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*Research Article*

# A Kinetic Approach to Relativistic Shocks in Astrophysics

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Stochastic acceleration of charged particles across highly relativistic shock is often considered as the main source for observed emission. Here is shown that the derivation of the appropriate quasilinear equation describing particle transport across such shocks depends on the assumptions made for the power spectra in the upstream region ahead of the shock. For both an ambient magnetic field perpendicular to the shock front and for an oblique magnetic field derivation is given of the quasilinear diffusion equation for particle transport appropriate to both sides of the shock. There is both pitch angle diffusion and energy diffusion; the relative strengths of the two processes depends on the assumptions made concerning the upstream wave power spectra. Transformations of the diffusion equations into the frame where the shock is stationary are given for the upstream and downstream regions including both energy diffusion and pitch angle scattering. The remaining outstanding concern is the determination of the continuity of the transport equations across the shock. This latter problem has yet to be solved fully in even the simple case of assumed pitch angle scattering only. Including energy diffusion and pitch angle scattering presumably makes the determination of the correct continuity behaviour more difficulty.

## 1. Introduction

Highly collimated, highly relativistic, bulk flows from the cores of Active Galactic Nuclei (AGN) are about the only method known that will allow one to account for the high $\gamma$-ray brightness observed in conjunction with the extremely rapid temporal fluctuations in brightness. Such bulk flows (with Lorentz $\Gamma$ values in the several hundred regime) imply massive acceleration processes in the AGNs in order to produce the beams. It is to be doubted that such a production would not also energize individual particles so that they, too, would have thermal energies with Lorentz factors in the several hundred region for protons and even higher for the much less massive electrons.

While the mechanisms for production of such beams remain to be worked out in detail, such beams exit into the interstellar, or intergalactic, medium of each AGN, and so produce relativistic shocks. On the upstream side of such a shock one has the cool interstellar medium, while on the downstream side one has the shocked, highly relativistic plasma. Viewed from a frame of reference in which the shock is stationary, the upstream particles approach the shock at a high Lorentz factor. The question of some considerable astrophysical interest is to describe the kinetic behaviour of waves and particles taken together under such a scenario with as few assumptions as possible so that one has a generic procedure for investigating consequences of such shocks. This question is of considerably more than just pure academic interest. The review by Kirk and Duffy [1] emphasizes the fact that, to date, the description of such relativistic shocks has been handled as an MHD problem with, therefore, a prescribed equation of state and with test particle diffusion only. Indeed, the corresponding diffusion coefficient is kinematically given and, so far, appears to involve only pitch angle scattering in a momentum independent manner.

Now we know from other astrophysical objects, such as supernovae remnants, that upstream and downstream wave generation plays a fundamental role in modifying the behaviour of particles that cross-shocks so that a full kinetic description is needed. There is no reason to suppose that similar modifications to the MHD picture for relativistic shocks associated with AGNs will not also produce major alterations to the treatments available to date. And, even if there is only a slight modification produced to the MHD picture by a more correct kinetic picture, then the conditions under which one can be sure the MHD picture is appropriate are then more sharply defined. In addition, use of an ad hoc diffusion coefficient, not tied to the wave spectra, makes it difficult to investigate the relevant particle acceleration or deceleration. And the justification for using a diffusion coefficient in the first place is based upon the quasilinear theory for homogeneous plasmas. The presence of the shock modifies the patterns of behaviour from those obtaining in a rigorously homogeneous medium so that one must describe the general problem de novo.

Accordingly, there are excellent astrophysical and physical reasons for considering the basic problem anew from a kinetic particle Vlasov type of approach. It is this development that is considered here.

There are two basic conditions that can exist for steady-state shock propagation with respect to an upstream uniform ambient magnetic field. One can find a frame of reference for which the shock is stationary and for which one has an upstream magnetic field at the de Hoffman-Teller angle with respect to the shock front. Alternatively, one can find a frame of reference in which the shock is stationary and the ambient magnetic is perpendicular to the shock front. As noted by Kirk and Duffy [1], the de Hoffman-Teller reference frame is likely not to be of much direct application to the problem of highly relativistic beams because the allowable angles for the upstream magnetic field, as viewed from the rest frame of the shock, are typically $O(1/\Gamma)$ with respect to the shock front plane. Thus, the probability is that the remaining part of the angular regime ahead of the shock $[O(\pi - 1/\Gamma)]$ is the most appropriate situation, for which the ambient magnetic field can always be arranged to be perpendicular to the shock front. It is precisely that situation that is discussed directly; the case where the ambient magnetic field is not so describable will form the second major part of this paper.

## 2. Part I—Perpendicular Ambient Magnetic Field

There are several background information categories that are extremely useful in handling the problem of particle transport in a relativistic shock regime. This section of the paper sets up the nomenclature so that the development of the corresponding transport equations (to be handled in the next section) is facilitated. The categories are: Definitions in frames of references; Relevant Lorentz transformations and symbolism within the various frames of reference, Fields and Distribution Function structures in the different reference frames; and General operator structure of the ensemble-averaged transport equations. Consider each in turn.

*2.1. Frames of Reference.* Three different frames of reference are extremely useful in handling the basic problem of particle behaviour coupled to field fluctuations. First is the frame of reference in which the shock is stationary (this frame of reference is denoted $S$). Second is the frame of reference in which the upstream particles have zero bulk velocity (this frame of reference is denoted $U$). Third is the frame of reference in which the downstream particles have no bulk velocity (this frame of reference is denoted $D$). Figure 1 shows a sketch of the situation in the shock frame of reference $S$.

For particles of charge $e$, rest mass $m$, under the influence of fluctuating electric ($\mathbf{E}$) and magnetic fields ($\mathbf{B}$), and also under the influence of the uniform magnetic field $\mathbf{B}_0$ that points in the $z$ direction, one can write the particle distribution function, $f$, for each species of particles as the solution to the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + e\left[\mathbf{E} + \frac{\mathbf{vxB}}{c} + \frac{\mathbf{vxB}_0}{c}\right] \cdot \frac{\partial f}{\partial \mathbf{p}} = 0, \quad (1)$$

where $\mathbf{v}$ is the particle velocity, $\mathbf{p}$ the particle momentum with $m\mathbf{v} = \mathbf{p}/(1 + \mathbf{p} \cdot \mathbf{p}/(mc)^2)^{1/2}$. Equation (1) holds in any of the frames $S$, $D$, and $U$, but care has to be exercised in connecting definitions of quantities on one reference frame to another, as will be shown a little later.

In addition to (1), one has the corresponding Maxwell equations:

$$\frac{\partial E_i}{\partial x_i} = \Sigma e_\alpha \int f_\alpha d^3\mathbf{p}, \quad (2a)$$

$$\varepsilon_{ijk} \frac{\partial E_j}{\partial x_k} = -\left(\frac{1}{c}\right) \frac{\partial B_i}{\partial t}, \quad (2b)$$

$$\frac{\partial B_i}{\partial x_i} = 0, \quad (2c)$$

$$\varepsilon_{ijk} \frac{\partial B_j}{\partial x_k} = \left(\frac{1}{c}\right) \frac{\partial E_i}{\partial t} + 4\pi \Sigma e_\alpha \int \mathbf{v} f_\alpha d^3 \frac{\mathbf{p}}{c}, \quad (2d)$$

where the Einstein double summation convention over repeated Latin indices is used and where the summations are taken over all particle species. Equations (2a)–(2d) hold in any of the frames of reference and, in addition, $\int f_\alpha d^3\mathbf{p} = n_\alpha$ represents the number density of particles of type $\alpha$ measured in a given frame of reference.

