Hindawi Publishing Corporation
International Journal of Stochastic Analysis
Volume 2010, Article ID 697257, 22 pages
doi:10.1155/2010/697257

Research Article

Portfolio Selection with Jumps under Regime Switching

Lin Zhao

Department of Mathematics, University of Wales Swansea, Swansea SA2 8PP, UK

Correspondence should be addressed to Lin Zhao, malz@swansea.ac.uk

Received 23 February 2010; Accepted 10 June 2010

Academic Editor: Hideo Nagai

We investigate a continuous-time version of the mean-variance portfolio selection model with jumps under regime switching. The portfolio selection is proposed and analyzed for a market consisting of one bank account and multiple stocks. The random regime switching is assumed to be independent of the underlying Brownian motion and jump processes. A Markov chain modulated diffusion formulation is employed to model the problem.

1. Introduction

The jump diffusion process has come to play an important role in many branches of science and industry. In their book [1], Øksendal and Sulem have studied the optimal control, optimal stopping, and impulse control for jump diffusion processes. In mathematical finance theory, many researchers have developed option pricing theory, for example, Merton [2] was the first to use the jump processes to describe the stock dynamics, and Bardhan and Chao [3] were amongst the first authors to consider market completeness in a discontinuous model. The jump diffusion models have been discussed by Chan [4], Föllmer and Schweizer [5], El Karoui and Quenez [6], Henderson and Hobson [7], and Merculio and Runggaldier [8], to name a few.

On the other hand, regime-switching models have been widely used for price processes of risky assets. For example, in [9] the optimal stopping problem for the perpetual American put has been considered, and the finite expiry American put and barrier options have been priced. The asset allocation has been discussed in [10], and Elliott et al. [11] have investigated volatility problems. Regime-switching models with a Markov-modulated asset have already been applied to option pricing in [12–14] and references therein. Moreover, Markowitz's mean-variance portfolio selection with regime switching has been studied by Yin and Zhou [15], Zhou and Yin [16], and Zhou and Li [17].

Portfolio selection is an important topic in finance; multiperiod mean-variance portfolio selection has been studied by, for example, Samuelson [18], Hakansson [19], and Pliska [20] among others. Continuous-time mean-variance hedging problems were attacked by Duffie and Richardson [21] and Schweizer [22] where optimal dynamic strategies were derived, based on the projection theorem, to hedge contingent claims in incomplete markets.

In this paper, we will extend the results of Yin and Zhou [15] to SDEs with jumps under regime switching. After dealing with the difficulty from the jump processes, we obtain similar results to those of Yin and Zhou [15].

2. SDEs under Regime Switching with Jumps

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space on which it is defined a standard d-dimensional Brownian motion $W(t) \equiv (W_1(t), \ldots, W_d(t))'$ and a continuous-time stationary Markov chain $\alpha(t)$ taking value in a finite state space $\mathbb{S} = \{1, 2, \ldots, l\}$. Let $N(t, z)$ be as n-dimensional Poisson process and denote the compensated Poisson process by

$$\begin{aligned}\widetilde{N}(dt, dz) &= \left(\widetilde{N}_1(dt, dz_1), \ldots, \widetilde{N}_n(dt, dz_n)\right)' \\ &= (N_1(dt, dz_1) - \nu_1(dz_1)dt, \ldots, N_n(dt, dz_n) - \nu(dz_n)dt)',\end{aligned} \tag{2.1}$$

where N_j, $j = 1, \ldots, n$, are independent 1-dimensional Poisson random measures with characteristic measure ν_j, $j = 1, \ldots, n$, coming from n independent 1-dimensional Poisson point processes. We assume that $W(t)$, $\alpha(t)$, and $N(dt, dz)$ are independent. The Markov chain $\alpha(t)$ has a generator $Q = (q_{ij})_{l\times l}$ given by

$$P\{\alpha(t+\Delta) = j \mid \alpha(t) = i\} = \begin{cases} q_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + q_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases} \tag{2.2}$$

where $\Delta > 0$. Here $q_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$q_{ii} = -\sum_{j \neq i} q_{ij}, \tag{2.3}$$

and stationary transition probabilities

$$p_{ij}(t) = P(\alpha(t) = j \mid \alpha(0) = i), \quad t \geq 0, \ i, j = 1, 2, \ldots, l. \tag{2.4}$$

Define $\mathcal{F}_t = \sigma\{W(s), \alpha(s), N(s, \cdot) \ : \ 0 \leq s \leq t\}$. Let $|\cdot|$ denote the Euclidean norm as well as the matrix trace norm and M' the transpose of any vector or matrix M. We denote by $L^2_{\mathcal{F}(0,T;\mathbb{R}^m)}$ the set of all $\mathbb{R}^m$-valued, measurable stochastic processes $f(t)$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, such that $\mathbb{E}\int_0^T |f(t)|^2 dt < +\infty$.

Consider a market in which $d+1$ assets are traded continuously. One of the assets is a bank account whose price $P_0(t)$ is subject to the following stochastic ordinary differential equation:

$$\begin{gathered} dP_0(t) = r(t,\alpha(t))P_0(t)dt, \quad t \in [0,T], \\ P_0(0) = p_0 > 0, \end{gathered} \tag{2.5}$$

where $r(t,i) \geq 0, i = 1,2,\ldots,l$, are given as the interest rate process corresponding to different market modes. The other d assets are stocks whose price processes $P_m(t)$, $m = 1,2,\ldots,d$, satisfy the following system of stochastic differential equations (SDEs):

$$\begin{aligned} dP_m(t) &= P_m(t)\left\{ b_m(t,\alpha(t))dt + \sum_{n=1}^{d}\sigma_{mn}(t,\alpha(t))dW_n(t) + \sum_{j=1}^{n}\int_{\mathbb{R}} \rho_{mj}(t,\alpha(t),z_j)\widetilde{N}_j(dt,dz_j) \right\}, \\ &\qquad t \in [0,T], \\ P_m(0) &= p_m > 0, \end{aligned} \tag{2.6}$$

where for each $i = 1,2,\ldots,l$, $b : [0,T] \times \mathbb{S} \to \mathbb{R}^{d\times 1}$, $\sigma : [0,T] \times \mathbb{S} \to \mathbb{R}^{d\times d}$, $\rho : [0,T] \times \mathbb{S} \times \mathbb{R}^n \to \mathbb{R}^{d\times n}$ is the appreciation rate process, and $\sigma_m(t,i) := (\sigma_{m1}(t,i),\ldots,\sigma_{md}(t,i))$ are adapted processes such that the integrals exist. And each column $\rho^{(k)}$ of the $d \times n$ matrix $\rho = [\rho_{ij}]$ depends on z only through the kth coordinate z_k, that is,

$$\rho^{(k)}(t,i,z) = \rho^{(k)}(t,i,z_k), \quad z = (z_1,\ldots,z_n) \in \mathbb{R}^n. \tag{2.7}$$

Remark 2.1. Generally speaking, one uses noncompensated Poisson processes in a jump diffusion model (see Kushner [23]). However, we use compensated Poisson processes in (2.6) instead of using noncompensated Poisson processes, this is because firstly, using relationship (2.1) we can easily transform a jump diffusion model driven by noncompensated Poisson processes into a jump diffusion model driven by compensated Poisson processes; secondly, using compensated Poisson processes we can keep the Riccati Equation (4.2) similar to that of a diffusion model without jump processes, and then $H(t,i)$ in (4.3) has a financial interpretation.

Define the volatility matrix, for each $i = 1, \ldots, l$,

$$\sigma(t,i) := \begin{pmatrix} \sigma_1(t,i) \\ \vdots \\ \sigma_d(t,i) \end{pmatrix} \equiv (\sigma_{mn}(t,i))_{d\times d},$$

$$b(t,i) = \begin{pmatrix} b_1(t,i) \\ \vdots \\ b_d(t,i) \end{pmatrix} \in \mathbb{R}^{d\times 1}, \tag{2.8}$$

$$\rho(t,i,z) = \begin{pmatrix} \rho_1(t,i,z) \\ \vdots \\ \rho_d(t,i,z) \end{pmatrix} \in \mathbb{R}^{d\times n},$$

where

$$\rho_m(t,i,z) = \big(\rho_{m1}(t,i,z), \ldots, \rho_{mn}(t,i,z)\big). \tag{2.9}$$

We assume throughout this paper that the following nondegeneracy condition:

$$\sigma(t,i)\sigma(t,i)' \geq \delta I, \quad \forall t \in [0,T],\ i = 1,2,\ldots,l, \tag{2.10}$$

is satisfied for some $\delta > 0$. We also assume that all the functions $r(t,i)$, $b_m(t,i)$, and $\sigma_{mn}(t,i)$, $\rho_{mn}(t,i,z)$ are measurable and uniformly bounded in t.

