Research Article
Weakly Injective BCK-Modules

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We introduce the notion of weakly injective BCK-module and show that Baer's criterion holds for weakly injective BCK-modules but not for injective BCK-modules in general. We also provide examples and counterexamples of weakly injective BCK-modules.

## 1. Introduction

Inspired by Meredith's BCK-systems, Iséki and Imai introduced the notion of BCK-algebra in 1966. These pioneers developed major aspects of the theory in the late 1960s and the 1970s. They were soon joined by many other researchers to develop various aspects of the BCKalgebra theory. Since then, BCK-algebras have been a subject of intense research. The main approach of this development has been trying to build a theory that is parallel to the standard ring theory. In this order of ideas, Noetherian and Artinian BCK-algebras [1], BCK-modules [2], injective and projective BCK-modules [2], and fractions BCK-algebras [3] have recently been treated. So far, the only articles on BCK-modules have been [2,4]. Considering the topics covered by these two articles, it is quite clear that very little is known about the theory of modules over BCK-algebras. For instance, the notion of injective modules over BCK-algebras was introduced in [2], but not a single example was treated. In classical ring theory, injective modules are studied using Baer's criterion and divisible modules. Unfortunately, as we will show, this criterion does not hold for injective BCK-modules, and there are no natural notion of divisible modules over BCK-algebras.

The main goal of this work is to shed some light on the notion of injective modules over BCK-algebras. We do this by introducing a new class of modules (weakly injective modules) that strictly contains the above class and obtain a Baer's criterion for this class. In order to achieve this goal, we found ourselves imposing a new axiom to BCK-modules.

Recall that the notion of left module over a bounded commutative BCK-algebra $(X, *, 0,1)$ was first introduced in 1994 by Abujabal et al. [4]. We consider the class of left BCK-modules that satisfy the following axiom in addition to the axioms of [4],

$$
\begin{equation*}
(x+y) m=x m+y m \tag{1.1}
\end{equation*}
$$

for all $x, y \in X$ and $m \in M$ where $x \dot{+} y=(x * y) \vee(y * x)$.
We will refer to BCK-modules of this class as BCK-modules of type 2 . The consideration of this class is motivated not only by the fact that it makes BCK-modules more in line with modules over rings, but also the fact that the main results obtained by the previous authors remain valid for this class. Using this class of BCK-modules, we introduce weakly injective BCK-modules. We prove that weakly injective BCK-modules are Characterized by Baer's criterion, which we use to prove that over principal bounded implicative BCK's, every module is weakly injective [Corollary 3.9]. We use these characterizations to build examples of (weakly) injective modules over BCK-algebras and also find examples that prove that our Baer's criterion is the sharpest we can get.

## 2. Generalities on BCK-Modules

Recall that a BCK-algebra is an algebra $(X, *, \leq, 0)$ satisfying for all $x, y, z \in X$
(i) $(x * z) *(x * y) \leq y * z$,
(ii) $x *(x * y) \leq y$,
(iii) $0 \leq x$,
(iv) $x \leq x$,
(v) $x \leq y$ and $y \leq x$ implies $x=y$,
(vi) $x \leq y$ if and only if $x * y=0$.

In addition, if there exists an element 1 in $X$ such that $x \leq 1$ for all $x \in X$, then $X$ is said to be bounded and we write $N x$ for $1 * x$. Also, if $x *(x * y)=y *(y * x)$ for all $x, y \in X, X$ is said to be commutative. In addition, $X$ is called implicative if $x *(y * x)=x$ for all $x, y \in X$. As proved in [5, Theorem 10], implicative BCK-algebras are commutative. A subset $I$ of a BCK-algebra $X$ is called an ideal of $X$ if it satisfies (i) $0 \in I$ and (ii) for every $x, y \in X$ such that $y \in I$ and $x * y \in I$, then $x \in I$.