Interest centers on determining the downstream distribution functions together with the downstream electric and magnetic fields, and the downstream bulk velocity, $\mathrm{U}_D$, given the upstream distribution functions far from the shock, given the upstream electric and magnetic fields, and also given the upstream flow velocity $\mathrm{U}_U$. Both $\mathrm{U}_D$ and $\mathrm{U}_U$ are in the $z$ direction and, based on the sketch of Figure 1, can be written as $\mathrm{U}_D = -\mathbf{1}_z \mathrm{U}_D$ and $\mathrm{U}_U = -\mathbf{1}_z \mathrm{U}_U$ where $\mathrm{U}_D$ and $\mathrm{U}_U$ are positive constants as measured in the frame $S$.

*2.2. Lorentz Transformations.* Denote by $p_{U,z}$ the $z$ component of momentum of a particle measured in the upstream frame $U$. Then in the frame $S$ of the shock one measures $p_{S,z}$ with the connection

$$p_{U,z} = \left(p_{S,z} + \frac{E_S U_U}{c^2}\right)\Gamma_U, \quad (3)$$

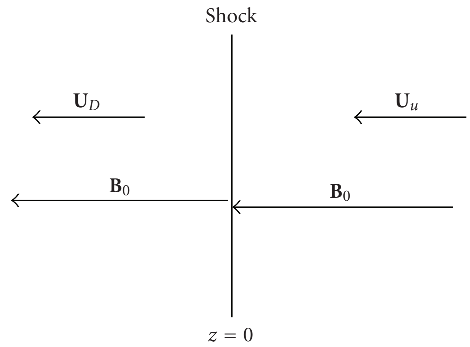


FIGURE 1: Sketch of the geometry of a plane shock with a perpendicular ambient magnetic field.

where $\Gamma_U = (1 - {U_U}^2/c^2)^{-1/2}$, $E_S^2 = (mc^2)^2[1 + (p_{S,z}^2 + \mathbf{p}_{S,\perp} \cdot \mathbf{p}_{S,\perp})/(mc)^2]$ for particles of rest mass $m$. One also has $\mathbf{p}_{S,\perp} = \mathbf{p}_{U,\perp}$ for the momentum components perpendicular to the shock. Equally, in the downstream region, one has the connection between quantities measured in the frames $D$ and $S$ as

$$p_{D,z} = \left(p_{S,z} + \frac{E_S U_D}{c^2}\right)\Gamma_D, \tag{4}$$

where $\Gamma_D = (1 - U_D^2/c^2)^{-1/2}$ , $E_S^2 = (mc^2)^2[1 + (p_{S,z}^2 + \mathbf{p}_{S,\perp} \cdot \mathbf{p}_{S,\perp})/(mc)^2]$.

For angular frequency $\omega$ and vector wavenumber $\mathbf{k}$ one has the transformation pair

$$\omega_S = (\omega_U - k_{U,z}U_U)\Gamma_U, \tag{5a}$$

$$k_{S,z} = \left(k_{U,z} - \frac{U_U \omega_U}{c^2}\right)\Gamma_U, \tag{5b}$$

together with $\mathbf{k}_{S,\perp} = \mathbf{k}_{U,\perp}$ for the wavenumber components perpendicular to the shock. Thus, for illustration, consider an Alfven wave with speed $V_A$ as measured in the frame $U$, so that $\omega_U = \pm V_A k_{U,z}$, then the corresponding relation between $\omega_S$ and $k_{S,z}$ measured in the frame $S$ is

$$\frac{\omega_S}{k_{S,z}} = \frac{\left[-U_U(1 - V_A^2/c^2) \pm V_A/\Gamma_U^2\right]}{1 - (V_A U_U/c^2)^2}. \tag{6}$$

Corresponding relations hold also in the downstream frame $D$.

For the electric and magnetic fields, the connections are that

$$E_{S,z} = E_{U,z}, \qquad B_{S,z} = B_{U,z}, \tag{7a}$$

$$\mathbf{E}_{S,\perp} = \Gamma_U\left(\mathbf{E}_{U,\perp} + \mathbf{U}_U \mathbf{x} \frac{\mathbf{B}_U}{c}\right), \tag{7b}$$

$$\mathbf{B}_{S,\perp} = \Gamma_U\left(\mathbf{B}_{U,\perp} - \mathbf{U}_U \mathbf{x} \frac{\mathbf{E}_U}{c}\right), \tag{7c}$$

which relates fields $\mathbf{E}_S$ and $\mathbf{B}_S$ measured in the frame $S$ to the fields $\mathbf{E}_U$ and $\mathbf{B}_U$ measured in the frame $U$. Again, corresponding relations hold also in the downstream frame $D$.

Note further that a wave phase $\Phi = \mathbf{k} \cdot \mathbf{x} - \omega t$ is a Lorentz invariant so that if one were to take the Fourier transform of (7a)–(7c) then

$$\begin{aligned} E_{S,z}(\mathbf{k}_S, \omega_S) &= E_{U,z}(\mathbf{k}_U, \omega_U), \\ B_{S,z}(\mathbf{k}_S, \omega_S) &= B_{U,z}(\mathbf{k}_S, \omega_S), \end{aligned} \tag{8a}$$

$$\mathbf{E}_{S,\perp}(\mathbf{k}_S, \omega_S) = \Gamma_U\left(\mathbf{E}_{U,\perp}(\mathbf{k}_U, \omega_U) + \mathbf{U}_U \mathbf{x} \frac{\mathbf{B}_U(\mathbf{k}_U, \omega_U)}{c}\right), \tag{8b}$$

$$\mathbf{B}_{S,\perp}(\mathbf{k}_S, \omega_S) = \Gamma_U\left(\mathbf{B}_{U,\perp}(\mathbf{k}_U, \omega_U) - \mathbf{U}_U \mathbf{x} \frac{\mathbf{E}_U(\mathbf{k}_U, \omega_U)}{c}\right). \tag{8c}$$

Hence if the upstream components of the electric and magnetic fields are known in the frame $U$ as functions of $\mathbf{k}_U$ and $\omega_U$, then one can immediately write down the corresponding components as measured in the frame $S$. The corresponding relations hold between the frames $S$ and $D$. In addition one has the invariant $d^3\mathbf{k}_U d\omega_U = d^3\mathbf{k}_S d\omega_S$.

*2.3. Distribution Function Considerations.* Because the plane $z = 0$ is taken to be the steady-state shock front as measured in the frame $S$, some care has to be taken in setting up and solving the Vlasov equation to obtain the downstream behaviour of particles and waves in relation to the prescribed (on $z = \infty$) input of particles and waves.

The general argument operates as follows. First, on $z = \infty$ one specifies the upstream distribution functions and all electric and magnetic fields including the wave spectrum. As the upstream particles are convected to the shock front (at a velocity $-\mathbf{1}_z U_U$) measured in the frame $S$, both the known far upstream particle distribution functions plus wave spectrum are altered near the shock by the multiple shock crossings the particles can make and also by the resulting downstream particles and fields.

Second, the downstream particles at $z = -\infty$ are taken to have a steady-state component to their distribution functions plus a fluctuating component, both as measured in the frame $S$. The "jump" conditions across the shock at $z = 0$ should then provide the necessary conditions to connect upstream and downstream behaviors, and should also relate the upstream modulation of the particles and waves to the shock reflection and transmission properties.

In order to obtain equations allowing the connections to be made, it is conventional to make a weak turbulence argument (to be described in detail below). However, note that one cannot assume that the background medium is homogeneous everywhere (usually one of the linchpins that permits derivation of a convection-diffusion model from the weak turbulence assumption) because of the presence of the shock. It is, therefore, appropriate to consider the relevant quasilinear equations for the shock-influenced distribution functions in the upstream, downstream, and shock frames, including all assumptions and approximations made.