Suppose that the initial market mode $\alpha(0) = i_0$. Consider an agent with an initial wealth $x_0 > 0$. These initial conditions are fixed throughout the paper. Denote by $x(t)$ the total wealth of the agent at time $t \geq 0$. Assume that the trading of shares takes place continuously and that transaction cost and consumptions are not considered. Suppose the right portfolio $(\pi_0(t), \pi_1(t), \ldots, \pi_d(t))$ exists, where $\pi_0(t)$ is the money invested in the bond, and $\pi_i(t)$ is the money invested in the ith stock. Then

$$\begin{gathered} x(t) = \sum_{i=0}^{d} \pi_i(t) = \sum_{i=0}^{d} \eta_i(t) P_i(t), \\ x(0) = x_0, \end{gathered} \tag{2.11}$$

where $\eta_0(t)$ is the number of bond units bought by the investor, and $\eta_i(t)$ is the amount of units for the ith stock. We call $x(t)$ the wealth process for this investor in the market. Now let us derive intuitively the stochastic differential equation (SDE) for the wealth process as follows. Suppose the portfolio is self-financed, that is, in a short time dt the investor does not put in or withdraw any money from the market. Let the money $x(t)$ change in the market due to the market own performance, that is, self-finance produces

$$dx(t) = \eta_0(t)dP_0(t) + \sum_{i=1}^{d}\eta_i(t)dP_i(t). \tag{2.12}$$

Now substituting (2.5) and (2.6) into the above equation, after a simple calculation we arrive at

$$\begin{aligned} dx(t) &= r(t,\alpha(t))x(t)dt + \sum_{m=1}^{\mathrm{d}}\pi_m(t)(b_m(t,\alpha(t)) - r(t,\alpha(t)))dt \\ &\quad + \sum_{m=1}^{d}\sum_{n=1}^{d}\pi_m(t)\sigma_{mn}(t,\alpha(t))dW_n(t) \\ &\quad + \sum_{m=1}^{d}\sum_{j=1}^{n}\int_{\mathbb{R}}\pi_m(t)\rho_{mj}(t,\alpha(t),z_j)\widetilde{N}_j(dt,dz_j), \\ x(0) &= x_0 > 0, \quad \alpha(0) = i_0, \end{aligned} \tag{2.13}$$

where $\pi(t) = (\pi_1(t),\ldots,\pi_d(t))'$ which we call a portfolio of the agent. And $\pi_m(t)$ is the total market value of the agent's wealth in the mth asset, $m = 0,1,\ldots,d$, at time t.

Setting

$$B(t,i) := (b_1(t,i) - r(t,i),\ldots,b_d(t,i) - r(t,i)), \quad i = 1,2,\ldots,l, \tag{2.14}$$

we can rewrite the wealth equation (2.13) as

$$\begin{aligned} dx(t) &= r(t,\alpha(t))x(t)dt + B(t,\alpha(t))\pi(t)dt + \pi'(t)\sigma(t,\alpha(t))dW(t) \\ &\quad + \int_{\mathbb{R}^n}\pi'(t)\rho(t,\alpha(t),z)\widetilde{N}(dt,dz), \\ x(0) &= x_0 > 0, \quad \alpha(0) = i_0. \end{aligned} \tag{2.15}$$

Definition 2.2. A portfolio $\pi(\cdot)$ is said to be admissible if $\pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ and the SDE (2.15) has a unique solution $x(\cdot)$ corresponding to $\pi(\cdot)$. In this case, we refer to $(x(\cdot),\pi(\cdot))$ as an admissible (wealth, portfolio) pair.

Remark 2.3. Most works in the literature define a portfolio, say $\pi(\cdot)$, as the fractions of wealth allocated to different stocks. That is,

$$u(t) = \frac{\pi(t)}{x(t)}, \quad t \in [0,T]. \tag{2.16}$$

With this definition, (2.15) can be rewritten as

$$\begin{aligned} dx(t) &= x(t)[r(t,\alpha(t)) + B(t,\alpha(t))u(t)]dt \\ &\quad + x(t)u(t)'\sigma(t,\alpha(t))dW(t) \\ &\quad + \int_{\mathbb{R}^n} x(t)u(t)'\rho(t,\alpha(t),z)\widetilde{N}(dt,dz), \\ x(0) &= x_0 > 0, \quad \alpha(0) = i_0. \end{aligned} \tag{2.17}$$

It is well known that this equation has a unique solution (see [1, page 10, Theorem 1.19]. We can use the same method in [18, Example 1.15, page 8] to show positivity of the solution of (2.17) if the initial wealth x_0 is positive and $u(t)'\rho(t,i,z) > -1$. A wealth process with possible zero or negative values is sensible at least for some circumstances. The nonnegativity of wealth process is better imposed as an additional constraint, rather than as a built-in feature. In our formulation, a portfolio is well defined even if the wealth is zero or negative, and the nonnegativity of the wealth could be a constraint.

The agent's objective is to find an admissible portfolio $\pi(\cdot)$ among all the admissible portfolios whose expected terminal wealth is $\mathbb{E}x(T) = \zeta$ for some given $\zeta \in \mathbb{R}^1$, so that the risk measured by the variance of the terminal wealth

$$\operatorname{Var} x(T) \equiv \mathbb{E}[x(T) - \mathbb{E}x(T)]^2 = \mathbb{E}[x(T) - \zeta]^2 \tag{2.18}$$

is minimized. Finding such a portfolio $\pi(\cdot)$ is referred to as the mean-variance portfolio selection problem. Specifically, we have the following formulation.

Definition 2.4. The mean-variance portfolio selection is a constrained stochastic optimization problem, parameterized by $\zeta \in \mathbb{R}^1$:

$$\begin{aligned} &\text{minimize} \quad J_{MV}(x_0, i_0, \pi(\cdot)) := \mathbb{E}[x(T) - \zeta]^2, \\ &\text{subject to} \quad \begin{cases} \mathbb{E}x(T) = \zeta, \\ (x(\cdot), \pi(\cdot)) \quad \text{admissible}. \end{cases} \end{aligned} \tag{2.19}$$

Moreover, the problem is called feasible if there is at least one portfolio satisfying all the constraints. The problem is called finite if it is feasible and the infimum of $J_{MV}(x_0, i_0, \pi(\cdot))$ is finite. Finally, an optimal portfolio to the above problem, if it ever exists, is called an efficient portfolio corresponding to ζ; the corresponding $(\operatorname{Var} x(T), \zeta) \in \mathbb{R}^2$ and $(\sigma_{x(T)}, \zeta) \in \mathbb{R}^2$ are interchangeably called an efficient point, where $\sigma_{x(T)}$ denotes the standard deviation of $x(T)$. The set of all the efficient points is called the efficient frontier.

For more details of mean-variance portfolio selection see [15, 16], We need more notations; let Δ_{ij} be consecutive, left closed, right open intervals of the real line each having length γ_{ij} such that

$$\begin{aligned}
\Delta_{12} &= [0, q_{12}), \\
\Delta_{13} &= [q_{12}, q_{12} + q_{13}), \\
&\vdots \\
\Delta_{1l} &= \left[\sum_{j=2}^{l-1} q_{1j}, \sum_{j=2}^{l} q_{1j}\right), \\
\Delta_{21} &= \left[\sum_{j=2}^{l} q_{1j}, \sum_{j=2}^{l} q_{1j} + q_{21}\right), \\
\Delta_{23} &= \left[\sum_{j=2}^{l} q_{1j} + q_{21}, \sum_{j=2}^{l} q_{1j} + \gamma_{21} + q_{23}\right), \\
&\vdots \\
\Delta_{2l} &= \left[\sum_{j=2}^{l} q_{1j} + \sum_{j=1, j\neq 2}^{l-1} q_{2j}, \sum_{j=2}^{l} q_{1j} + \sum_{j=1, j\neq 2}^{l} q_{2j}\right).
\end{aligned} \tag{2.20}$$

For future use, we cite the generalized Itô lemma (see [1, 24, 25]) as the following lemma.