As defined in [4, Definition 2.1], a left module over a bounded commutative BCKalgebra $(X, *, 0,1)$ is an Abelian group $(M,+)$ with a multiplication $(x, m) \mapsto x m$ satisfying
(i) $(x \wedge y) m=x(y m)$ for all $x, y \in X$ and $m \in M$,
(ii) $x(m+n)=x m+x n$ for all $x \in X$ and $m, n \in M$,
(iii) $0 m=0$ for all $m \in M$,
(iv) $1 m=m$ for all $m \in M$,
where $x \wedge y=y *(y * x)$.
If in addition, $M$ satisfy the axiom (v) below, we call $M$ an $X$-module of type 2 .
(v) $(x+y) m=x m+y m$ for all $x, y \in X$ and $m \in M$. Where $x+y=(x * y) \vee(y * x)$.

Our terminology type 2 is motivated by the fact that every $X$-module satisfying $(v)$ is as Abelian group, of exponent 2 .

Recall [4, Lemma 2.4] that if $X$ is a bounded implicative BCK-algebra, then $(X, \dot{+}, \wedge)$ is a commutative ring. Therefore, $X$-module of type 2 are modules over the ring $(X, \dot{+}, \wedge)$.

If $M$ is a left $X$-module, a subset $S$ of $M$ is a submodule if $(S,+)$ is a subgroup of ( $M,+$ ) such that $x m \in S$ whenever $x \in X$ and $m \in S$.

Given two left $X$-modules $M$ and $N$, an $X$-module homomorphism from $M$ to $N$ is a map $f: M \rightarrow N$ satisfying
(i) $f\left(m+m^{\prime}\right)=f(m)+f\left(m^{\prime}\right)$ for all $m, m^{\prime} \in M$,
(ii) $f(x m)=x f(m)$ for all $x \in X$ and $m \in M$.

The set of all $X$-module homomorphisms from $M$ to $N$ is denoted by $\operatorname{Hom}_{X}(M, N)$ which has a natural structure of X -module via the multiplication $(x f)(m)=x f(m)$.

We introduce the following definition.
Definition 2.1. A left $X$-module $Q$ is weakly injective if for every left $X$-module $M, N$ so that $N$ is of type 2, every injective homomorphism $f: M \rightarrow N$ and every homomorphism $g: M \rightarrow Q$, there exists a homomorphism $\phi: N \rightarrow Q$ such that $\phi \circ f=g$.

Note that injective X-modules as defined in [2] are clearly weakly injective. We have the following lemma whose some parts have been proved by other authors, but which we offer a proof here for the convenience of the reader.

Lemma 2.2. Let $X$ be a bounded implicative $B C K$-algebra with unit 1. Then, for all $x, y, z \in X$,
(i) $x \wedge y=x * N y$,
(ii) $x *(x \wedge y)=x * y$,
(iii) $x \wedge(y * z)=(x \wedge y) *(x \wedge z)$,
(iv) $(x * y)+(y * x)=x+y$,
(v) $(x+y) \wedge z=(x \wedge z)+(x \wedge z)$.

Proof. (i) From [5, Proposition 6], we have $x \wedge N y \leq x * y$. In addition, $x * y \leq x$ and $x * y \leq$ $1 * y=N y$, so $x * y \leq x \wedge N y$. Thus, $x \wedge N y=x * y$. Therefore, $x \wedge y=x \wedge N N y=x * N y$.

For (ii), let $x, y \in X$. By (i), we do have

$$
\begin{equation*}
x *(x \wedge y)=x *(x * N y)=N y \wedge x=x * y . \tag{2.1}
\end{equation*}
$$

For (iii), let $x, y, z \in X$. We first prove that

$$
\begin{equation*}
x \wedge(y * z)=(x \wedge y) * z . \tag{2.2}
\end{equation*}
$$

In fact,

$$
\begin{align*}
x \wedge(y * z) & =(y * z) * N x \\
& =(y * N x) * z  \tag{2.3}\\
& =(x \wedge y) * z .
\end{align*}
$$