*2.3.1. Formal Manipulations.* The formal path to obtaining a quasilinear equation for the time-independent (in the shock frame $S$) component of the distribution function operates as follows. In the Vlasov equation (1) set $f_S = F_S + \delta f_S$ where $F_S$ is the time independent component and $\delta f_S$ is the time dependent component of the distribution function. Then, quite generally, one has

$$\mathbf{v}\cdot\frac{\partial F_S}{\partial \mathbf{x}} + \left[\frac{\mathbf{vxB}_0}{c}\right]\cdot\frac{\partial F_S}{\partial \mathbf{p}} = -\left\langle e\left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial \delta f_S}{\partial \mathbf{p}}\right\rangle \quad (9)$$

together with

$$\frac{\partial \delta f_S}{\partial t} + \mathbf{v}\cdot\frac{\partial \delta f_S}{\partial \mathbf{x}} + e\left[\frac{\mathbf{vxB}_0}{c}\right]\cdot\frac{\partial \delta f_S}{\partial \mathbf{p}} = -e\left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial F_S}{\partial \mathbf{p}} - e\left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial \delta f_S}{\partial \mathbf{p}} + e\left\langle\left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial \delta f_S}{\partial \mathbf{p}}\right\rangle, \quad (10)$$

where angular brackets denote a conventional time-averaging.

The canonical assumption made in deriving the quasilinear behaviour of $F_S$ is that only the first terms on the right hand side of (10) need to be retained so that

$$\mathcal{L}\delta f_S = -e\left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial F_S}{\partial \mathbf{p}}, \quad (11)$$

where

$$\mathcal{L}\delta f_S = \frac{\partial \delta f_S}{\partial t} + \mathbf{v}\cdot\frac{\partial \delta f_S}{\partial \mathbf{x}} + e\left[\frac{\mathbf{vxB}_0}{c}\right]\cdot\frac{\partial \delta f_S}{\partial \mathbf{p}}. \quad (12)$$

Equation (11) has the solution

$$\delta f_S = \mathcal{L}^{-1}\left(-e\left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial F_S}{\partial \mathbf{p}}\right) \quad (13)$$

with $\mathcal{L}^{-1}$ the inverse operator to $\mathcal{L}$, and is worked out in detail on Appendix A. Then (9) can be written

$$\mathbf{v}\cdot\frac{\partial F_S}{\partial \mathbf{x}} + \left[\frac{\mathbf{vxB}_0}{c}\right]\cdot\frac{\partial F_S}{\partial \mathbf{p}} = e^2 < \left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\left[\mathcal{L}^{-1}\left(\left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\right)\right] > F_S \quad (14)$$

which relates the spatial and momentum evolution of the steady-state component of the distribution function to quadratic products of fluctuations in the electric and magnetic fields. Because there is an unique relationship between such fields for each wave type, the evolution of $F_S$ is thus tied to the power spectrum of waves in both the upstream and downstream regions. Equation (14) is the general quasilinear approximation to the steady-state distribution function.

Now in the case of a planar shock the spatial dependence of $F_S$ can be a function of $z$ only and not of spatial coordinates parallel to the shock front so that (14) takes the simplified form

$$\frac{v_z\partial F_S}{\partial z} + \frac{\Omega\partial F_S}{\partial \varphi} = e^2 < \left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\left[\mathcal{L}^{-1}\left(\left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\right)\right] > F_S, \quad (15)$$

where $\Omega = eB_0/(mc\gamma)$ with $\gamma = (1+(p_\perp^2+p_z^2)/(mc)^2)^{1/2}$, and where $p_x = p_\perp \sin\varphi$, $p_y = p_\perp\cos\varphi$, with $\varphi$ the phase angle.

One further approximation to the general quasilinear equation is to be concerned only with the component of the quasilinear equation that is independent of the *phase* angle the particles make around the mean magnetic field $B_0$, so that one singles out just that component from the general equation (15), which depends on the particle momentum, the pitch angle of the particles, and on the spatial coordinates.

Then one writes

$$F_S = \sum_n F_n \exp(\mathrm{i}n\varphi), \quad (16)$$

where the sum over $n$ is $-\infty < n < \infty$, and one is interested in the component with $n = 0$. The lowest order equation for $F_0$ is then given through

$$\frac{v_z\partial F_0}{\partial z} = (2\pi)^{-1}\left[\int_0^{2\pi} d\varphi e^2\left\langle\left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\left[\mathcal{L}^{-1}\left(\left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\right)\right]F_0(z,p_\perp,p_z)\right\rangle\right] \quad (17)$$

so that the evolution of $F_0$ is clearly seen to depend on the averaging of the electromagnetic spectrum of waves. In (17) we have spelled out the dependence of $F_0$ in the right hand side factor so that one can see immediately that the averaging over the phase angle and over time are with respect to the waves alone.

The general structure of (17) is of the form

$$\frac{v_z\partial F_0}{\partial z} = \frac{\partial}{\partial p_z}\left(\frac{A\partial F_0}{\partial p_z}\right) + \frac{\partial}{\partial p_z}\left(\frac{B\partial F_0}{\partial p_\perp}\right) + \frac{\partial}{\partial p_\perp}\left(\frac{C\partial F_0}{\partial p_z}\right) + \frac{\partial}{\partial p_\perp}\left(\frac{D\partial F_0}{\partial p_\perp}\right), \quad (18)$$

where all factors are measured in the frame $S$. The coefficients $A$, $B$, $C$, and $D$ are evaluated in Appendix B and are dependent on the wave spectra assumed and also on $p_\perp$ and $p_z$.

*2.3.2. Conditions for Different Frames of Reference*

*(1) The Fields.* Because the coefficients $A$, $B$, $C$ and $D$ in (18) depend on the electric and magnetic fields fluctuations as measured in frame $S$ (see Appendix B), as well as on $p_z$ and $p_\perp$, also measured in the frame $S$, and because the electric and magnetic field fluctuations are considered known far upstream from the shock and far downstream from the shock in the frames $U$ and $D$ (where there is no bulk motion of the plasma in the upstream or downstream regions, resp.) one needs to write the components $\mathbf{E}$ and $\mathbf{B}$, as measured in frame $S$, in terms of the field components measured in the frames $U$ and/or $D$. Such is accomplished with the standard transformations given in (7a) through (7c). Accordingly, in the frame $S$ the corresponding components of the power spectra tensor for the electromagnetic fields can be written down in terms of the equivalent values measured in either the upstream or downstream frames $U$ and $D$, respectively. And, again in turn, such means one can write the four coefficients $A$, $B$, $C$, and $D$ in terms of the upstream and downstream wave spectra measured in the rest frames $U$ and $D$, and in terms of $p_z$ and $p_\perp$ together with the bulk transport speeds $U_U$ and $U_D$.

*(2) The Distribution Function.* (i) *General Remarks.* It is often assumed that in either or both of the frames $U$ and $D$ (i.e., far upstream or far downstream from the shock) the distribution function is effectively isotropic with changes from isotropy being brought about in the vicinity of the shock by the shock passage through the plasma medium. More generally, in either of the frames $U$ or $D$ the distribution function must be a symmetric function of the $z$ component of momentum measured in the frame because in neither of the frames $U$ or $D$ does the plasma have a bulk speed.

In the frame $S$ the distribution function cannot be symmetric in the $z$ component of momentum measured in the frame $S$ because particles far upstream must approach the shock at a speed $U_U$ and, far downstream of the shock particles must recede from the shock at a speed $U_D$. The distribution function, $F_0(z, p_\perp, p_z)$, measured in terms of quantities in the shock frame $S$, must have an anisotropic component. In the frames $U$ or $D$ the plasma has no bulk speed and so can be a function only of $z, p_\perp$, and $p_z^2$ as measured in the frames $U$ or $D$. Thus this sort of condition requires that one write the distribution function as measured in the shock frame in terms of the momentum components measured in the frames $U$ and/or $D$, respectively. This aspect is now considered.