Lemma 2.5. *Given a d-dimensional process $y(\cdot)$ satisfying*

$$dy(t) = f(t, y(t), \alpha(t))dt + g(t, y(t), \alpha(t))dW(t) + \int_{\mathbb{R}^n} \gamma(t, y(t), \alpha(t), z)\widetilde{N}(dt, dz), \tag{2.21}$$

where f, g, and γ satisfy Lipschitz condition with appropriate dimensions, each column $\gamma^{(k)}$ of the matrix $\gamma = [\gamma_{ij}]$ depends on z only through the kth coordinate z_k. Let $\varphi(t, x, i) \in C^{1,2}([0, T] \times \mathbb{R}^n \times S; \mathbb{R})$, one then has

$$\begin{aligned}
d\varphi(t, y(t), \alpha(t)) = {}& \Gamma\varphi(t, y(t), \alpha(t))dt + \varphi_x(t, y(t), \alpha(t))' g(t, y(t), \alpha(t))dW(t) \\
&+ \sum_{k=1}^{n} \int_{\mathbb{R}} \Big\{\varphi\Big(t, y(t) + \gamma^{(k)}(t, \alpha(t), z_k), \alpha(t)\Big) - \varphi(t, y(t), \alpha(t)) \\
&\qquad - \varphi_x(t, y(t), \alpha(t))' \gamma^{(k)}(t, \alpha(t), z)\Big\}\nu(dz_k)dt \\
&+ \sum_{k=1}^{n} \int_{\mathbb{R}} \Big\{\varphi\Big(t, y(t) + \gamma^{(k)}(t, \alpha(t), z), \alpha(t)\Big) - \varphi(t, y(t), \alpha(t))\Big\}\widetilde{N}_k(dt, dz_k) \\
&+ \int_{\mathbb{R}} \Big(\varphi\Big(t, y(t), \alpha(0) + h\Big(\alpha(t), \bar{l}\Big)\Big) - \varphi(t, y(t), \alpha(t))\Big)\mu\Big(dt, d\bar{l}\Big),
\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}
\Gamma\varphi(t,x,i) :=\ & \varphi_t(t,x,i) + \varphi_x(t,x,i)' f(t,x,i) \\
& + \frac{1}{2}\operatorname{trace}\left[\mathrm{g}(\mathrm{t},\mathrm{x},\mathrm{i})'{}'_{\mathrm{xx}}(\mathrm{t},\mathrm{x},\mathrm{i})\mathrm{g}(\mathrm{t},\mathrm{x},\mathrm{i})\right] + \sum_{\mathrm{j}=1}^{\mathrm{l}} \mathrm{q}_{\mathrm{ij}}{}'(\mathrm{t},\mathrm{x},\mathrm{j}) \\
& + \sum_{k=1}^{n}\int_{\mathbb{R}}\Big\{\varphi\Big(t, y(t)+\gamma^{(k)}(t,\alpha(t),z_k),\alpha(t)\Big) - \varphi(t,y(t),\alpha(t)) \\
& \qquad -\varphi_x(t,y(t),\alpha(t))'\gamma^{(k)}(t,\alpha(t),z)\Big\}\nu(dz_k),
\end{aligned} \tag{2.23}$$

where μ is a martingale measure,

$$h(i,y) = \begin{cases} j-i, & \textit{if } y \in \Delta_{ij}, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.24}$$

and $\mu(dt,d\bar{l}) = \gamma(dt,d\bar{l}) - \mu(d\bar{l})dt$ is a martingale measure. And $\gamma(dt,dy)$ is a Poisson random measure with intensity $dt \times \mu(dy)$, in which μ is the Lebesgue measure on $\mathbb{R}$.

3. Feasibility

Since the problem (2.19) involves a terminal constraint $\mathbb{E}x(T) = \zeta$, in this section, we derive conditions under which the problem is at least feasible. First of all, the following generalized Itô lemma [25] for Markov-modulated processes is useful.

The associated wealth process $x^0(\cdot)$ satisfies

$$\begin{aligned}
dx^0(t) &= r(t,\alpha(t))x^0(t)dt, \\
x^0(0) &= x_0 > 0, \quad \alpha(0) = i_0,
\end{aligned} \tag{3.1}$$

with its expected terminal wealth

$$\zeta^0 := \mathbb{E}x^0(T) = \mathbb{E}e^{\int_0^T r(s,\alpha(s))ds}x_0. \tag{3.2}$$

Lemma 3.1. *Let $\psi(\cdot,i)$, $i = 1,2,\ldots,l$, be the solutions to the following system of linear ordinary differential equations (ODEs):*

$$\begin{aligned}
\dot{\psi}(t,i) &= -r(t,i)\psi(t,i) - \sum_{j=1}^{l} q_{ij}\psi(t,j), \\
\psi(T,i) &= 1, \quad i = 1,2,\ldots,l.
\end{aligned} \tag{3.3}$$

Then the mean-variance problem (2.19) *is feasible for every* $\zeta \in \mathbb{R}^1$ *if and only if*

$$\varrho := \mathbb{E}\int_0^T \left|\psi(t,\alpha(t))B(t,\alpha(t))\right|^2 dt > 0. \tag{3.4}$$

Proof. To prove the "if" part, construct a family of admissible portfolios $\pi^\beta(\cdot) = \beta\pi(\cdot)$ for $\beta \in \mathbb{R}^1$ where

$$\pi(t) = B(t,\alpha(t))'\psi(t,\alpha(t)). \tag{3.5}$$

Assume that $x^\beta(t)$ is the solution of (2.15). Let $x^\beta(t) = x^0(t) + \beta y(t)$, where $x^0(\cdot)$ satisfies (3.1), and $y(\cdot)$ is the solution to the following equation:

$$\begin{aligned} dy(t) &= \left[r(t,\alpha(t))y(t) + B(t,\alpha(t))\pi(t)\right]dt + \pi(t)'\sigma(t,\alpha(t))dW(t) \\ &\quad + \int_{\mathbb{R}^n} \pi(t)'\rho(t,\alpha(t),z)\widetilde{N}(dt,dz), \\ y(0) &= 0, \quad \alpha(0) = i_0. \end{aligned} \tag{3.6}$$

Therefore, problem (2.19) is feasible for every $\zeta \in \mathbb{R}^1$ if there exists $\beta \in \mathbb{R}$ such that $\zeta = \mathbb{E}x^\beta(T) \equiv \mathbb{E}x^0(T) + \beta\mathbb{E}y(T)$. Equivalently, (2.19) is feasible for every $\zeta \in \mathbb{R}$ if $\mathbb{E}y(T) \neq 0$. Applying the generalized Itô formula (Lemma 2.5) to $\varphi(t,x,i) = \psi(t,i)x$, we have

$$\begin{aligned} &d\left[\psi(t,\alpha(t))y(t)\right] \\ &\quad = \dot{\psi}(t,\alpha(t))y(t)dt + \psi(t,\alpha(t))\left[r(t,\alpha(t))y(t) + B(t,\alpha(t))\pi(t)\right]dt \\ &\qquad + \sum_{j=1}^{l} q_{\alpha(t)j}\psi(t,j)y(t)dt + \pi(t)'\sigma(t,\alpha(t))dW(t) \\ &\qquad + \sum_{k=1}^{n}\int_{\mathbb{R}} \Big\{\psi(t,\alpha(t))\Big(y(t) + \pi(t)'\rho^{(k)}(t,\alpha(t),z)\Big) - \psi(t,\alpha(t))y(t) \\ &\qquad\qquad -\psi(t,\alpha(t))\pi(t)'\rho^{(k)}(t,\alpha(t),z)\Big\}\nu(dz)dt \\ &\qquad + \sum_{k=1}^{n}\int_{\mathbb{R}} \Big\{\psi(t,\alpha(t))\Big(y(t) + \pi(t)'\rho^{(k)}(t,\alpha(t),z)\Big) \\ &\qquad\qquad -\psi(t,\alpha(t))\pi(t)'\rho^{(k)}(t,\alpha(t),z)\Big\}\widetilde{N}_k(dt,dz_k) \\ &\qquad + \int_{\mathbb{R}} \Big\{\psi\Big(t,\alpha(0) + h\Big(\alpha(t),\bar{l}\Big)\Big)y(t) - \psi(t,\alpha(t))y(t)\Big\}\mu\Big(dt,d\bar{l}\Big) \end{aligned}$$

