon}{k^2(1+\varepsilon)}\left[\frac{E(k)}{K(k)} - \left(1 - \frac{k^2}{2}\right)\right]\right\}, \quad (19)$$

where Ω is a nonlocal gyro-frequency (also seen in [4]).

Lamalle [10] seems to prove that the frequencies for the local and nonlocal scenarios in the resonances are compatible. If the momentum, P_i, is constants of motion, the frequency is $\omega_i = dq_i/dt = \partial H/\partial P_i$ which is bounce-averaged frequency [4]. The coordinates are $q_i = \omega_i t + \text{const.}$ where q_i could represent η, ξ, or α [2]. The new momenta Π, P_α, and P_ζ are conjugate to the coordinates η, α, and ζ in which $P_\zeta = -(e/c)\Psi_0$ is actually a position variable [8].

In the extended phase space the Hamiltonian is written as follows [2]:

$$\overline{H}(\overline{p}, \overline{q}) = H(p, q, t) - H, \quad (20)$$

where $\overline{p}_{n+1} = -H, \overline{q}_{n+1} = t$, H is the particle energy.

According to Liouville's theorem, the distribution function, f, satisfies Vlasov's equation:

$$\frac{\partial f}{\partial t} + \dot{\eta}\frac{\partial f}{\partial \eta} + \dot{\varsigma}\frac{\partial f}{\partial \varsigma} + \dot{\alpha}\frac{\partial f}{\partial \alpha} + \dot{\Pi}\frac{\partial f}{\partial \Pi} + \dot{P}_\varsigma\frac{\partial f}{\partial P_\varsigma} + \dot{P}_\alpha\frac{\partial f}{\partial P_\alpha} + \dot{H}\frac{\partial f}{\partial H} = 0, \quad (21)$$

where f can be divided in two parts, the averaged part and oscillatory part:

$$f = \overline{f} + \widetilde{f}. \quad (22)$$

The energy perturbation H_1 from (1) is

$$H_1 = e\Phi_1 - e\vec{v}_\perp \cdot \vec{A}_\perp - ev_\phi A_\phi, \quad (23)$$

where considering (9) and (10), Φ_1 and $\vec{A}_1$ are

$$\left[\Phi_1, \vec{A}_1\right] = \left[\Phi_k, \vec{A}_k\right] e^{i(\vec{k}\cdot\vec{r} - \omega t)},$$
$$\vec{k}\cdot\vec{r} = \vec{k}_\perp \cdot \left[(R_c - \rho\cos\alpha)\hat{R} + (Z_c - \rho\sin\alpha)\hat{Z}\right] + n\phi. \quad (24)$$

The linear solution of (21) is

$$\widetilde{f} = -\frac{\left(\omega H_1\left(\partial\overline{f}/\partial H\right) + nH_1\left(\partial\overline{f}/\partial P_\varsigma\right) + \mathfrak{C}\right)}{\left(\omega - n\omega_\varsigma - m\omega_\eta - l\Omega\right)}, \quad (25)$$

where $\mathfrak{C}$ denotes $mH_1(\partial\overline{f}/\partial\Pi) + lH_1(\partial\overline{f}/\partial P_\alpha)$, and $\overline{f}$ in (22) is a function of H, P_α, and P_ζ only, independent of η, ς, α, and Π and satisfies the equation:

$$\frac{\partial\overline{f}}{\partial t} - i\left(n\frac{\partial\widetilde{f}^\bullet}{\partial P_\varsigma} + l\frac{\partial\widetilde{f}^\bullet}{\partial P_\alpha} + \omega\frac{\partial\widetilde{f}^\bullet}{\partial H}\right)H_1 = 0 \quad (26)$$

from (21), where we have used Hamiltonian equations, $\dot{P}_i = -\partial\overline{H}/\partial q_i$, P_i and q_i are canonical variables, and $\widetilde{f}^\bullet$ is the conjugate to $\widetilde{f}$. If H_1 is a slow function of P_ς, P_α, and H, we get the quasi-linear equation:

$$\frac{\partial\overline{f}}{\partial\tau} + \hat{L}D\hat{L}\overline{f} = C\left(\overline{f}\right), \quad (27)$$

where $\tau = t\nu$, $\hat{L} = \partial/\partial H + (n/\omega)(\partial/\partial P_\varsigma) + (l/\omega)(\partial/\partial P_\alpha)$ is differential operator of actions. Here, P_ς represents spatial diffusion while H and P_α represent velocity space diffusion, C is a collision operator normalized to ν which can take the form given by [11], ν is collision frequency, and D is diffusion coefficient:

$$D = \pi H_1^2 \delta\left(\omega - n\omega_\varsigma - m\omega_\eta - l\Omega\right\}, \quad (28)$$

which is similar to the one obtained in [4]. We have used the well-known Plemelj formula called as Landau's rule in [4]:

$$\frac{1}{\omega - m\omega_\eta - n\omega_\xi - l\Omega} = P\frac{1}{\omega - m\omega_\eta - n\omega_\xi - l\Omega} - i\pi\delta\left(\omega - m\omega_\eta - n\omega_\xi - l\Omega\right). \quad (29)$$