(ii) *Frame of Reference Considerations for the Distribution Function Dependence.* Denote by subscript $U$ or $D$ the momentum components measured in frames $U$ or $D$ respectively so that, for example, $p_{U,z}$ refers to the $z$ component of momentum measured in the upstream frame $U$. Then $F_0(z, p_\perp, p_z)$ must be a function of the form

$$F = F_0\left(z, p_\perp, \Gamma^2 (p_{U,z} - mU\gamma_S)^2\right), \tag{19}$$

where $\gamma_0 = (1 + (p_{U,z}^2 + p_\perp^2)/(mc)^2)^{1/2}$; $p_{S,z} = \Gamma(p_{U,z} - mU\gamma_S)$; $\gamma_S = \Gamma(\gamma_0 - Up_{U,z}/c)$; and also $\gamma_S = (1 + (p_{S,z}^2 + p_\perp^2)/(mc)^2)^{1/2}$. The flux of particles across the shock as measured in the shock frame $S$ is then given by

$$\text{Flux} = \int v_z F_0 d^2 p_\perp dp_z \tag{20a}$$

which, using (19), can easily be shown to be also give by

$$\text{Flux} = -U \int F_0\left(z, p_\perp, \Gamma^2 p_{U,z}^2\right) d_{p\perp}^2 dp_{U,z}. \tag{20b}$$

with the usual identification of the number density, $n_S$, measured in the shock frame $S$ as

$$n_S = \int F_0\left(z, p_\perp, \Gamma^2 p_{U,z}^2\right) d^2 p_\perp dp_{U,z}. \tag{20c}$$

one then has the constant flux of particles as Flux $= -Un_S$ as required by conservation.

Thus in both the upstream and downstream frames $F_0$ takes on different functional behaviours due to $U_D$ and $U_U$ being different.

The corresponding diffusion (18) can then be written either as is, but then the diffusion coefficients that are given in the frames $U$ and $D$ must be transformed to coordinates appropriate to the shock frame and one must also use the fact that $F_0$ is represented solely as expressed in (19), or can transform the diffusion equation to "mixed" momentum coordinates $p_\perp$, $p_{U,z}$ and then one has the diffusion coefficients given in the frames $U$ or $D$ together with the representation of $F_0$ as a function of $p_\perp$, $p_{U,z}^2$. Equally, in the downstream frame $D$ one can write a similar expression with the replacement of subscripts $U$ for subscripts $D$. While such a mixed momentum coordinate system simplifies enormously a discussion of the basic diffusion equation, the transformation to such a mixed momentum coordinate system also complicates the conditions the distribution function must obey at the shock $z = 0$.

Written in terms of the mixed momentum coordinates $p_\perp$, $p_{U,z}$ and with $w_z = p_{U,z}/(m\gamma_0)$, $w_\perp = p_{U,\perp}/(m\gamma_0)$, the diffusion equation (18) takes on the form

$$\begin{aligned}
&\frac{\Gamma(w_z - U)\partial F_0}{\partial_z} \\
&= \frac{\partial}{\partial p_{U,z}} \left\{ A_S \Gamma^{-2} \left(1 - \frac{Uw_z}{c^2}\right)^{-1} \frac{\partial F_0}{\partial p_{U,z}} \right\} \\
&\quad + \frac{\partial}{\partial p_{U,z}} \left\{ \frac{B_S \Gamma^{-1} \partial F_0}{\partial p_\perp} + w_\perp U c^{-2} \left(1 - \frac{Uw_z}{c^2}\right)^{-1} \frac{B_S\, \Gamma^{-1} \partial F_0}{\partial p_{U,z}} \right\} \\
&\quad + \left\{ \left(1 - \frac{Uw_z}{c^2}\right) \frac{\partial}{\partial p_\perp} + \frac{w_\perp U c^{-2} \partial}{\partial p_{U,z}} \right\} \\
&\quad \times \left\{ \left(C_S \Gamma^{-1} + D_S w_\perp U c^{-2}\right) \left(1 - \frac{Uw_z}{c^2}\right)^{-1} \frac{\partial F_0}{\partial p_{U,z}} + \frac{D_S \partial F_0}{\partial p_\perp} \right\},
\end{aligned} \tag{21}$$

where the four diffusion coefficients, $A, B, C, D$, must also be prescribed in terms of the mixed momentum coordinates. And the distribution function must be a function solely in the form $F_0(z, p_\perp, (\Gamma p_{U,z})^2)$ in the upstream frame $U$ in order that the steady state particle flux across the shock, as measured in the frame $S$, be $-Un_S$. With the replacement of upstream subscript $U$ with downstream subscript $D$ the diffusion equation (21) is then of generic validity.

## 3. Conditions across the Shock at $z = 0$

At the stage where one has written down the diffusion (21) in terms of the mixed momentum coordinates, the precise structure of the four diffusion coefficients, $A, B, C, D$, becomes important in respect of their dependences on $p_\perp$ and $p_{U,z}$ (or $p_{D,z}$); the dependences are also influenced by the connections one assumes for the waves doing the scattering for these waves determine the interaction of electric and magnetic fields with the particles.

Accordingly, the general solution of (21), and the matching of conditions across the shock depend on precisely what one assumes for the fluctuating waves in both the upstream and downstream regions. In a general sense several factors stand out. First note that (21) involves diffusion in both parallel and perpendicular momentum and also note that that the respective diffusion coefficients are, in general, different. Thus one cannot have only pitch angle diffusion without also having energy diffusion. Second note that the respective diffusion coefficients are dependent on $p_\perp$ and $p_{U,z}$ (or $p_{D,z}$) and on the bulk velocity on each side of the shock. Thus the four diffusion coefficients are prescribed once the wave spectra entering the definitions of $A$ through $D$ are given. One cannot, arbitrarily, assume particular functional forms for $A$ through $D$ in order to achieve some preconceived and desired result because any such choice must be shown to be consistent with the power spectral information that is allowed for individual wave types. Third, note the presence of two sorts of mixed terms in (21) involving cross derivatives with respect to $p_\perp$ and $p_{U,z}$ (or $p_{D,z}$), with a difference in coefficients that occurs pre and post the partial derivative operators, so that significant differences ensue, particularly in the highly relativistic regime where $(1 - Uw_z/c^2) \approx 0$.

## 4. Part II—Oblique Ambient Magnetic Field

As noted in Part I Kirk and Duffy [1] have stated that the de Hoffman-Teller reference frame is likely not to be of much direct application to the problem of highly relativistic beams because the allowable angles for the upstream magnetic field, as viewed from the rest frame of the shock, are typically $O(1/\Gamma)$ with respect to the shock front plane. Thus, the probability is that the remaining part of the angular regime ahead of the shock $[O(\pi - 1/\Gamma)]$ is the most appropriate situation, for which the ambient magnetic field can always be arranged to be perpendicular to the shock front. Precisely that situation was the focus of Part I; the case where the ambient magnetic field is not so describable will form the subject of this section.

Consider then that the shock makes an angle $\theta$ with respect to a background magnetic field, of magnitude $B_U$ as measured in the upstream rest frame, as sketched in Figure 2. Then resolve the background magnetic field into components perpendicular and parallel to the shock front with $B_{U,z} = B_U \cos\theta$, and a parallel component (taken without loss of generality to point in the $x$-direction) with $B_{U,\perp} = B_U \sin\theta$. In the shock frame $S$ one then has the measured field components $B_{S,\perp} = \Gamma_U B_U \sin\theta$, and $B_{S,z} = B_U \cos\theta$. Define an angle $\varsigma$ by $\tan\varsigma = B_{S,\perp}/B_{S,z} = \Gamma_U \tan\theta$ so that for highly relativistic shocks ($\Gamma_U \gg 1$) the angle $\varsigma$ will be almost $\pi/2$ unless $\theta \ll 1/\Gamma_U$, which is the usual division between "parallel" and "perpendicular" regimes. Note also that in the shock frame $S$ there is now a background electric field given through $\mathbf{E}_{S,\perp} = \Gamma_U \mathbf{U}_U \mathbf{x} \mathbf{B}_U/c$, provided there is no background electric field in the frame $U$.