$$\begin{aligned}
&= -r(t,\alpha(t))\varphi(t,\alpha(t))y(t)dt - \sum_{j=1}^{l} q_{\alpha(t)j}\varphi(t,j)y(t)dt \\
&\quad + r(t,\alpha(t))\varphi(t,\alpha(t))y(t)dt + B(t,\alpha(t))\pi(t)\varphi(t,\alpha(t))dt \\
&\quad + \sum_{j=1}^{l} q_{\alpha(t)j}\varphi(t,j)y(t)dt + \pi(t)'\sigma(t,\alpha(t))dW(t) + \sum_{k=1}^{n}\int_{\mathbb{R}}\{\varphi(t,\alpha(t))y(t)\}\widetilde{N}_k(dt,dz_k) \\
&\quad + \int_{\mathbb{R}}\left\{\varphi\left(t,\alpha(0)+h\left(\alpha(t),\bar{l}\right)\right)y(t) - \varphi(t,\alpha(t))y(t)\right\}\mu\left(dt,d\bar{l}\right) \\
&= B(t,\alpha(t))\pi(t)\varphi(t,\alpha(t))dt + \pi(t)'\sigma(t,\alpha(t))dW(t) + \sum_{k=1}^{n}\int_{\mathbb{R}}\{\varphi(t,\alpha(t))y(t)\}\widetilde{N}_k(dt,dz_k) \\
&\quad + \int_{\mathbb{R}}\left\{\varphi\left(t,\alpha(0)+h\left(\alpha(t),\bar{l}\right)\right)y(t) - \varphi(t,\alpha(t))y(t)\right\}\mu\left(dt,d\bar{l}\right).
\end{aligned} \tag{3.7}$$

Integrating from 0 to T, taking expectation, and using (3.5), we obtain

$$\begin{aligned}
\mathbb{E}y(T) &= \mathbb{E}\int_0^T \varphi(t,\alpha(t))B(t,\alpha(t))\pi(t)dt \\
&= \mathbb{E}\int_0^T \left|\varphi(t,\alpha(t))B(t,\alpha(t))\right|^2 dt.
\end{aligned} \tag{3.8}$$

Consequently, $\mathbb{E}y(T) \neq 0$ if (3.4) holds.

Conversely, suppose that problem (2.19) is feasible for every $\zeta \in \mathbb{R}^1$. Then for each $\zeta \in \mathbb{R}$, there is an admissible portfolio $\pi(\cdot)$ so that $\mathbb{E}x(T) = \zeta$. However, we can always decompose $x(t) = x^0(t) + y(t)$ where $y(\cdot)$ satisfies (3.6). This leads to $\mathbb{E}x^0(T) + \mathbb{E}y(T) = \zeta$. However, $\mathbb{E}x^0(T) \equiv \zeta^0$ is independent of $\pi(\cdot)$; thus it is necessary that there is a $\pi(\cdot)$ with $\mathbb{E}y(T) \neq 0$. It follows then from (3.8) that (3.4) is valid. □

Theorem 3.2. *The mean-variance problem* (2.19) *is feasible for every* $\zeta \in \mathbb{R}$ *if and only if*

$$\mathbb{E}\int_0^T |B(t,\alpha(t))|^2 dt > 0. \tag{3.9}$$

Proof. By virtue of Lemma (3.1), it suffices to prove that $\varphi(t,i) > 0$ $\forall t \in [0,T]$, $i = 1,2,\ldots,l$. To this end, note that (3.3) can be rewritten as

$$\begin{aligned}
&\dot{\varphi}(t,i) = \left[-r(t,i) - q_{ii}\right]\varphi(t,i) - \sum_{j\neq i}^{l} q_{ij}\varphi(t,j), \\
&\varphi(T,i) = 1, \quad i = 1,2,\ldots,l.
\end{aligned} \tag{3.10}$$

Treating this as a system of terminal-valued ODEs, a variation-of-constant formula yields

$$\psi(t,i) = e^{-\int_t^T [-r(s,i)-q_{ii}]ds} + \int_t^T e^{-\int_t^s [-r(\tau,i)-q_{ii}]d\tau} \sum_{j\neq i}^{l} q_{ij}\psi(s,j)ds, \quad i = 1,2,\ldots,l. \tag{3.11}$$

Construct a sequence $\psi^{(k)}(\cdot,i)$ (known as the Picard sequence) as follows:

$$\begin{aligned}
&\psi^{(0)}(t,i) = 1, \quad t \in [0,T],\ i = 1,2,\ldots,l, \\
&\psi^{(k+1)}(t,i) = e^{-\int_t^T [-r(s,i)-q_{ii}]ds} + \int_t^T e^{-\int_t^s [-r(\tau,i)-q_{ii}]d\tau} \sum_{j\neq i}^{l} q_{ij}\psi^{(k)}(s,j)ds, \\
&\qquad\qquad t \in [0,T],\ i = 1,2,\ldots,l,\ k = 0,1,\ldots.
\end{aligned} \tag{3.12}$$

Noting that $q_{ij} \geq 0$ for all $j \neq i$, we have

$$\psi^{(k)}(t,i) \geq e^{-\int_t^T [-r(s,i)-q_{ii}]ds} > 0, \quad k = 0,1,\ldots. \tag{3.13}$$

On the other hand, it is well known that $\psi(t,i)$ is the limit of the Picard sequence $\psi^{(k)}(t,i)$ as $k \to \infty$. Thus $\psi(t,i) > 0$. This proves the desired result. □

Corollary 3.3. *If* (3.9) *holds, then for any $\zeta \in \mathbb{R}$, an admissible portfolio that satisfies $\mathbb{E}x(T) = \zeta$ is given by*

$$\pi(t) = \frac{\zeta - \zeta^0}{\varrho} B(t,\alpha(t))' \psi(t,\alpha(t)), \tag{3.14}$$

where x^0 and ϱ are given by (3.2) *and* (3.4)*, respectively.*

Proof. This is immediate from the proof of the "if" part of Lemma (3.1)

$$\begin{aligned}
\mathbb{E}x(T) &= \zeta \\
&= x^0(T) + \mathbb{E}y(T), \\
\zeta - \zeta^0 &= \mathbb{E}y(T) \\
&= \mathbb{E}\int_0^T \psi(t,\alpha(t))B(t,\alpha(t))\pi(t)dt.
\end{aligned} \tag{3.15}$$

Then one has

$$\pi(t) = \frac{\zeta - \zeta^0}{\varrho} B(t,\alpha(t))' \psi(t,\alpha(t)). \tag{3.16}$$

□

Corollary 3.4. *If* $\mathbb{E}\int_0^T |B(t,\alpha(t))|^2 dt = 0$, *then any admissible portfolio* $\pi(\cdot)$ *results in* $\mathbb{E}x(T) = \zeta^0$.

Proof. This is seen from the proof of the "only if" part of Lemma (3.1)

$$\begin{aligned} \mathbb{E}x(T) &= \mathbb{E}x^0(T) + \mathbb{E}y(T) \\ &= \zeta^0 + \psi(t,\alpha(t))B(t,\alpha(t))\pi(t)dt \\ &= \zeta^0 \end{aligned} \tag{3.17}$$

since $\mathbb{E}\int_0^T |B(t,\alpha(t))|^2 dt = 0$. □

Remark 3.5. Condition (3.9) is very mild. For example, (3.9) holds as long as there is one stock whose appreciation-rate process is different from the interest-rate process at any market mode, which is obviously a practically reasonable assumption. On the other hand, if (3.9) fails, then Corollary (3.4) implies that the mean-variance problem (2.19) is feasible only if $\zeta = \zeta^0$. This is pathological and trivial case that does not warrant further consideration. Therefore, from this point on we will assume that (3.9) holds or, equivalently, the mean-variance problem (2.19) is feasible for any ζ.

Having addressed the issue of feasibility, we proceed with the study of optimality. The mean-variance problem (2.19) under consideration is a dynamic optimization problem with a constraint $\mathbb{E}x(T) = \zeta$. To handle this constraint, we apply the Lagrange multiplier technique. Define

$$\begin{aligned} J(x_0, i_0, \pi(\cdot), \lambda) :&= \mathbb{E}\left\{|x(T) - \zeta|^2 + 2\lambda[x(T) - \zeta]\right\} \\ &= \mathbb{E}[x(T) + \lambda - \zeta]^2 - \lambda^2, \quad \lambda \in \mathbb{R}. \end{aligned} \tag{3.18}$$

Our first goal is to solve the following unconstrained problem parameterized by the Lagrange multiplier λ:

$$\begin{aligned} &\text{minimize} \quad J(x_0, i_0, \pi(\cdot), \lambda) = \mathbb{E}[x(T) + \lambda - \zeta]^2 - \lambda^2, \\ &\text{subject to} \quad (x(\cdot), \pi(\cdot)) \text{ admissible}. \end{aligned} \tag{3.19}$$

This turns out to be a Markov-modulated stochastic linear-quadratic optimal control problem, which will be solved in the next section.