Now, we use (ii) and (2.2) to show (iii)

$$
\begin{align*}
(x \wedge y) *(x \wedge z) & =(x * N y) *(x \wedge z) \\
& =(x *(x \wedge z)) * N y \\
& =(x * z) * N y \quad \text { by (ii) } \\
& =(x * N y) * z  \tag{2.4}\\
& =(x \wedge y) * z \\
& =x \wedge(y * z) \quad \text { by }(2.2) .
\end{align*}
$$

(iv) We have

$$
\begin{align*}
(x * y)+(y * x) & =[(x * y) *(y * x)] \vee[(y * x) *(x * y)] \\
& =((x *(y * x)) * y) \vee((y *(x * y)) * x) \quad \text { by [5, Equation (3)] }  \tag{2.5}\\
& =(x * y) \vee(y * x) \quad \text { since } X \text { is implicative } \\
& =x+y .
\end{align*}
$$

(v) Recall that being a bounded implicative BCK-algebra, $(X, \vee, \wedge)$ is a distributive lattice [5, Theorem 12]. We have

$$
\begin{align*}
(x+y) \wedge z & =((x * y) \vee(y * x)) \wedge z \\
& =((x * y) \wedge z) \vee((y * x) \wedge z) \\
& =((x \wedge z) *(y \wedge z)) \vee((y \wedge z) *(x \wedge z)) \quad \text { by }(\text { ii })  \tag{2.6}\\
& =(x \wedge z)+(y \wedge z) .
\end{align*}
$$

Proposition 2.3. Every bounded implicative BCK-algebra has a natural structure of X-module of type 2. Furthermore, under this structure, every ideal of $X$ is a submodule of $X$.

Proof. Consider the operational system $(X, \dot{+})$. Then, by [4, Proposition 2.5], $(X, \dot{+})$ is an Abelian group and together with the multiplication $(x, y) \mapsto x y:=x \wedge y$ satisfying (i), (ii), (iii), and (iv) of the definition of $X$-module above. It remains to verify that ( $X, \dot{+}$ ) satisfies (v). But this is straightforward from the definition of the multiplication and Lemma 2.2 (v).

As for the proof that an ideal of $X$ is a submodule, the argument is identical to the one of [2, Theorem 2.1].

Example 2.4. Consider the bounded implicative $\mathrm{BCK}, \mathrm{X}=p(\mathbb{Z})$ with the standard operations. Consider $M_{1}=\operatorname{Maps}(\mathbb{Z}, \mathbb{Z})$ and $M_{2}=\operatorname{Maps}\left(\mathbb{Z}, \mathbb{Z}_{2}\right)$, then under the multiplication $(A, f) \mapsto$ $1_{A} f$ (where $1_{A}$ is the characteristic function of $A$ ), $M_{1}$ is an X-module that is not of type 2 while $M_{2}$ is an $X$-module of type 2 .

Example 2.5. For every $X$-modules $M$ and $N$ so that $N$ is of type 2 , the $X$-module $\operatorname{Hom}_{X}(M, N)$ is also of type 2. In particular, for every $X$-module $M, \operatorname{Hom}_{X}(M, X)$ is of type 2.

Remark 2.6. $(X, \dot{+})$ is an Abelian group of exponent 2; therefore, finite bounded implicative BCK-algebras have order a power of 2 . This is not surprising as such BCKs are Boolean algebras [5, Theorem 12].

## 3. Injective BCK-Modules and Baer's Criterion

$X$ will denote a bounded implicative BCK-algebra with unit 1 . In addition, the term $X$-module will refer to left $X$-module. We start by the following lemma which is crucial for the proof of Baer's criterion.