This difference between the situation of a background magnetic field exactly perpendicular to the shock and one making an angle $\theta$ (as measured in the frame $U$) means that, should one wish to obtain a diffusion equation for particle behaviour in the presence of fluctuating electromagnetic fields, then it is simplest to operate in the upstream frame $U$ where only a background *magnetic* field is present and not a background electric field, and then to transform the final results to the shock frame $S$. However, the price that must be paid for such a simplification of the equations is the complexity of the boundary conditions that, in the upstream (downstream) frame $U(D)$, are now time-dependent. The reason is that, as seen in the frame $U$, the shock moves through the upstream plasma at speed $U_U$ in the $z$-direction. The strategic question is, then, whether it is simpler to work in the frame $U$ with complex boundary conditions that are also time-dependent, or whether it is simpler to work in the frame $S$ with more complex equations involving background magnetic and electric fields as measured in $S$ but with static boundary conditions. We have found it simpler to work with static boundary conditions and so for the remainder of this section of the paper the development of the quasilinear equation will be considered in the frame $S$.

With the background electric and magnetic fields in the frame $S$, in the Vlasov equation for particles one has the factor $\mathbf{E}_S + \mathbf{v}\mathbf{x}\mathbf{B}_S/c$. Noting that $\mathbf{E}_{S,\perp} = \Gamma_U \mathbf{U}_U \mathbf{x} \mathbf{B}_U/c$, that $B_{S,\perp} = \Gamma_U B_U \sin\theta$, and $B_{S,z} = B_U \cos\theta$ and that $\mathbf{U}_u = -U_u \mathbf{1_z}$ where $\mathbf{1_z}$ is a unit vector in the $z$ direction, one can write

$$\mathbf{E}_S + \frac{\mathbf{v}\mathbf{x}\mathbf{B}_S}{c} = \frac{\mathbf{1_y}\Gamma_U B_U \sin\theta (v_z + U_U)}{c} - \frac{\mathbf{1_z} v_y \Gamma_U B_U \sin\theta}{c} + \frac{B_U \cos\theta \left(v_y \mathbf{1_x} - v_x \mathbf{1_y}\right)}{\mathbf{c}}. \tag{22}$$

If the upstream components of the fluctuating electric and magnetic fields are known in the frame $U$ as functions of $\mathbf{k}_U$ and $\omega_U$, then one can immediately write down the corresponding components as measured in the frame $S$.

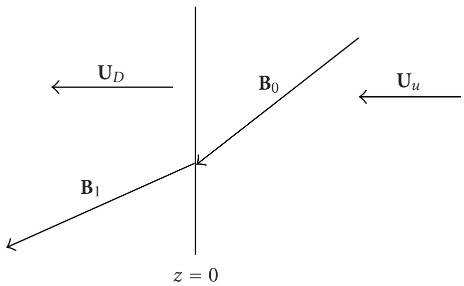


FIGURE 2: Sketch of the geometry of a plane shock with an oblique ambient magnetic field.

Corresponding relations hold between the frames $S$ and $D$. In addition one has the invariant $d^3\mathbf{k}_U d\omega_U = d^3\mathbf{k}_S d\omega_S$.

The formal path to obtaining a quasilinear equation for the time-independent (in the shock frame $S$) component of the distribution function operates as follows. In the Vlasov equation (but with the addition of the background electric field $\mathbf{E}_S$ in the shock frame) set $f_S = F_S + \delta f_S$ where $F_S$ is the time independent component and $\delta f_S$ is the time dependent component of the distribution function. Then, quite generally, one has

$$\mathbf{v}\cdot\frac{\partial F_S}{\partial \mathbf{x}} + \left[\mathbf{E}_S + \frac{\mathbf{vxB}_S}{c}\right]\cdot\frac{\partial F_S}{\partial \mathbf{p}} = -\left\langle e\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial \delta f_S}{\partial \mathbf{p}}\right\rangle \tag{23}$$

together with

$$\begin{aligned}\frac{\partial \delta f_S}{\partial t} &+ \mathbf{v}\cdot\frac{\partial \delta f_S}{\partial \mathbf{x}} + e\left[\mathbf{E}_S + \frac{\mathbf{vxB}_S}{c}\right]\cdot\frac{\partial \delta f_S}{\partial \mathbf{p}}\\ &= -e\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial F_S}{\partial \mathbf{p}} - e\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial \delta f_S}{\partial \mathbf{p}}\\ &\quad + e\left\langle\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial \delta f_S}{\partial \mathbf{p}}\right\rangle,\end{aligned} \tag{24}$$

where angular brackets denote a conventional time-averaging. Here the background magnetic field in the shock frame has the two components given through $B_{S,\perp} = \Gamma_U B_U \sin\theta$, and $B_{S,z} = B_U \cos\theta$, and the background electric field in the shock frame is given through $\mathbf{E}_S = \Gamma_U \mathbf{U}_U \mathbf{xB}_U/c$.

The canonical assumption made in deriving the quasi-linear behaviour of $F_S$ is that only the first terms on the right hand side of (24) need to be retained so that

$$\mathcal{M}\delta f_S = -e\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial F_S}{\partial \mathbf{p}}, \tag{25}$$

where

$$\mathcal{M}\delta f_S = \frac{\partial \delta f_S}{\partial t} + \mathbf{v}\cdot\frac{\partial \delta f_S}{\partial \mathbf{x}} + e\left[\mathbf{E}_S + \frac{\mathbf{vxB}_S}{c}\right]\cdot\frac{\partial \delta f_S}{\partial \mathbf{p}}. \tag{26}$$

Equation (11) has the solution

$$\delta f_S = \mathcal{M}^{-1}\left(-e\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial F_S}{\partial \mathbf{p}}\right) \tag{27}$$

with $\mathcal{M}^{-1}$ the inverse operator to $\mathcal{M}$. Then (9) can be written

$$\begin{aligned}\mathbf{v}\cdot\frac{\partial F_S}{\partial \mathbf{x}} &+ e\left[\mathbf{E}_S + \frac{\mathbf{vxB}_S}{c}\right]\cdot\frac{\partial F_S}{\partial \mathbf{p}}\\ &= e^2\left\langle\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\left[\mathcal{M}^{-1}\left(\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\right)\right]\right\rangle F_S\end{aligned} \tag{28}$$

which relates the spatial and momentum evolution of the steady-state component of the distribution function to quadratic products of fluctuations in the electric and magnetic fields. Because there is an unique relationship between such fields for each wave type, the evolution of $F_S$ is thus tied to the power spectrum of waves in both the upstream and downstream regions. Equation (14) is the general quasilinear approximation to the steady-state distribution function.

Now in the case of a planar shock the spatial dependence of $F_S$ can be a function of $z$ only and not of spatial coordinates parallel to the shock front so that (14) then takes the simplified form

$$\begin{aligned}\frac{v_z\partial F_S}{\partial z} &+ e\left[\mathbf{E}_S + \frac{\mathbf{vxB}_S}{c}\right]\cdot\frac{\partial F_S}{\partial \mathbf{p}}\\ &= e^2\left\langle\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\left[\mathcal{M}^{-1}\left(\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\right)\right]\right\rangle F_S.\end{aligned} \tag{29}$$

One further approximation to the general quasilinear equation is to be concerned only with the component of the quasilinear equation that is independent of the *phase* angle the particles make around the mean magnetic field measured in the upstream coordinates, so that one singles out just that component from the general (29), which depends on the particle momentum, the pitch angle of the particles, and on the spatial coordinates. This process is technically involved when the upstream magnetic field is other than perpendicular to the shock front. Specifically, the distribution function component of interest can then be a function of only $p'_z$ and $p'_\perp$ where

$$\begin{aligned}p'_z &= p_{Uz}\cos\theta + p_{Ux}\sin\theta,\\ p'_y &= p_{Uy} = p_{Ux}\cos\theta - p_{Uz}\sin\theta,\\ p'_\perp &= \left\{p_{Uy}^2 + \left(p_{Ux}\cos\theta - p_{Uz}\sin\theta\right)^2\right\}^{1/2}.\end{aligned} \tag{30a}$$