4. Solution to the Unconstrained Problem

In this section we solve the unconstrained problem (3.19). Firstly define

$$\gamma(t,i) := B(t,i)\left[\sigma(t,i)\sigma(t,i)' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)\right]^{-1} B(t,i)', \quad i = 1, 2, \ldots, l. \tag{4.1}$$

Consider the following two systems of ODEs:

$$\begin{aligned}
&\dot{P}(t,i) = [\gamma(t,i) - 2r(t,i)]P(t,i) - \sum_{j=1}^{l} q_{ij}P(t,j), \quad 0 \le t \le T, \\
&P(T,i) = 1, \quad i = 1,2,\ldots,l,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\dot{H}(t,i) = r(t,i)H(t,i) - \frac{1}{P(t,i)}\sum_{j=1}^{l} q_{ij}P(t,j)[H(t,j) - H(t,i)], \quad 0 \le t \le T, \\
&H(T,i) = 1, \quad i = 1,2,\ldots,l.
\end{aligned} \tag{4.3}$$

The existence and uniqueness of solutions to the above two systems of equations are evident as both are linear with uniformly bounded coefficients.

Proposition 4.1. *The solutions of* (4.2) *and* (4.3) *must satisfy* $P(t,i) > 0$ *and* $0 < H(t,i) \le 1$, $\forall t \in [0,T]$, $i = 1,2,\ldots,l$. *Moreover, if for a fixed* i, $r(t,i) > 0$, *a.e.,* $t \in [0,T]$, *then* $H(t,i) < 1$, $\forall t \in [0,T)$.

Proof. The assertion $P(t,i) > 0$ can be proved in exactly the same way as that of $\psi(t,i) > 0$; see the proof of Theorem 3.2. Having proved the positivity of $P(t,i)$, one can then show that $H(t,i) > 0$ using the same argument because now $P(t,j)/P(t,i) > 0$.

To prove that $H(t,i) \le 1$, first note that the following system of ODEs:

$$\begin{aligned}
&\frac{d}{dt}\widetilde{H}(t,i) = -\frac{1}{P(t,i)}\sum_{j=1}^{l} q_{ij}P(t,j)\left[\widetilde{H}(t,j) - \widetilde{H}(t,i)\right], \\
&\widetilde{H}(T,i) = 1, \quad i = 1,2,\ldots,l,
\end{aligned} \tag{4.4}$$

has the only solutions $\widetilde{H}(t,i) \equiv 1$, $i = 1,2,\ldots,l$, due to the uniqueness of solutions. Set

$$\widehat{H}(t,i) := \widetilde{H}(t,i) - H(t,i) \equiv 1 - H(t,i), \tag{4.5}$$

which solves the following equations:

$$\begin{aligned}
\frac{d}{dt}\widehat{H}(t,i) &= r(t,i)\widehat{H}(t,i) - r(t,i) - \frac{1}{P(t,i)}\sum_{j=1}^{l} P(t,j)\left[\widehat{H}(t,j) - \widehat{H}(t,i)\right] \\
&= \left[r(t,i) + \frac{1}{P(t,i)}\sum_{j\ne i}^{l} P(t,j)\right]\widehat{H}(t,i) - r(t,i) - \frac{1}{P(t,i)}\sum_{j=1}^{l} P(t,j)\widehat{H}(t,j), \\
\widehat{H}(T,i) &= 0, \quad i = 1,2,\ldots,l.
\end{aligned} \tag{4.6}$$

A variation-of-constant formula leads to

$$\widehat{H}(t,i) = \int_t^T e^{-\int_t^s [r(\tau,i)+(1/P(\tau,i))\sum_{j\neq i}^l P(\tau,j)]d\tau} \left[r(s,i) + \frac{1}{P(s,i)} \sum_{j=1}^{l} P(s,j)\widehat{H}(s,j) \right] ds. \tag{4.7}$$

A similar trick using the construction of Picard's sequence yields that $\widehat{H}(t,i) \geq 0$. In addition, $\widehat{H}(t,i) > 0$, $\forall t \in [0,T)$, if $r(t,i) > 0$, a.e., $t \in [0,T]$. The desired result then follows from the fact that $\widehat{H}(t,i) = 1 - H(t,i)$. $\square$

Remark 4.2. Equation (4.2) is a Riccati type equation that arises naturally in studying the stochastic LQ control problem (3.19) whereas (4.3) is used to handle the nonhomogeneous terms involved in (3.19); see the proof of Theorem 4.3. On the other hand, $H(t,i)$ has a financial interpretation: for fixed (t,i), $H(t,i)$ is a deterministic quantity representing the risk-adjusted discount factor at time t when the market mode is i (note that the interest rate itself is random).

Theorem 4.3. *Problem* (3.19) *has an optimal feedback control*

$$\pi^*(t,x,i) = -\left[\sigma(t,i)\sigma(t,i)' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)\right]^{-1} B(t,i)'[x + (\lambda - \zeta)H(t,i)]. \tag{4.8}$$

Moreover, the corresponding optimal value is

$$\begin{aligned} &\inf_{\pi(\cdot)\ admissible} J(x_0, i_0, \pi(\cdot), \lambda) \\ &\quad = \left[P(0,i_0)H(0,i_0)^2 + \theta - 1\right](\lambda - \zeta)^2 + 2[P(0,i_0)H(0,i_0)x_0 - \zeta](\lambda - \zeta) + P(0,i_0)x_0^2 - \zeta^2, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} \theta &:= \mathbb{E}\int_0^T \sum_{j=1}^{l} q_{\alpha(t)j} P(t,j)\left[H(t,j) - H(t,\alpha(t))\right]^2 dt \\ &= \sum_{i=1}^{l}\sum_{j=1}^{l}\int_0^T P(t,j)p_{i_0 i}(t)q_{ij}\left[H(t,j) - H(t,i)\right]^2 dt \geq 0, \end{aligned} \tag{4.10}$$

with the transition probabilities $p_{i_0 i}(t)$ given by (2.4).

Proof. Let $\pi(\cdot)$ be any admissible control and $x(\cdot)$ the corresponding state trajectory of (2.15). Applying the generalized Itô formula (Lemma 2.5) to