Lemma 3.1. Let $N$ be an $X$-module of type 2 and $M$ a submodule of $N$. For every $n \in N$, define

$$
\begin{equation*}
I_{n}=\{x \in X \mid x n \in M\} \tag{3.1}
\end{equation*}
$$

Then, $I_{n}$ is an ideal of $X$ for all $n \in N$.
Proof. Let $n \in N$, then
(i) $0 n=0 \in M$ as $M$ is a submodule; therefore, $0 \in I_{n}$.
(ii) Let $x, y \in X$ such that $x * y \in I_{n}$ and $y \in I_{n}$. As $X$ is implicative, $y * x=y \wedge N x=$ $N x \wedge y$ by Lemma 2.2 (i). Hence, $(y * x) n=(N x \wedge y) n=N x(y n)$ which is in $M$ as $y n \in M$ and $M$ is a submodule, thus $y * x \in I_{n}$. Therefore, $x * y, y * x, y$ are all in $I_{n}$. Thus $(x * y) n,(y * x) n$ and $y n$ are all in $M$, hence as $M$ is a submodule, then $(x * y) n+(y * x) n+y n \in M$. But from the axiom (v) of X-module, it follows that

$$
\begin{align*}
(x * y) n+(y * x) n+y n & =((x * y)+(y * x) \dot{+} y) n \\
& =(x+y \dot{+} y) n \quad \text { by Lemma } 2.2 \text { (iv) }  \tag{3.2}\\
& =x n \quad \text { since } y \dot{+} y=0 .
\end{align*}
$$

Thus, $x n \in M$, consequently $x \in I_{n}$ as desired.
Whence $I_{n}$ is an ideal of $X$ as stated.
Remark 3.2. Given an $X$-module $M$ and $m \in M$, then the set $X m:=\{x m \mid x \in X\}$ is a submodule of $M$, the submodule generated by $m$.

Theorem 3.3 (Baer's Criterion). Let $Q$ be an $X$ - module.
Then, $Q$ is weakly injective if and only if for every ideal I of $X$, every $X$-module homomorphism from $I \rightarrow Q$ extends to a homomorphism from $X \rightarrow Q$.

Proof. $\Rightarrow)$ : This direction is obvious as $X$ is an $X$-module of type 2, and every ideal of $X$ is an X-module [Proposition 2.3].
$\Leftarrow)$ : Assume that for every ideal $I$ of $X$, every $X$-module homomorphism from $I \rightarrow Q$ extends to a homomorphism from $X \rightarrow Q$. Consider $0 \rightarrow M \xrightarrow{f} N$ and $M \xrightarrow{g} Q$, where $N$ is type 2. Let X-Mod be the set of X-modules. Consider

$$
\begin{equation*}
\sum=\left\{(C, \phi): C \in X-\operatorname{Mod}, M \subseteq C \subseteq N ; \phi: C \longrightarrow Q ;\left.\phi\right|_{M}=g\right\} \tag{3.3}
\end{equation*}
$$

First, note that $\sum \neq \emptyset$ since $(M, g) \in \sum$. Define on $X$ the relation $\preccurlyeq$ by $\left(C_{1}, \phi_{1}\right) \preccurlyeq\left(C_{1}, \phi_{1}\right)$ if $C_{1} \subseteq C_{2}$ and $\left.\phi_{2}\right|_{C_{1}}=\phi_{1}$. Then, $\preccurlyeq$ is easily verified to be an order on $\sum$. The usual argument also show that every chain in $(\Sigma, \preccurlyeq)$ has an upper bound, and therefore, by the Zorn's lemma, $(\Sigma, \preccurlyeq)$ has a maximal element $(D, \varphi)$.

We show that $D=N$, and therefore, $\varphi$ would be the required extension of $g$.
By definition, we have $D \subseteq N$. Conversely, let $n \in N$, then by Lemma 3.1, as $N$ is type 2, the set $I_{n}=\{x \in X \mid x n \in D\}$ is an ideal of $X$. Define, $\alpha: I_{n} \rightarrow Q$ by $\alpha(x)=\varphi(x n)$. Then $\alpha(x \dot{+} y)=\varphi((x \dot{+} y) n)=\varphi(x n+y n)=\varphi(x n)+\varphi(y n)=\alpha(x)+\alpha(y)$. In addition $\alpha(x y)=\alpha(x \wedge y)=\varphi((x \wedge y) n)=\varphi(x(y n))=x \varphi(y n)=x \alpha(y)$. Hence, $\alpha$ is an X-module homomorphism, and by hypothesis $\alpha$ extends to $\beta: X \rightarrow Q$.

