Research Article

Best Approximation of Data Distributed in a Band

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We study the problem to approximate a data set which are affected in a such way that they present us as a band in the plane. We introduce a deviation measure, and we research the asymptotic behavior of the best approximants when the band shrink in some sense.

1. Introduction

In some situations, we find us with the problem to approximate a given function of physical origin which is contaminated by different causes. For example, it occurs when we receive a signal, and we observe at the screen of an electronic oscilloscope a band produced by noise or other factors. Here, a criterion of selecting is necessary in order to approximate to that band. More precisely, we must choose a measure of deviation from one band to a given approximant class. A way could be to approximate a segment value multivalued function using the Hausdorff metric in the plane (see [1]); another could be to consider the best simultaneous approximation to the set of every functions whose graphics live in the band determined by them (see [2–4]). In this paper, we give an alternative deviation measure, and we establish a relation with the best simultaneous approximation.

Let $1 \leq p < \infty$, and $F : [a, b] \rightarrow 2^{\mathbb{R}}$, be a multivalued function with $F(x)$ a Lebesgue measurable set for all $x \in [a, b]$. Given an approximant class $\mathcal{S}$, we consider the following function as measure of deviation of F to $Q \in \mathcal{S}$:

$$\Phi_{[a,b]}(F, Q) = \left(\int_a^b \int_{F(x)} |y - Q(x)|^p dy dx \right)^{1/p}. \quad (1.1)$$

Let $D := \{(x, y) : x \in [a, b], y \in F(x)\}$. It is easy to see that (1.1) is a special case to approximate a function I from a given class $\mathcal{C}$ with a norm $\|\cdot\|$ over the space of functions

defined on D . In fact, if $\mathcal{C} = \{H_Q : Q \in \mathcal{S}\}$, where $H_Q(x, y) = Q(x)$, then $\Phi_{[a,b]}(F, Q) = \|I - H_Q\|$, where $I(x, y) = y$ and

$$\|G\| = \left(\iint_D |G(x, y)|^p dy dx \right)^{1/p}. \quad (1.2)$$

As usual, if $1 \leq q \leq \infty$, $A \subset [a, b]$ is a Lebesgue measurable set and w is a nonnegative integrable function on A , then $L_w^q(A)$ denotes the space of Lebesgue measurable functions f satisfying

$$\|f\|_{q,w,A} := \left(\int_A |f(x)|^q w(x) dx \right)^{1/q} < \infty, \quad (1.3)$$

with the usual understanding if $q = \infty$. If $w = 1$, we write $L^q(A)$ and $\|f\|_{q,A}$ for $L_w^q(A)$ and $\|f\|_{q,w,A}$, respectively.

Given two functions $f, g : [a, b] \rightarrow \mathbb{R}$, $f \leq g$, we only consider in this paper the multivalued function F defined by $F(x) = [f(x), g(x)]$ for each $x \in [a, b]$.

Our main goal is to study the asymptotic behavior of those Q which minimize (1.1) when the band shrinks in some sense, and the approximant class $\mathcal{S}$ is a finite dimensional linear subspace. In this case, $\mathcal{C}$ is a finite dimensional linear subspace, and the existence of such a Q is well known (see [5]).

We consider that the band shrinks to a curve in two situations:

- (i) The functions f and g are replaced by a family of functions f_ϵ, g_ϵ , where f_ϵ, g_ϵ converge to a function h , as ϵ tends to 0. That is, *the band shrinks vertically*.
- (ii) The interval $[a, b]$ is substituted by $[x_0 - \epsilon, x_0 + \epsilon]$, where ϵ tends to 0; that is, *the band shrinks horizontally*.

If there exists the limit of Q minimizing (1.1) when the band shrinks to a curve, as such, it provides useful qualitative and approximation analytic information concerning the approximants on small bands, which is difficult to obtain from a strictly numerical treatment. The existence of the limit of Q is close to the best local approximation problem (see [6–8]).

In Section 2, we prove that if the band shrinks vertically to a given function, then the set of closure points of Q is contained in the set of best approximants to that function, with a suitable seminorm. Moreover, we see that the limit of Q exists when $p > 1$.

In Section 3, we prove that if the band shrinks horizontally, the limit of Q is the mean of the Taylor polynomials of f and g at x_0 . We also show that this approximation problem is related with the subject of best simultaneous local approximation which was studied in [8].