$$\varphi(t,x,i) = P(t,i)[x + (\lambda - \zeta)H(t,i)]^2, \tag{4.11}$$

we obtain

$$
\begin{aligned}
& d\Big\{P(t,\alpha(t))[x(t)+(\lambda-\zeta)H(t,\alpha(t))]^2\Big\} \\
&\quad = \dot{P}(t,\alpha(t))[x(t)+(\lambda-\zeta)H(t,\alpha(t))]^2 dt \\
&\qquad + 2P(t,\alpha(t))(\lambda-\zeta)[x(t)+(\lambda-\zeta)H(t,\alpha(t))]\dot{H}(t,\alpha(t))dt \\
&\qquad + 2\{r(t,\alpha(t))x(t)+B(t,\alpha(t))\pi(t)\} \\
&\qquad \times P(t,\alpha(t))[x(t)+(\lambda-\zeta)H(t,\alpha(t))]dt \\
&\qquad + \sum_{j=1}^{l} q_{\alpha(t)j} P(t,j)\left[x(t)+(\lambda-\zeta)H(t,j)\right]^2 dt \\
&\qquad + \frac{1}{2} 2P(t,\alpha(t))\pi(t)'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']\pi(t)dt \\
&\qquad + P(t,\alpha(t))\pi(t)'\left\{\int_{\mathbb{R}^n}\rho(t,\alpha(t),z)\rho(t,\alpha(t),z)'\nu(dz)\right\}\pi(t)dt \\
&\qquad + 2P(t,\alpha(t))x(t)^2\pi(t)'\sigma(t,\alpha(t))dW(t) \\
&\qquad + \sum_{k=1}^{n}\int_{\mathbb{R}} P(t,\alpha(t))\Big\{2[x(t)+(\lambda-\zeta)H(t,\alpha(t))]\rho^{(k)}(t,\alpha(t),z)+\rho^{(k)}(t,\alpha(t),z)^2\Big\} d\widetilde{N}(dt,dz) \\
&\qquad + \int_{\mathbb{R}}\Big\{P\Big(t,\alpha(0)+h\Big(\alpha(t),\bar{l}\Big)\Big)\Big[x(t)+(\lambda-\zeta)H(t,\alpha(0)+h(\alpha(t),\bar{(l)}))\Big]^2 \\
&\qquad\qquad -P(t,\alpha(t))[x(t)+(\lambda-\zeta)H(t,\alpha(t))]^2\Big\}\mu\Big(dt,d\bar{l}\Big) \\
&\quad = P(t,\alpha(t))\Big\{\pi(t)'\Big[\sigma(t,\alpha(t))\sigma(t,\alpha(t))'+\int_{\mathbb{R}^n}\rho(t,\alpha(t),z)\rho(t,\alpha(t),z)'\nu(dz)\Big]\pi(t) \\
&\qquad\qquad + 2\pi(t)'B(t,\alpha(t))'[x(t)+(\lambda-\zeta)H(t,\alpha(t))] \\
&\qquad\qquad + \gamma(t,\alpha(t))[x(t)+(\lambda-\zeta)H(t,\alpha(t))]\Big\}dt \\
&\qquad + (\lambda-\zeta)^2\sum_{j=1}^{l} q_{\alpha(t)j} P(t,j)\left[H(t,j)-H(t,i)\right]^2 dt \\
&\qquad + 2P(t,\alpha(t))x(t)^2\pi(t)'\sigma(t,\alpha(t))dW(t) \\
&\qquad + \sum_{k=1}^{n}\int_{\mathbb{R}} P(t,\alpha(t))\Big\{2[x(t)+(\lambda-\zeta)H(t,\alpha(t))]\rho^{(k)}(t,\alpha(t),z)+\rho^{(k)}(t,\alpha(t),z)^2\Big\} d\widetilde{N}(dt,dz) \\
&\qquad + \int_{\mathbb{R}}\Big\{P\Big(t,\alpha(0)+h\Big(\alpha(t),\bar{l}\Big)\Big)\Big[x(t)+(\lambda-\zeta)H(t,\alpha(0)+h(\alpha(t),\bar{(l)}))\Big]^2 \\
&\qquad\qquad -P(t,\alpha(t))[x(t)+(\lambda-\zeta)H(t,\alpha(t))]^2\Big\}\mu\Big(dt,d\bar{l}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= P(t,\alpha(t))[\pi(t)-\pi^*(t,x(t),\alpha(t))]'\left[\sigma(t,\alpha(t))\sigma(t,\alpha(t))'+\int_{\mathbb{R}^n}\rho(t,\alpha(t))\rho(t,\alpha(t),z)'\nu(dz)\right] \\
&\quad\times[\pi(t)-\pi^*(t,x(t),\alpha(t))]dt \\
&\quad+(\lambda-\zeta)^2\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)\,[H(t,j)-H(t,i)]^2dt \\
&\quad+2P(t,\alpha(t))x(t)^2\pi(t)'\sigma(t,\alpha(t))dW(t) \\
&\quad+\sum_{k=1}^{n}\int_{\mathbb{R}}P(t,\alpha(t))\left\{2[x(t)+(\lambda-\zeta)H(t,\alpha(t))]\rho^{(k)}(t,\alpha(t),z)+\rho^{(k)}(t,\alpha(t),z)^2\right\}d\widetilde{N}(dt,dz) \\
&\quad+\int_{\mathbb{R}}\left\{P\Big(t,\alpha(0)+h\Big(\alpha(t),\bar{l}\Big)\Big)\Big[x(t)+(\lambda-\zeta)H(t,\alpha(0)+h(\alpha(t),\bar{(l)}))\Big]^2\right. \\
&\qquad\left.-P(t,\alpha(t))[x(t)+(\lambda-\zeta)H(t,\alpha(t))]^2\right\}\mu\Big(dt,d\bar{l}\Big),
\end{aligned}
\tag{4.12}
$$

where $\pi^*(t,x,i)$ is defined as the right-hand side of (4.8). Integrating the above from 0 to T and taking expectations, we obtain

$$
\begin{aligned}
&\mathbb{E}[x(T)+\lambda-\zeta]^2 \\
&\quad= P(0,i_0)[x_0+(\lambda-\zeta)H(0,i_0)]^2+\theta(\lambda-\zeta)^2 \\
&\qquad+\mathbb{E}\int_0^T P(t,\alpha(t))[\pi(t)-\pi^*(t,x(t),\alpha(t))]' \\
&\qquad\times\left[\sigma(t,\alpha(t))\sigma(t,\alpha(t))'+\int_{\mathbb{R}^n}\rho(t,\alpha(t),z)\rho(t,\alpha(t),z)'\nu(dz)\right] \\
&\qquad\times[\pi(t)-\pi^*(t,x(t),\alpha(t))]dt.
\end{aligned}
\tag{4.13}
$$

Consequently,

$$
\begin{aligned}
&J(x_0,i_0,\pi(\cdot),\lambda) \\
&\quad=\mathbb{E}[x(T)+\lambda-\zeta]^2-\lambda^2 \\
&\quad=[P(0,i_0)H(0,i_0)+\theta-1](\lambda-\zeta)^2 \\
&\qquad+2[P(0,i_0)H(0,i_0)x_0-\zeta](\lambda-\zeta)+P(0,i_0)x_0^2-\zeta^2 \\
&\qquad+\mathbb{E}\int_0^T P(t,\alpha(t))[\pi(t)-\pi^*(t,x(t),\alpha(t))]' \\
&\qquad\times\left[\sigma(t,\alpha(t))\sigma(t,\alpha(t))'+\int_{\mathbb{R}^n}\rho(t,\alpha(t),z)\rho(t,\alpha(t),z)'\nu(dz)\right] \\
&\qquad\times[\pi(t)-\pi^*(t,x(t),\alpha(t))]dt.
\end{aligned}
\tag{4.14}
$$

Since $P(t,\alpha(t)) > 0$ by Proposition (4.1), it follows immediately that the optimal feedback control is given by (4.8) and the optimal value is given by (4.9), provided that the corresponding equation (2.15) under the feedback control (4.8) has a solution. But under (4.8), the system (2.15) is a nonhomogeneous linear SDE with coefficients modulated by $\alpha(t)$. Since all the coefficients of this linear equation are uniformly bounded and $\alpha(t)$ is independent of $W(t)$, the existence and uniqueness of the solution to the equation are straightforward based on a standard successive approximation scheme.

Finally, since

$$\theta := \mathbb{E}\int_0^T \sum_{j\neq i}^{l} q_{\alpha(t)j} P(t,j)\left[H(t,j) - H(t,\alpha(t))\right]^2 dt \tag{4.15}$$

and $q_{ij} \geq 0$ for all $i \neq j$, we must have $\theta \geq 0$. This completes the proof. □

5. Efficient Frontier

In this section we proceed to derive the efficient frontier for the original mean-variance problem (2.19).

Theorem 5.1 (efficient portfolios and efficient frontier). *Assume that* (3.9) *holds. Then one has*

$$P(0,i_0)H(0,i_0)^2 + \theta - 1 < 0. \tag{5.1}$$

Moreover, the efficient portfolio corresponding to z, as a function of the time t, the wealth level x, and the market mode i, is

$$\pi^*(t,x,i) = -\left[\sigma(t,i)\sigma(t,i)' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)\right]^{-1} B(t,i)'\left[x + (\lambda^* - \zeta)H(t,i)\right], \tag{5.2}$$

where

$$\lambda^* = \frac{\zeta - P(0,i_0)H(0,i_0)x_0}{P(0,i_0)H(0,i_0)^2 + \theta - 1} + \zeta. \tag{5.3}$$

Furthermore, the optimal value of $\operatorname{Var} x(T)$*, among all the wealth processes $x(\cdot)$ satisfying $\mathbb{E}x(T) = \zeta$, is*

$$\begin{aligned}\operatorname{Var} x^*(T) = {} & \frac{P(0,i_0)H(0,i_0)^2 + \theta}{1 - \theta - P(0,i_0)H(0,i_0)^2}\left[\zeta - \frac{P(0,i_0)H(0,i_0)}{P(0,i_0)H(0,i_0)^2 + \theta}x_0\right]^2 \\ & + \frac{P(0,i_0)\theta}{P(0,i_0)H(0,i_0)^2 + \theta}x_0^2.\end{aligned} \tag{5.4}$$