Define $\varphi^{\prime}: D+X n \rightarrow Q$ by $\varphi^{\prime}(d+x n)=\varphi(d)+\beta(x)$. We need to verify that $\varphi^{\prime}$ is a well-defined homomorphism (which clearly extends $\varphi$ ).

For the well-definition, suppose $d_{1}+x_{1} n=d_{2}+x_{2} n$, then $d_{1}-d_{2}=x_{1} n+x_{2} n=\left(x_{1}+x_{2}\right) n$. So, $\left(x_{1} \dot{+} x_{2}\right) n \in D$, hence $\left(x_{1} \dot{+} x_{2}\right) \in I_{n}$. Now using the fact that $\varphi$ is a homomorphism, we obtain

$$
\begin{equation*}
\varphi\left(d_{1}\right)-\varphi\left(d_{2}\right)=\varphi\left(\left(x_{1} \dot{+} x_{2}\right) n\right)=\alpha\left(x_{1} \dot{+} x_{2}\right)=\alpha\left(x_{1}\right)+\alpha\left(x_{2}\right)=\beta\left(x_{1}\right)+\beta\left(x_{2}\right) . \tag{3.4}
\end{equation*}
$$

Hence, $\varphi\left(d_{1}\right)+\beta\left(x_{1}\right)=\varphi\left(d_{1}\right)+\beta\left(x_{1}\right)$, because $\beta\left(x_{1}\right)+\beta\left(x_{1}\right)=0$. Therefore, $\varphi^{\prime}\left(d_{1}+x_{1} n\right)=$ $\varphi^{\prime}\left(d_{2}+x_{2} n\right)$ and $\varphi^{\prime}$ is well defined. Next, we check that $\varphi^{\prime}$ is a homomorphism.

Let $d, d^{\prime} \in D$ and $x, x^{\prime} \in X$, then

$$
\begin{align*}
\varphi^{\prime}\left((d+x n)+\left(d^{\prime}+x^{\prime} n\right)\right) & =\varphi^{\prime}\left(\left(d+d^{\prime}\right)+\left(x n+x^{\prime} n\right)\right) \\
& =\varphi^{\prime}\left(\left(d+d^{\prime}\right)+\left(x+x^{\prime}\right) n\right) \\
& =\varphi\left(d+d^{\prime}\right)+\beta\left(x+x^{\prime}\right) \\
& =\varphi(d)+\varphi\left(d^{\prime}\right)+\beta(x)+\beta\left(x^{\prime}\right) \\
& =\varphi^{\prime}(d+x n)+\varphi^{\prime}\left(d^{\prime}+x^{\prime} n\right),  \tag{3.5}\\
\varphi^{\prime}\left(x^{\prime}(d+x n)\right) & =\varphi^{\prime}\left(\left(x^{\prime} d\right)+x^{\prime}(x n)\right) \\
& =\varphi^{\prime}\left(x^{\prime} d+\left(x \wedge x^{\prime}\right) n\right) \\
& =\varphi\left(x^{\prime} d\right)+\beta\left(x^{\prime} \wedge x\right) \\
& =x^{\prime} \varphi(d)+x^{\prime} \beta(x) \\
& =x^{\prime} \varphi^{\prime}(d+x n) .
\end{align*}
$$

Thus, $\varphi^{\prime}$ is a homomorphism as needed. Whence $(D, \varphi) \preccurlyeq\left(D+X n, \varphi^{\prime}\right)$ and by the maximality of $(D, \varphi)$, we obtain $D=D+X n$, so $n \in D$ which shows that $D=N$ as required.

The theory of modules over BCK-algebras displays some real pathologies as the remark below explains. Before the remark, a couple of definitions.