One also has

$$p_{Sz} = \left(p_{U,z} - \frac{E_U U_U}{c^2}\right)\Gamma_U, \qquad p_{S\perp} = p_{U\perp}. \tag{30b}$$

Then one writes

$$F_S = \sum_n F_n \exp(\mathrm{i}n\varphi), \tag{30c}$$

where the sum over $n$ is $-\infty < n < \infty$, and one is interested in the component with $n = 0$. Here the phase angle $\varphi$ is defined through $\tan\varphi = p'_y/p'_x$. The lowest order equation for $F_0$ is then obtained by writing all components of $\mathbf{v}$ and $\mathbf{p}$, as measured in the shock frame, in terms of the upstream values $p'_z$, $p'_\perp$, and $\varphi$. One then performs an average over the phase angle in $0 < \varphi < 2\pi$, and then converts back to the shock momentum components through (30a)–(30b) so that one then has the lowest order quasilinear equation in the form needed with static boundary conditions at the shock front on $z = 0$. This procedure is not difficult to effect but is extremely tedious in the extreme, which is, presumably, why one conventionally considers just the case of an exactly perpendicular magnetic field.

In addition, it has been known since the work of Hudson [2, 3], over forty years ago, that the case of an uniform magnetic field making an angle with respect to the shock front produces extremely different behaviours for the particle distribution function than in the case of an exactly perpendicular magnetic field even in the absence of electromagnetic fluctuations. The difference arises because, for an oblique magnetic field, the phase angle motion of the particles around the ambient magnetic field is not precisely parallel to the shock front. Particles then intersect the shock front preferentially on one side or the other of the oblique direction, leading to preferential energy and momentum variations—something that does not occur at all for a magnetic field precisely perpendicular to the shock front. So even in the case where electromagnetic fluctuations are ignored, or are absent, the symmetry breaking of the oblique ambient magnetic field produces variations of the transmitted and reflected particles from the shock front that are not recoverable from the situation when one deals with a magnetic field exactly perpendicular to the shock front. Including electromagnetic waves complicates the situation even further, of course.

In short, the distribution function for the particles must have a phase-dependent component that is of fundamental significance unlike the exactly perpendicular magnetic field situation where one can ignore completely any such variation on symmetry grounds (at least at the quasilinear level of approximation) as shown in Part I.

In order to preserve the asymmetric effect of the oblique background magnetic field and, at the same time, to handle the corresponding electromagnetic wave scattering of particles, a procedure that allow for both is the following.

In the frame where the shock is stationary write the operator

$$e\left[\mathbf{E}_S + \frac{\mathbf{vxB}_S}{c}\right]\cdot\frac{\partial F_S}{\partial \mathbf{p}} = \mathcal{N}F_S \tag{31a}$$

and then substitute

$$F_S = F_0 \exp G \tag{31b}$$

into (29), where one also takes the function $F_0$ to satisfy precisely the background field equation in the form

$$\frac{v_z \partial F_S}{\partial z} + \mathcal{N}F_S = 0. \tag{32}$$

It then follows that the function $G$ satisfies

$$\begin{aligned} &\frac{v_z \partial G}{\partial_z} + \mathcal{N}G \\ &= \left(\frac{1}{F_0}\right)\exp(-G)e^2 \\ &\times \left\langle \left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\left[\mathcal{M}^{-1}\left(\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\right)\right]\right\rangle \\ &\times F_0 \exp(G). \end{aligned} \tag{33}$$

The initial/boundary conditions on $F_0$ and $G$ are chosen so that $F_0$ represents the influence of solely the background fields in perturbing the distribution function, while $G$ is chosen so that if the electromagnetic fluctuations are set to zero then $G$ is chosen to be zero also. This separation serves to show the influence of the two sorts of fields on the distribution function and also enables one to handle the basic problem in the shock frame $S$ without the complex evaluation that would otherwise be necessary as referred to above.

Because one has considered fluctuating electromagnetic field correlations of no higher order than pair-wise in the derivation of the basic quasilinear equation as indicated above, the same assumption is appropriate here so that, to the same order, one can set the exponential factors involving $\exp G$ (or $\exp -G$) to unity in the right hand side of (33). One then has

$$\begin{aligned} &\frac{v_z \partial G}{\partial z} + \mathcal{N}G \\ &= \left(\frac{1}{F_0}\right)e^2\left\langle\left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\left[\mathcal{M}^{-1}\left(\left[\mathbf{E}+\frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\right)\right]\right\rangle F_0 \end{aligned} \tag{34}$$

with formal solution

$$\begin{aligned} G = {}&\mathcal{W}^{-1}\left(\frac{1}{F_0}\right)e^2 \\ &\times \left\langle\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\left[\mathcal{M}^{-1}\left(\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\right)\right]\right\rangle F_0, \end{aligned} \tag{35}$$

where $\mathcal{W}^{-1}$ is the inverse operator to $\mathcal{W}Q = v_z \partial Q/\partial z + \mathcal{N}Q$. Basically one notes that $\mathcal{W}^{-1}$ is just $\mathcal{M}^{-1}$ evaluated at zero frequency. In addition one can add homogeneous solutions satisfying $\mathcal{W}Q = 0$ to the particular solution given by (35). At the quasilinear level of approximation the general solution is then $F_S = F_0 \exp G$. Note also that $G$ depends on the structural form of $F_0$ with respect to the momentum components $\mathbf{p}$ but does not depend on the magnitude of $F_0$. It is this dependence that makes difficult the problem of obtaining a match across the shock. The point is that while one knows how the fields transform across the shock the structure of the distribution function across the shock is precisely what one wishes to obtain. Because the structure of $F_0$ enters the behavior of $G$ then the determination of the balanced shock conditions is a very complex problem. Indeed, even in the MHD approach as discussed in Kirk and Duffy [1] the formidable problem of obtaining a relevant shock crossing balance was only recently addressed in a significant manner by Blasi and Vietri [4].

There is no good reason to suppose that the full boundary problem, including electric field fluctuations should be any easier—although future numerical calculations may point the way to a good approximation scheme. Such procedures would also have to incorporate the basic nonperpendicular particle kinetic development given here.

Two tasks remain to carry through in detail. First, one must evaluate the structure of the operator fields $\mathcal{W}^{-1}$ and $\mathcal{M}^{-1}$ as they act on the fluctuating electromagnetic fields, so that the spectral types of electromagnetic fields need to be specified in both the upstream and downstream regions. Second, one must then apply the boundary conditions at the shock front plane ($z = 0$) to both the upstream and downstream components and, simultaneously, effect a smoothly continuous transition for the distribution function as one crosses the shock.

## 5. Part III—Discussion and Conclusion

A complete solution of the distribution function equation in the cases of either a perpendicular (tothe shock) ambient magnetic field or an oblique ambient magnetic field across the shock would seem to require a sophisticated mixture of analytical procedures in conjunction with equally sophisticated numerical methods. Indeed, in even the seemingly simple case of a postulated scattering in pitch angle alone in the presence of a perpendicular ambient magnetic field, Kirk and Schneider [5] and Kirk and Duffy (1999) had to resort to left-handed and right-handed eigenfunctions to effect even an approximate solution and the final approximate analysis had then to be performed numerically. An investigation of this pitch angle scattering situation has been given in considerable generality by Vietri [6] and Blasi and Vietri [4] and represents the most advanced solution of the shock diffusion equation available to date under the pitch angle scattering assumption.