Proof. By assumption (3.9) and Theorem 3.2, the mean-variance problem (2.19) is feasible for any $\zeta \in \mathbb{R}^1$. Moreover, using exactly the same approach in the proof of Theorem 4.3, one can show that problem (2.19) without the constraint $\mathbb{E}x(T) = \zeta$ must have a finite optimal value, hence so does the problem (2.19). Therefore, (2.19) is finite for any $\zeta \in \mathbb{R}^1$. Now we need to prove that $J_{MV}(x_0, i_0, \pi(\cdot))$ is strictly convex in $\pi(\cdot)$. We can easily get

$$\begin{gathered}\mathbb{E}(2x_1x_2) \leq \mathbb{E}\left(x_1^2 + x_2^2\right), \\ \mathbb{E}(2\kappa(1-\kappa)x_1x_2) \leq \mathbb{E}\left(\kappa(1-\kappa)x_1^2 + \kappa(1-\kappa)x_2^2\right), \\ \mathbb{E}\left(\kappa^2x_1^2 + (1-\kappa)^2x_2^2 + 2\kappa(1-\kappa)x_1x_2\right) \leq \mathbb{E}\left(\kappa x_1^2 + (1-\kappa)x_2^2\right), \\ \mathbb{E}(\kappa x_1 + (1-\kappa)x_2 - \zeta)^2 \leq \mathbb{E}\left(\kappa(x_1-\zeta)^2\right) + \mathbb{E}\left((1-\kappa)(x_2-\zeta)^2\right),\end{gathered} \tag{5.5}$$

where $\kappa \in [0,1]$. So, we obtain

$$\mathbb{E}(\kappa x_1 - \kappa\zeta + (1-\kappa)x_2 - (1-\kappa)\zeta)^2 \leq \mathbb{E}\left(\kappa(x_1-\zeta)^2\right) + \mathbb{E}\left((1-\kappa)(x_2-\zeta)^2\right), \tag{5.6}$$

which proves $J_{MV}(x_0, i_0, \pi(\cdot))$ is strictly convex in $\pi(\cdot)$. that Affine space means the complement of points at infinity. It can also be viewed as a vector space whose operations are limited to those linear combinations whose coefficients sum to one. Since $J_{MV}(x_0, i_0, \pi(\cdot))$ is strictly convex in $\pi(\cdot)$ and the constraint function $\mathbb{E}x(T) - \zeta$ is affine in $\pi(\cdot)$, we can apply the well-known duality theorem (see [26, page 224, Theorem 1]) to conclude that for any $\zeta \in \mathbb{R}^1$, the optimal value of (2.19) is

$$\begin{aligned}&\sup_{\lambda\in\mathbb{R}^1}\inf_{\pi(\cdot)\text{admissible}} J(x_0, i_0, \pi(\cdot), \lambda) \\ &\quad = \max_{\zeta\in\mathbb{R}^1}\inf_{\pi(\cdot)\text{admissible}} (J(x_0, i_0, \pi(\cdot), \lambda) + \langle \zeta, \zeta^* \rangle) \\ &\quad > -\infty.\end{aligned} \tag{5.7}$$

By Theorem 4.3, $\inf_{\pi(\cdot)\text{admissible}} J(x_0, i_0, \pi(\cdot), \lambda)$ is a quadratic function (4.9) in $\lambda - \zeta$. It follows from the finiteness of the supremum value of this quadratic function that

$$P(0, i_0)H(0, i_0)^2 + \theta - 1 \leq 0. \tag{5.8}$$

Now if

$$P(0, i_0)H(0, i_0)^2 + \theta - 1 = 0, \tag{5.9}$$

then again by Theorem 4.3 and (5.7) we must have

$$P(0, i_0)H(0, i_0)x_0 - \zeta = 0, \tag{5.10}$$

for every $\zeta \in \mathbb{R}^1$, which is a contradiction. This proves (5.1). On the other hand, in view of (5.7), we maximize the quadratic function (4.9) over $\lambda - \zeta$ and conclude that the maximizer is given by (5.3) whereas the maximum value is given by the right-hand side of (5.4). Finally, the optimal control (5.2) is obtained by (4.8) with $\lambda = \lambda^*$.The efficient frontier (5.4) reveals explicitly the tradeoff between the mean (return) and variance (risk) at the terminal. Quite contrary to the case without Markovian jumps [17], the efficient frontier in the present case is no longer a perfect square (or, equivalently, the efficient frontier in the mean-standard deviation diagram is no more a straight line). As a consequence, one is not able to achieve a risk-free investment. This, certainly, is expected since now the interest rate process is modulated by the Markov chain, and the interest rate risk cannot be perfectly hedged by any portfolio consisting of the bank account and stocks [27], because the Markov chain is independent of the Brownian motion. Nevertheless, expression (5.4) does disclose the minimum variance, namely, the minimum possible terminal variance achievable by an admissible portfolio, along with the portfolio that attains this minimum variance. □

Theorem 5.2 (minimum variance). *The minimum terminal variance is*

$$\operatorname{Var} x^*_{\min}(T) = \frac{P(0,i_0)\theta}{P(0,i_0)H(0,i_0)^2+\theta}x_0^2 \geq 0 \tag{5.11}$$

with the corresponding expected terminal wealth

$$\zeta_{\min} := \frac{P(0,i_0)H(0,i_0)}{P(0,i_0)H(0,i_0)^2+\theta}x_0 \tag{5.12}$$

and the corresponding Lagrange multiplier $\lambda^*_{\min} = 0$. Moreover, the portfolio that achieves the above minimum variance, as a function of the time t, the wealth level x, and the market mode i, is

$$\pi^*_{\min}(t,x,i) = -\left[\sigma(t,i)\sigma(t,i)' + \int_{\mathbb{R}^n}\rho(t,i,z)\rho(t,i,z)'\nu(dz)\right]^{-1}B(t,i)'\left[x-\zeta_{\min}H(t,i)\right]. \tag{5.13}$$

Proof. The conclusions regarding (5.11) and (5.12) are evident in view of the efficient frontier (5.4). The assertion $\lambda^*_{\min} = 0$ can be verified via (5.3) and (5.12). Finally, (5.13) follows from (5.2). □

Remark 5.3. As a consequence of the above theorem, the parameter s can be restricted to $\zeta \geq \zeta_{\min}$ when one defines the efficient frontier for the mean-variance problem (2.19).

Theorem 5.4 (mutual fund theorem). *Suppose that an efficient portfolio $\pi_1^*(\cdot)$ is given by* (5.2) *corresponding to $\zeta = \zeta_1 > \zeta_{\min}$. Then a portfolio $\pi^*(\cdot)$ is efficient if and only if there is a $\mu \geq 0$ such that*

$$\pi^*(t) = (1-\mu)\pi^*_{\min}(t) + \mu\pi_1^*(t), \quad t \in [0,T], \tag{5.14}$$

where $\pi^*_{\min}(\cdot)$ is the minimum variance portfolio defined in Theorem 5.2.

Proof. We first prove the "if" part. Since both $\pi^*_{\min}(\cdot)$ and $\pi_1^*(\cdot)$ are efficient, by the explicit expression of any efficient portfolio given by (5.2), $\pi^*(t) = (1-\mu)\pi_0^*(\cdot)+\mu\pi_1^*(t)$ must be in the form of (5.2) corresponding to $\zeta = (1-\mu)\zeta_{\min} + \mu\zeta_1$ (also noting that $x^*(\cdot)$ is linear in $\pi^*(\cdot)$). Hence $\pi^*(t)$ must be efficient.