Definition 3.4. As defined in [6], an element $x$ of a BCK-algebra $X$ is called a zero-divisor if there exists a nonzero element $y$ in $X$ such that $x \wedge y=0$. If $X$ has nontrivial zero-divisors, then $X$ is called cancellative. These correspond to domains in ring theory.

Remark 3.5. The natural approach for understanding injective modules over rings consists of establishing the relationship with divisible modules. Unfortunately, what should be the natural equivalent of divisible modules over BCK-algebras turns out to be useless. In fact, it is straightforward to see that the only cancellative implicative BCK-algebra is $\{0,1\}$ so that every module over such is always divisible.

Remark 3.6. Recall [7, Theorem 3] that if $X$ is a BCK-algebra (not necessarily implicative) and $a \in X$, the ideal of $X$ generated by $a$ is denoted by $\langle a\rangle$ is given by $\left\{x \in X \mid \exists n>0 ; x * a^{n}=0\right\}$. In the case when $X$ is implicative, this simplifies to $\langle a\rangle=\{x \in X \mid x * a=0\}=\{x \in X \mid x \leq a\}$.

Definition 3.7. A BCK-algebra $X$ is principal if every ideal of $X$ is generated by one element.
Example 3.8. (1) $X=\{0,1,2, \ldots, \omega\}$ as defined in [5, Example 1] is a principal BCK-algebra. In fact, it is easy to see that the only ideals of $X$ are $0, X$ and $\{0,1,2, \ldots\}$. Note that $X$ is not implicative.
(2) The BCK-algebra $B_{4-2-3}$ from [8, Appendix] is bounded implicative and principal.

We now deduce from the above Baer's criterion that all modules over principal bounded implicative BCK-algebras are weakly injective.

Corollary 3.9. Let $M$ be an $X$-module and suppose that $X$ is bounded implicative and principal. Then, $M$ is weakly injective. In particular, every bounded implicative and principal BCK-algebra is weakly injective as a module over itself.

Proof. Let $M$ be an $X$-module, with $X$ bounded implicative and principal, $I$ an ideal of $X$, and $f: I=\langle a\rangle \rightarrow M$ an $X$-homomorphism. Define $h: X \rightarrow M$ by $h(x)=f(a \wedge x)$. It is clear that $h$ is an $X$-homomorphism; in fact, using Lemma 2.2 (v), we obtain

$$
\begin{gather*}
h\left(x_{1} \dot{+} x_{2}\right)=f\left(a \wedge\left(x_{1}+x_{2}\right)\right)=f\left(\left(a \wedge x_{1}\right)+\left(a \wedge x_{2}\right)\right)=f\left(a \wedge x_{1}\right)+f\left(a \wedge x_{2}\right)=h\left(x_{1}\right)+h\left(x_{2}\right), \\
h(\gamma \wedge x)=f(a \wedge(\gamma \wedge x))=f(\gamma \wedge(a \wedge x))=\gamma f(a \wedge x)=\gamma h(x) . \tag{3.6}
\end{gather*}
$$

We claim that $h$ extends $f$. In fact, let $x \in I=\langle a\rangle$. Then, $x=a \wedge x$. So, $h(x)=f(a \wedge x)=f(x)$. Hence, $M$ is weakly injective by Baer's criterion.

Example 3.10. Consider $B_{4-2-3}$ as above which is a bounded implicative and principal BCKalgebra. By Corollary 3.9, $B_{4-2-3}$ is weakly injective as a module over itself.

## 4. Examples

This section is devoted to constructing examples.

Example 4.1 (A non weakly injective BCK-Module). Consider $S$ any infinite set and the BCKalgebra $\mathbb{X}=P(S)$ with the natural operations. Consider

$$
\begin{equation*}
I=\{X \subseteq S:|X|<\infty\} \tag{4.1}
\end{equation*}
$$

Then, $I$ is clearly an ideal of $\mathbb{X}$ and, therefore, an $\mathbb{X}$-module by Proposition 2.3.
Claim 4.2. I is not weakly injective.
To see this, it is enough to produce an $\mathbb{X}$-module homomorphism $\varphi: I \rightarrow I$ that does not extend to $\mathbb{X}$.