For one harmonic from (24) and (29) we get

$$H_{1l} = \left[\left(e\Phi_k - \frac{e}{c}v_\varsigma A_{\varphi k}\right)J_l(k_\perp\rho) - \frac{e}{c}v_\perp A_{Zk}\frac{J_{l-1} + J_{l+1}}{2} - \frac{e}{c}v_\perp A_{Rk}\frac{J_{l-1} - J_{l+1}}{2i}\right], \quad (30)$$
$$i^l \exp\left(i\vec{k}\cdot\vec{r}_c - il\alpha_0\right\}\exp(in\varsigma + im\eta + l\Omega - i\omega t),$$

where $\vec{A}$ is the vector potential of incident electromagnetic wave, $\vec{r}_c$ is the guiding center position, and α_0 is the angle between $\vec{k}$ and $-\mathbf{R}$ directions in the cylindrical coordinates. Equations (27), (28), and (30) are the main results in this paper.

For the relativistic circulating particles the resonant term in (28) is written as

$$\delta\left[\frac{\omega}{\gamma}\left(\gamma - \frac{l\gamma\Omega}{\omega} - N_{//}\frac{qR_0\gamma\omega_b}{c}\right)\right] = \frac{c}{qR_0\omega\omega_b}\delta(N_{\text{res}} - N_{//}), \quad (31)$$

where γ is the relativistic factor, and $N_{\text{res}} = (2K(k)/\pi)(\gamma - \gamma\Omega/\omega)/(\beta_{\text{th}}P_{//})$ which is consistent with the code developed in [6] if k approaches zero. However, there are differences for trapped or barely circulating particles.

From (28) and (31) we get

$$D = \frac{2}{\pi}K(k)\frac{P_{\perp}^2}{\gamma P_{//}}D_{\perp}\delta(N_{\text{res}} - N_{//}), \tag{32}$$

where $P_{\perp}$ and $P_{//}$ are normalized with respect to the thermal momentum $P_t = (m_0 T)^{0.5}$. The steady state is described by a relativistic Maxwellian with temperature T_t. From the definition of the δ-function, when $\Delta N_{//} \to 0$,

$$\delta(N_{\text{res}} - N_{//}) = \pi^{-0.5}\frac{1}{\Delta N_{//}}\exp\left[-\frac{(N_{\text{res}} - N_{//})^2}{\Delta N_{//}^2}\right], \tag{33}$$

$$\begin{aligned} D_{\perp} &= \frac{\pi\omega e^2 A_{\perp}^2 \beta_{th}}{\nu T^2}, \\ \hat{L} &= \frac{\gamma\partial}{P\partial P} + \frac{nT_t}{\omega}\frac{\partial}{\partial\Psi^{\bullet}} + \frac{l\Omega_0}{\omega(1+\varepsilon)}\frac{\partial}{P^2\mu\partial\mu}, \end{aligned} \tag{34}$$

where the chosen variables are related to the momentum P and $\mu = \cos\theta$, with θ being the pitch angle.

4. Summary

The action and angle variables initiated by Kaufman are used [1]. The area-conserved transformation is employed [2]. The bounce-averaged quasi-linear Fokker-Planck equation for the relativistic particles is rigorously derived, including momentum, pitch angle, and spatial diffusion, that is, the three-dimensional diffusion. Equations (27), (28), and (30) are the main results of the work. The diffusion coefficient expressed in (28) is similar to the one obtained by Eriksson and Helander [4]. Equation (34) is only for circulating particles. Most of the past derivations of the quasi-linear operator assume that the wave field in the vicinity of the resonances can be represented in a Eikonal form. Here, the field is supposed, instead, to be given as a superposition of toroidal and poloidal Fourier modes. This is the representation used to solve Maxwell equation in tokamak plasmas with spectral full-wave code, for example, TORIC [5]. The formalism is special for tokamas; so it is simple and suitable for simulation of cyclotron heating, current drive, and radio-frequency wave-induced radial transport in ITER.

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