Conversely, suppose that $\pi^*(\cdot)$ is efficient corresponding to a certain $\zeta \geq \zeta_{\min}$. Write $\zeta = (1-\mu)\zeta_{\min} + \mu\zeta_1$ with some $\mu \geq 0$. Multiplying

$$\begin{aligned}&\pi^*_{\min}(t)\\ &= -\left[\sigma(t,\alpha(t))\sigma(t,\alpha(t))' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)\right]^{-1} B(t,\alpha(t))'\left[x^*_{\min}(t) - \zeta_{\min} H(t,\alpha(t))\right]\end{aligned} \tag{5.15}$$

by $(1-\mu)$, multiplying

$$\begin{aligned}&\pi^*_1(t)\\ &= -\left[\sigma(t,\alpha(t))\sigma(t,\alpha(t))' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)\right]^{-1} B(t,\alpha(t))'\left[x^*_1(t) + \left(\lambda^*_1 - \zeta_1\right) H(t,\alpha(t))\right]\end{aligned} \tag{5.16}$$

by μ, and summing them up, we obtain that $(1-\mu)\pi^*_{\min}(t) + \mu\pi^*_1(t)$ is represented by (5.2) with $x^*(t) = (1-\mu)x^*_{\min}(t) + \mu x^*_1(t)$ and $\zeta = (1-\mu)\zeta_{\min} + \mu\zeta_1$. This leads to (5.14). □

Remark 5.5. The above mutual fund theorem implies that any investor needs only to invest in the minimum variance portfolio and another prespecified efficient portfolio in order to achieve the efficiency. Note that in the case where all the market parameters are deterministic [17], the corresponding mutual fund theorem becomes the one-fund theorem, which yields that any efficient portfolio is a combination of the bank account and a given efficient risky portfolio (known as the tangent fund). This is equivalent to the fact that the fractions of wealth among the stocks are the same among all efficient portfolios. However, in the present Markov-modulated case this feature is no longer available.

Since the wealth processes $x(\cdot)$ are with jumps, it is more complicated when we solve the unconstrained problem (3.19). Firstly, we aim to derive conditions of feasibility. It is not hard to prove feasibility of the constrained stochastic optimization problem (2.19), which we get the unconstrained problem (3.19) from. Then we solve the unconstrained problem (3.19). If we assume that

$$\begin{aligned}\gamma(t,i) &:= B(t,i)\left[\sigma(t,i)\sigma(t,i)'\right]^{-1}B(t,i)', \quad i = 1,2,\ldots,l,\\ \pi^*(t,x,i) &:= -\left[\sigma(t,i)\sigma(t,i)'\right]^{-1}B(t,i)'[x + (\lambda-\zeta)H(t,i)],\end{aligned} \tag{5.17}$$

we have

$$\begin{aligned}\inf_{\pi(\cdot)\text{admissible}} J(x_0,i_0,\pi(\cdot),\lambda) = &\left[P(0,i_0)H(0,i_0)^2 + \theta - 1\right](\lambda-\zeta)^2\\ &+ 2[P(0,i_0)H(0,i_0)x_0 - \zeta](\lambda-\zeta) + P(0,i_0)x_0^2 - \zeta^2,\end{aligned} \tag{5.18}$$

where

$$\theta := \mathbb{E}\left\{\int_0^T \sum_{j=1}^{l} q_{\alpha(t)j} P(t,j)\left[H(t,j) - H(t,\alpha(t))\right]^2 dt \frac{1}{(\lambda-\zeta)^2} + P(t,\alpha(t))\pi(t)'\left\{\int_{\mathbb{R}^n} \rho(t,\alpha(t),z)\rho(t,\alpha(t),z)'\nu(dz)\right\}\pi(t)dt\right\}. \tag{5.19}$$

So, we added one item $\int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)$ in optimal feedback control $\pi^*(t,x,i)$ (see (3.19)) to simplify the calculation.

Acknowledgments

The author would like to thank Dr. C. Yuan for his helpful comments and discussions. He would like to thank the referees for the careful reading of the first version of this paper.

References

[1] B. ∅ksendal and A. Sulem, *Applied Stochastic Control of Jump Diffusions*, Universitext, Springer, Berlin, Germany, 2005.
[2] R. C. Merton, "Option pricing when underlying stock returns are discontinuous," *Journal of Financial Economics*, vol. 3, no. 1-2, pp. 125–144, 1976.
[3] I. Bardhan and X. Chao, "Pricing options on securities with discontinuous returns," *Stochastic Processes and Their Applications*, vol. 48, no. 1, pp. 123–137, 1993.
[4] T. Chan, "Pricing contingent claims on stocks driven by Lévy processes," *The Annals of Applied Probability*, vol. 9, no. 2, pp. 504–528, 1999.
[5] H. Föllmer and M. Schweizer, "Hedging of contingent claims under incomplete information," in *Applied Stochastic Analysis (London, 1989)*, M. H. A. Davis and R. J. Elliott, Eds., vol. 5 of *Stochastics Monogr.*, pp. 389–414, Gordon and Breach, New York, NY, USA, 1991.
[6] N. El Karoui and M.-C. Quenez, "Dynamic programming and pricing of contingent claims in an incomplete market," *SIAM Journal on Control and Optimization*, vol. 33, no. 1, pp. 29–66, 1995.
[7] V. Henderson and D. Hobson, "Coupling and option price comparisons in a jump-diffusion model," *Stochastics and Stochastics Reports*, vol. 75, no. 3, pp. 79–101, 2003.
[8] F. Merculio and W. J. Runggaldier, "Option pricing for jump-diffusion: approximation and their interpretation," *Mathematical Finance*, vol. 3, pp. 191–200, 1993.
[9] A. Jobert and L. C. G. Rogers, "Option pricing with Markov-modulated dynamics," *SIAM Journal on Control and Optimization*, vol. 44, no. 6, pp. 2063–2078, 2006.
[10] R. J. Elliott and J. Van der Hoek, "An application of hidden Markov models to asset allocation problems," *Finance and Stochastics*, vol. 1, pp. 229–238, 1997.
[11] R. J. Elliott, W. P. Malcolm, and A. H. Tsoi, "Robust parameter estimation for asset price models with Markov modulated volatilities," *Journal of Economic Dynamics & Control*, vol. 27, no. 8, pp. 1391–1409, 2003.
[12] X. Guo, *Inside information and stock uctuations*, Ph.D. thesis, Rutgers University, 1999.
[13] X. Guo, "An explicit solution to an optimal stopping problem with regime switching," *Journal of Applied Probability*, vol. 38, no. 2, pp. 464–481, 2001.
[14] X. Guo, "Information and option pricings," *Quantitative Finance*, vol. 1, no. 1, pp. 38–44, 2001.
[15] G. Yin and X. Y. Zhou, "Markowitz's mean-variance portfolio selection with regime switching: from discrete-time models to their continuous-time limits," *IEEE Transactions on Automatic Control*, vol. 49, no. 3, pp. 349–360, 2004.
[16] X. Y. Zhou and G. Yin, "Markowitz's mean-variance portfolio selection with regime switching: a continuous-time model," *SIAM Journal on Control and Optimization*, vol. 42, no. 4, pp. 1466–1482, 2003.

[17] X. Y. Zhou and D. Li, "Continuous-time mean-variance portfolio selection: a stochastic LQ framework," *Applied Mathematics and Optimization*, vol. 42, no. 1, pp. 19–33, 2000.

[18] P. A. Samuelson, "Lifetime portfolio selection by dynamic stochastic programming," *Review of Economics and Statistics*, vol. 51, pp. 293–246, 1969.

[19] N. H. Hakansson, "Multi-period mean-variance analysis: toward a general theory of portfolio choice," *Journal of Finance*, vol. 26, pp. 857–884, 1971.

[20] S. R. Pliska, *Introduction to Mathematical Finance*, Basil Blackwell, Malden, Mass, USA, 1997.

[21] D. Duffie and H. R. Richardson, "Mean-variance hedging in continuous time," *The Annals of Applied Probability*, vol. 1, no. 1, pp. 1–15, 1991.

[22] M. Schweizer, "Approximation pricing and the variance-optimal martingale measure," *The Annals of Probability*, vol. 24, no. 1, pp. 206–236, 1996.

[23] H. J. Kushner, *Weak Convergence Methods and Singularly Perturbed Stochastic Control and Filtering Problems*, vol. 3 of *Systems & Control: Foundations & Applications*, Birkhäuser, Boston, Mass, USA, 1990.

[24] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College, London, UK, 2006.

[25] T. Björk, "Finite-dimensional optimal filters for a class of Itô-processes with jumping parameters," *Stochastics*, vol. 4, no. 2, pp. 167–183, 1980.

[26] D. G. Luenberger, *Optimization by Vector Space Methods*, John Wiley & Sons, New York, NY, USA, 1969.

[27] A. E. B. Lim and X. Y. Zhou, "Mean-variance portfolio selection with random parameters in a complete market," *Mathematics of Operations Research*, vol. 27, no. 1, pp. 101–120, 2002.