For this, consider any finite complement subset $A$ of $S$ and $\varphi: I \rightarrow I$ defined by $\varphi(X)=X \cap A$. Since $\cap$ distributes over $\triangle$ and $\cap$ is associative, it follows that $\varphi$ is an $\mathbb{X}$-module homomorphism.

We assert that there is no homomorphism $\bar{\varphi}: \mathbb{X} \rightarrow I$ such that $\left.\bar{\varphi}\right|_{I}=\varphi$. In fact, by contradiction, suppose there is such an extension. Then, for every $X \subseteq S$, we have $X=(X \cap$ A) $\Delta\left(X \cap A^{C}\right)$; therefore, since $\bar{\varphi}$ is a homomorphism, then

$$
\begin{align*}
\bar{\varphi}(X) & =\bar{\varphi}(X \cap A) \Delta \bar{\varphi}\left(X \cap A^{C}\right) \\
& =(X \cap \bar{\varphi}(A)) \Delta\left(X \cap \bar{\varphi}\left(A^{C}\right)\right)  \tag{4.2}\\
& =X \cap\left(\bar{\varphi}(A) \Delta \bar{\varphi}\left(A^{C}\right)\right) \\
& =X \cap \bar{\varphi}(A) \quad\left(\bar{\varphi}\left(A^{C}\right)=\varphi\left(A^{C}\right)=\emptyset\right)
\end{align*}
$$

Let $B=\bar{\varphi}(A)$, then since $A \cap A^{C}=\emptyset$ and $\bar{\varphi}$ is a homomorphism, then $A^{C} \cap B=\emptyset$, so $B \subseteq A$. Note that $B \subsetneq A$, because if $B=A$, then $\bar{\varphi}(A)=A \cap A=A$ and $A \notin I$. Therefore, there exists an element $a$ of $A$ that is not in $B$. We have $\bar{\varphi}(\{a\})=\varphi(\{a\})$, that is $\{a\} \cap A=\{a\} \cap B$ which is a contradiction.

Whence, $I$ is an $\mathbb{X}$-module that is not weakly injective (much less injective).
Example 4.3 (A weakly injective BCK-module that is not injective). $B_{2-1-1}$ denotes the unique BCK-algebra with two elements: 0,1.

First, observe that every Abelian group $(M,+)$ has a natural structure of $B_{2-1-1}$-module via $0 \cdot m=0$ and $1 \cdot m=m$ for all $m \in M$. We consider $\mathbb{Z}$ and $\mathbb{Q}$ in this view as $B_{2-1-1}$-modules. Consider the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ and $f: \mathbb{Z} \rightarrow B_{2-1-1}$ defined by $f(2 k)=0$ and $f(2 k+1)=1$. Then $f$ is a $B_{2-1-1}$-module homomorphism. If $B_{2-1-1}$ was injective, then, there would exist a homomorphism $\bar{f}: \mathbb{Q} \rightarrow B_{2-1-1}$ such that $\bar{f}(m)=f(m)$ for all $m \in \mathbb{Z}$. But such an extension would satisfy $1=f(1)=\bar{f}(1)=\bar{f}(1 / 2)+\bar{f}(1 / 2)=0$, which is a contradiction. Therefore, as a module over itself, $B_{2-1-1}$ is not injective.

On the other hand, $B_{2-1-1}$ is clearly weakly injective, as it has only two ideals making Baer's criterion obvious. Or more directly, use Corollary 3.9, since $B_{2-1-1}$ is principal.

Remark 4.4. The existence of weakly injective modules that are not injective (see Example 4.3), shows that Baer's criterion does not characterize injective BCK-modules. That is, being able to extend homomorphisms from ideals to the whole BCK-algebra is weaker than being injective.

Note also that the first example shows that not every type 2 module is weakly injective. Finally, the proof of Baer's criterion clearly works in the subcategory of X-modules of type 2 so that injective objects in this category are characterized by the criterion.

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