In a more general sense the following statements have to be adhered to far from the shock and at the shock. First: the perception of the dependence of the distribution function far from the shock in both the upstream and downstream regions (in respect of its variability with momentum components parallel and perpendicular to the shock front) is a crucial ingredient for such a prescription determines the behaviour of the distribution function as one nears the shock. But once this rather general condition is met, for solution of the diffusion equation across the shock, so that one can relate upstream conditions to downstream conditions, further factors need to be satisfied. Second, therefore, is the requirement that the distribution function be continuous across the shock coupled with the requirement that the distribution function be a function only of $z$, $p_\perp$ and $(p_{U,z})^2$(or $(p_{D,z})^2$) in the upstream (downstream) regions in $z > 0$ ($z < 0$). If this second condition is satisfied then the flux of particles across the shock is automatically satisfied as well.

Then two problems remain. Third, that one can actually effect solutions to the diffusion equation on each side of the shock, which solutions depend on the assignment given for the wave power spectra in the upstream and downstream regions, although once one has specified, say, the upstream wave spectra then the downstream wave spectra are determined-at least in principle. Fourth, that one can actually construct a match of the distribution function across the shock at $z = 0$ from the upstream to downstream regions with the correct continuity of the distribution function. It is the combination of these last two problems that produces considerable technical difficulty, as was already clear from the limited treatment presented in Kirk and Schneider [5] and given most generally by Vietri [6] and Blasi and Vietri [4] in the case of assumed pitch angle scattering only without connection to possible wave spectra. Even then, considerable difficulty ensued in obtaining a matched, smooth solution for the distribution function across $z = 0$, the shock plane location.

The major point to be made here is that, prior to investigation of the jump and continuity conditions across the shock, one must specify clearly the appropriate quasilinear distribution function equation that is valid on each side of the shock. Such a description requires that one be specific about the types of electromagnetic disturbances that are taken to exist upstream of the shock and with continuation of the electric and magnetic field components across the shock. In addition, once such a specification is made one must ensure that the evolution of the distribution function is correctly taken into account within the framework of such field distributions. In general, such a specification implies that there will be energy diffusion of the particles as well as pitch angle scattering-unless the magnetic field fluctuations upstream of the shock are carefully enough arranged to discount such energy diffusion. For fast magnetosonic waves, Alfven waves, and slow magnetosonic waves upstream of the shock there will be energy diffusion. But once one has obtained the appropriate diffusion equation either with perpendicular or oblique ambient magnetic fields-and such a description has been the purpose of the present paper- there remains the formidable problem of addressing the continuity of the distribution function across the shock. There seems to have been little headway achieved with this latter problem as of the present time, although the work of Vietri [6] and Blasi and Vietri [4] provides illustrations of the complexity of the continuity problem even for the simple case of pitch angle scattering only.

The determination of the appropriate diffusion equation for the particle distribution function is, in its own rite, complicated enough, as this paper has made clear. The shock continuity problem remains an outstanding challenge.

## Appendices

## A. The Inverse Operator $\mathcal{L}^{-1}$

From the main text one has

$$\mathcal{L}q = \frac{\partial q}{\partial t} + \mathbf{v} \cdot \frac{\partial q}{\partial \mathbf{x}} + e\left[\frac{\mathbf{vxB}_0}{c}\right] \cdot \frac{\partial q}{\partial \mathbf{p}} \equiv S. \tag{A.1}$$

Now the background magnetic field $\mathbf{B}_0$ is only in the $z$ direction with strength $\mathbf{B}_0$. So introduce coordinates $\Omega = eB_0/(mc\gamma)$ with $\gamma = (1 + (p_\perp^2 + p_z^2)/(mc)^2)^{1/2}$, and where $p_x = p_\perp \sin\varphi$, $p_y = p_\perp \cos\varphi$. It then follows that, after simple Fourier transforms in $\mathbf{x}$ and $t$, one can write

$$\begin{aligned} q = \mathcal{L}^{-1}S = -\Omega^{-1} \int d^3k d\omega d^3x' dt' (2\pi)^{-4} \\ \times \exp\{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - i\omega(t-t')\} \\ \times \int^{\varphi} d\varphi' S(\mathbf{x}', t', p_z, p_\perp, \varphi') x \\ \times \exp\{i\,\Omega^{-1}(k_z v_z - \omega)(\varphi - \varphi') \\ + i\,k_\perp p_\perp (m\gamma\Omega)^{-1} \\ (\sin(\varphi-\psi) - \sin(\varphi'-\psi))\}, \end{aligned} \tag{A.2}$$

where $k_x = k_\perp \cos\psi$, $k_y = k_\perp \sin\psi$. Equation (A.2) expresses the action of the inverse operator $\mathcal{L}^{-1}$ on an arbitrary function $S$, as needed in the main text and also in Appendix B.

## B. Simplification of the Right Hand Side of (17)

The main factor needing simplification on the right hand side of (17) is

$$\begin{aligned} \text{RHS} = \Big\langle \left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right] \\ \cdot \frac{\partial}{\partial \mathbf{p}}\left[\mathcal{L}^{-1}\left(\left[\mathbf{E} + \frac{\mathbf{vxB}}{c}\right] \cdot \frac{\partial}{\partial \mathbf{p}}\right)\right] F_0(z, p_\perp, p_z) \Big\rangle \end{aligned} \tag{B.1}$$

because the right hand side of (17) is then given by

$$(2\pi)^{-1}\int_0^{2\pi} d\varphi e^2 \text{RHS}. \tag{B.2}$$

In order to expedite evaluation of (B.1) it is most convenient to write the fluctuating fields $\mathbf{E}$ and $\mathbf{B}$ as Fourier transforms with

$$\begin{aligned} [\mathbf{E}_S(\mathbf{x}, t), \mathbf{B}_S(\mathbf{x}, t)] = \int d^3\mathbf{k}_S d\omega_S [\mathbf{E}_S(\mathbf{k}_S, \omega_S), \mathbf{B}_S(\mathbf{k}_S, \omega_S)] \\ \times \exp(i\mathbf{k}_S \cdot \mathbf{x} - i\omega_S t). \end{aligned} \tag{B.3}$$

In the conventional quasilinear approximation it is also taken that $F_S$ may depend on $p_z$ and $p_\perp$ but not on the phase angle. Under such conditions, one can then write

$$\begin{aligned} &(2\pi)^{-1}\int_0^{2\pi} d\varphi \ \text{RHS} \\ &= (2\pi)^{-1}\int_0 d\varphi < \int d^3\mathbf{k}_S d\omega_S \exp(i\mathbf{k}_S \cdot \mathbf{x} - i\omega_S t) \\ &\times \left[\mathbf{E}_S(\mathbf{k}_S, \omega_S) + \frac{\mathbf{v}(\varphi)\mathbf{xB}_S(\mathbf{k}_S, \omega_S)}{c}\right] \\ &\cdot \frac{\partial}{\partial \mathbf{p}}\Big[-\Omega^{-1}(2\pi)^{-4}\int d^3\mathbf{k} d\omega d^3\mathbf{x}' dt' \\ &\times \exp\left(i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{x}'\right) - i\omega(t-t')\right) d^3\mathbf{k}_s' d\omega_s' \int d\varphi' \\ &\times \left[\mathbf{E}_S\left(\mathbf{k}_S', \omega_S'\right) + \frac{\mathbf{v}(\varphi')\mathbf{xB}_S(\mathbf{k}_S, \omega_s')}{c}\right] \\ &\cdot \exp\left(i\mathbf{k}_S' \cdot \mathbf{x}' - i\omega_S' t'\right) \frac{\partial F_S(p_z, p_\perp, x', t')}{\partial \mathbf{p}x} \\ &\times \exp\left\{i\Omega^{-1}(k_z v_z - \omega)(\varphi - \varphi') + ik_\perp p_\perp (m\Omega\gamma)^{-1} \\ &\times (\sin(\varphi-\psi) - \sin(\varphi'-\psi))\right\}\Big], \end{aligned} \tag{B.4}$$

where $\psi$ is the angle the wave vector makes with the $z$ axis, and where the integral over $\varphi'$ is the indefinite integral extending to the upper limit of $\varphi$.