We assume conditions about the functions f and g in order to be (1.1) finite for all $Q \in \mathcal{S}$. Henceforward, $f, g \in L^q([a, b])$, $\mathcal{S} \subset L^q([a, b])$, and $g - f \in L^{q/(q-p)}([a, b])$, $p \leq q$. In this case, using Hölder's inequality, we have

$$\begin{aligned} \Phi_{[a,b]}^p(F, Q) &\leq \int_a^b \max\{|f(x) - Q(x)|, |g(x) - Q(x)|\}^p (g(x) - f(x)) dx \\ &\leq \|\max\{|f - Q|, |g - Q|\}\|_{q,[a,b]}^p \|g - f\|_{q/(q-p),[a,b]} < \infty, \end{aligned} \quad (1.4)$$

for all $Q \in \mathcal{S}$. If $q \geq p + 1$, the condition $g - f \in L^{q/(q-p)}([a, b])$ is automatically satisfied.

2. The Band Shrinks Vertically

Let g be a measurable nonnegative function in $L^q([a, b])$, $p \leq q$, and $\{f_\epsilon\}, \{g_\epsilon\}$, $\epsilon > 0$, two net of measurable functions such that $|f_\epsilon|, |g_\epsilon| \leq g$, a.e. on $[a, b]$. We write $F_\epsilon(x) = [f_\epsilon(x), g_\epsilon(x)]$, $x \in [a, b]$. Given $\mathcal{S} \subset L^q([a, b])$ a finite dimensional linear subspace, let $Q_\epsilon \in \mathcal{S}$ which minimizes $\Phi_{[a,b]}(F_\epsilon, Q)$, $Q \in \mathcal{S}$.

Theorem 2.1. Assume that there are two functions $W(\epsilon) : (0, \infty) \rightarrow (0, \infty)$ and $h \in L^{q/(q-p)}([a, b])$ such that almost everywhere on $[a, b]$

$$\frac{g_\epsilon - f_\epsilon}{W(\epsilon)} \leq h, \quad (2.1)$$

$$\lim_{\epsilon \rightarrow 0} \frac{g_\epsilon - f_\epsilon}{W(\epsilon)} = w, \quad (2.2)$$

with $|\{w > 0\}| > 0$. If f_ϵ and g_ϵ converge to f almost everywhere on $[a, b]$, then the set of closure points of Q_ϵ is a nonempty set, and it is contained in the set of best approximants to f from $\mathcal{S}$ with the seminorm $\|\cdot\|_{p,w,[a,b]}$. In particular, if $p > 1$, the net Q_ϵ converges to the unique best approximant to f , as $\epsilon \rightarrow 0$.

Proof. For $\epsilon > 0$, we denote $w_\epsilon^* = w_\epsilon / W(\epsilon)$, where $w_\epsilon = g_\epsilon - f_\epsilon$. Let

$$J_{Q,\epsilon}(x) := \int_{f_\epsilon(x)}^{g_\epsilon(x)} |y - Q(x)|^p dy, \quad x \in [a, b], \quad Q \in \mathcal{S}. \quad (2.3)$$

Let $Q \in \mathcal{S}$. By integral mean value theorem, for each $x \in [a, b]$, there exists

$$\alpha_\epsilon(x), \xi_\epsilon(x) \in [f_\epsilon(x), g_\epsilon(x)], \quad (2.4)$$

such that $J_{Q,\epsilon,\epsilon} = |\alpha_\epsilon - Q|^p w_\epsilon$ and $J_{Q,\epsilon} = |\xi_\epsilon - Q|^p w_\epsilon$. Now, the Fubini Theorem implies that $J_{Q,\epsilon,\epsilon}$ and $J_{Q,\epsilon}$ are measurable functions on $[a, b]$, and hence, α_ϵ and ξ_ϵ are measurable functions on $[a, b]$. Consequently, $\alpha_\epsilon, \xi_\epsilon \in L^q([a, b])$, because $|\alpha_\epsilon|, |\xi_\epsilon| \leq g$, a.e. on $[a, b]$.

On the other hand,

$$\begin{aligned} \|f - Q_\epsilon\|_{p,w_\epsilon,[a,b]} - \|\alpha_\epsilon - f\|_{p,w_\epsilon,[a,b]} &\leq \|\alpha_\epsilon - Q_\epsilon\|_{p,w_\epsilon,[a,b]} = \Phi_{[a,b]}(F_\epsilon, Q_\epsilon) \\ &\leq \Phi_{[a,b]}(F_\epsilon, Q) = \|\xi_\epsilon - Q\|_{p,w_\epsilon,[a,b]}. \end{aligned} \quad (2.5)$$