One now invokes a stationary statistical nature for the fluctuations in the electric and magnetic fields to write

$$\left\langle E_i(\mathbf{k}_1,\omega_1)E_j(\mathbf{k}_2,\omega_2)\right\rangle = P_{ij}(\mathbf{k}_1,\omega_1)\delta(\mathbf{k}_1+\mathbf{k}_2)\delta(\omega_1+\omega_2), \tag{B.5a}$$

$$\left\langle E_i(\mathbf{k}_1,\omega_1)B_j(\mathbf{k}_2,\omega_2)\right\rangle = Q_{ij}(\mathbf{k}_1,\omega_1)\delta(\mathbf{k}_1+\mathbf{k}_2)\delta(\omega_1+\omega_2), \tag{B.5b}$$

$$\left\langle B_i(\mathbf{k}_1,\omega_1)B_j(\mathbf{k}_2,\omega_2)\right\rangle = R_{ij}(\mathbf{k}_1,\omega_1)\delta(\mathbf{k}_1+\mathbf{k}_2)\delta(\omega_1+\omega_2), \tag{B.5c}$$

where $P_{ij}$, $Q_{ij}$, and $R_{ij}$ are the respective power spectra.

Then one can write

$$\begin{aligned}
&(2\pi)^{-1}\int_0^{2\pi} d\varphi \text{RHS} \\
&= (2\pi)^{-5}\int_0^{2\pi} d\varphi \cdot \frac{\partial}{\partial p_i\Omega^{-1}} \int d^3\mathbf{k}_S d\omega_S d^3\mathbf{x}' dt' \\
&\quad\times \exp\left(i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{x}'\right) - i\omega(t-t')\right) \\
&\quad\times \frac{\partial F_S(p_z,p_\perp,x',t')}{\partial p_\alpha} \\
&\quad\times \int d\varphi' \exp\left\{i\Omega^{-1}(k_z v_z-\omega)(\varphi-\varphi') \right.\\
&\qquad\left. +ik_{\perp p_\perp}(m\Omega\gamma)^{-1}(\sin(\varphi-\psi)-\sin(\varphi'-\psi))\right\} \\
&\quad\times \left[P_{i\alpha}(\mathbf{k}_S,\omega_S) + \frac{\varepsilon_{ijk}Q_{k\alpha}(\mathbf{k}_S,\omega_S)v_j(\varphi)}{c} \right.\\
&\qquad + \frac{\varepsilon_{\alpha\beta\gamma}Q_{\gamma i}(\mathbf{k}_S,\omega_S)v_\beta(\varphi')}{c} \\
&\qquad\left. + \frac{\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}R_{\gamma k}(\mathbf{k}_S,\omega_S)v_\beta(\varphi')v_j(\varphi)}{c_2}\right],
\end{aligned} \tag{B.6}$$

where the Einstein double summation convention over repeated Greek and Latin indices has been used. The steady-state assumption (in the frame $S$) for $F_S$ means that the integral over $t'$ returns the value $2\pi\delta(\omega)$; while in the standard quasilinear approximation one takes $F_S(\mathbf{x}')$ to vary on a spatial scale much longer than the field fluctuations and so one writes $F_S(\mathbf{x}') \approx F_S(\mathbf{x})$. The result is that the integral over $\mathbf{x}'$ then returns the value $(2\pi)^3\delta(\mathbf{k})$. Finally, then one can write the form

$$\begin{aligned}
&(2\pi)^{-1}\int_0^{2\pi} d\varphi \text{ RHS} \\
&= -(2\pi)^{-3}\int d^3\mathbf{k}_S d\omega_S \\
&\quad\times \int_0^{2\pi} d\varphi\left[\frac{\delta_{iz}\partial}{\partial p_z} + \left(\delta_{ix}\cos\varphi + \delta_{iy}\sin\varphi\right)\frac{\partial}{\partial p_\perp}\right.\\
&\qquad\left. -\left(\frac{1}{p_\perp}\right)\left(\delta_{ix}\sin\varphi - \delta_{iy}\cos\varphi\right)\frac{\partial}{\partial\varphi}\right] \\
&\quad\times\left[\Omega^{-1}\left(\frac{\delta_{iz}\partial F_S}{\partial p_z} + \left(\delta_{ix}\cos\varphi+\delta_{iy}\sin\varphi\right)\frac{\partial F_S}{\partial p_\perp}\right)\right.\\
&\qquad\times\left\{\left[P_{i\alpha}(\mathbf{k}_S,\omega_S) + c^{-1}\varepsilon_{ijk}Q_{k\alpha}(\mathbf{k}_S,\omega_S)\right.\right.\\
&\qquad\quad\left.\times\left(v_z\delta_{jz} + p_\perp(m\gamma)^{-1}\left(\delta_{jx}\cos\varphi+\delta_{jy}\sin\varphi\right)\right)\right]\varphi \\
&\qquad\quad + c^{-1}\varepsilon_{\alpha\beta\gamma}\left[Q_{\gamma i}(\mathbf{k}_S,\omega_S) + c^{-1}\varepsilon_{ijk}R_{\gamma k}(\mathbf{k}_S,\omega_S)\right.\\
&\qquad\qquad\left.\times\left(v_z\delta_{jz} + p_\perp(m\gamma)^{-1}\left(\delta_{jx}\cos\varphi+\delta_{jy}\sin\varphi\right)\right)\right] \\
&\qquad\quad\left.\left.\times\left[\varphi v_z\delta_{\beta z} + p_\perp(m\gamma)^{-1}\left(\delta_{\beta x}\sin\varphi - \delta_{\beta y}\cos\varphi\right)\right]\right\}\right].
\end{aligned} \tag{B.7}$$

The various integrals over $\varphi$ in (B.6) can be readily, if somewhat tediously, performed. The upshot is that the equation describing the evolution of $F_S$ takes on the form

$$\begin{aligned}
\frac{v_z\partial F_S}{\partial_z} &= \frac{\partial}{\partial p_z}\left(\frac{A\partial F_S}{\partial p_z}\right) + \frac{\partial}{\partial p_z}\left(\frac{B\partial F_S}{\partial p_\perp}\right) \\
&\quad + \frac{\partial}{\partial p_\perp}\left(\frac{C\partial F_S}{\partial p_z}\right) + \frac{\partial}{\partial p_\perp}\left(\frac{D\partial F_S}{\partial p_\perp}\right)
\end{aligned} \tag{B.8}$$

which is the structure given in text. The coefficients entering equation (B.8) are dependent on what one assigns for the power spectra of the wave fields in the upstream (and so also downstream) regions with the field connections given through the matching conditions for the fields at the shock. Once one specifies the upstream field fluctuations then the downstream fields are prescribed.

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## References


[1] J. G. Kirk and P. Duffy, "Particle acceleration and relativistic shocks," *Journal of Physics G*, vol. 25, no. 8, pp. R163–R194, 1999.

[2] P. D. Hudson, "Reflection of charged particles by plasma shocks," *Monthly Notices of the Royal Astronomical Society*, vol. 131, pp. 23–50, 1965.

[3] P. D. Hudson, "Perpendicular shocks in a medium containing cosmic rays," *Monthly Notices of the Royal Astronomical Society*, vol. 140, pp. 255–263, 1968.

[4] P. Blasi and M. Vietri, "On particle acceleration around shocks. II. A fully general method for arbitrary shock velocities and scattering media," *Astrophysical Journal*, vol. 626, no. 2, pp. 877–886, 2005.

[5] J. G. Kirk and P. Schneider, "On the acceleration of charged particles at relativistic shock fronts," *Astrophysical Journal*, vol. 315, pp. 425–433, 1987.

[6] M. Vietri, "On particle acceleration around shocks. I," *Astrophysical Journal*, vol. 591, no. 2, pp. 954–961, 2003.