In consequence, we get

$$\|f - Q_\epsilon\|_{p,w_\epsilon^*,[a,b]} \leq \|\alpha_\epsilon - f\|_{p,w_\epsilon^*,[a,b]} + \|\xi_\epsilon - Q\|_{p,w_\epsilon^*,[a,b]}. \quad (2.6)$$

As $|f| \leq g$ a.e. on $[a, b]$, from (2.1) and the Hölder inequality, we have

$$|\alpha_\epsilon - f|^p w_\epsilon^* \leq (2g)^p h \in L^1([a, b]), \quad |\xi_\epsilon - Q|^p w_\epsilon^* \leq (g + |Q|)^p h \in L^1([a, b]). \quad (2.7)$$

From (2.2) and the Lebesgue Dominated Convergence Theorem, it follows that

$$\lim_{\epsilon \rightarrow 0} \left(\|\alpha_\epsilon - f\|_{p, w_\epsilon^*, [a, b]} + \|\xi_\epsilon - Q\|_{p, w_\epsilon^*, [a, b]} \right) = \|f - Q\|_{p, w, [a, b]}. \quad (2.8)$$

By hypothesis, there is $k > 0$ satisfying $|\{w > k\}| > 0$. By the Egoroff Theorem (see [9]), there exists a set $A \subset \{w > k\}$, $|A| > 0$, where we have that w_ϵ^* uniformly converges to w on A . So, we can choose a positive constant ϵ_0 such that for all $x \in A$ and all $0 < \epsilon < \epsilon_0$, $w_\epsilon^*(x) \geq k/2$. Hence,

$$\|f - Q_\epsilon\|_{p, A} \leq \left(\frac{k}{2}\right)^{-1/p} \|f - Q_\epsilon\|_{p, w_\epsilon^*, [a, b]}, \quad 0 < \epsilon < \epsilon_0. \quad (2.9)$$

According to (2.6)–(2.9), $\{Q_\epsilon\}$ is a uniformly bounded net. Then, $\{Q_\epsilon\}$ have a subsequence that we again denote by $\{Q_\epsilon\}$ converging to $T \in \mathcal{S}$. Again, the Lebesgue dominated convergence theorem implies $\lim_{\epsilon \rightarrow 0} \|f - Q_\epsilon\|_{p, w_\epsilon^*, [a, b]} = \|f - T\|_{p, w, [a, b]}$. Finally, from (2.6) and (2.8), we conclude that

$$\|f - T\|_{p, w, [a, b]} \leq \|f - Q\|_{p, w, [a, b]}. \quad (2.10)$$

As $Q \in \mathcal{S}$ is arbitrary, the theorem immediately follows. $\square$

Suppose that $p > 1$. If we have two functions $W_i : (0, \infty) \rightarrow (0, \infty)$ and w_i , $i = 1, 2$, fulfilling the hypothesis of Theorem 2.1, we conclude that Q_ϵ converges to the best approximant to f from $\mathcal{S}$ when we consider $\|\cdot\|_{p, w_1, [a, b]}$ and $\|\cdot\|_{p, w_2, [a, b]}$, respectively. The next lemma shows that it is not surprising, because these norms differing by a constant.

Lemma 2.2. *Let (Ω, Σ, μ) be a finite measure space. Let $w_\epsilon : \Omega \rightarrow \mathbb{R}$, $\epsilon > 0$ be a net of nonnegative measurable functions. Assume that there are two functions $W_i : (0, \infty) \rightarrow (0, \infty)$, $i = 1, 2$ such that almost everywhere on Ω*

$$\lim_{\epsilon \rightarrow 0} \frac{w_\epsilon(x)}{W_i(\epsilon)} = w_i(x) \quad (2.11)$$

is finite. If $A_i := \{x \in \Omega : w_i(x) > 0\}$ and $\mu(A_i) > 0$, $i = 1, 2$, then there exists $k > 0$ satisfying $w_1 = kw_2$, a.e. on Ω .

Proof. We denote by I_i , $i = 1, 2$, the subsets of Ω , where w_i is finite, and we write $K = I_1 \cap I_2$. Clearly, $\mu(K) = \mu(\Omega)$. Let $J_i = A_i \cap K$.

If $\mu(J_1 \cap J_2) = 0$, then $\mu(J_1 - J_2) = \mu(A_1) > 0$ and $\mu(J_2 - J_1) = \mu(A_2) > 0$. We take $x_1 \in J_1 - J_2$ and $x_2 \in J_2 - J_1$. The equality

$$\frac{w_\epsilon(x)}{W_2(\epsilon)} = \frac{w_\epsilon(x)}{W_1(\epsilon)} \frac{W_1(\epsilon)}{W_2(\epsilon)}, \quad x \in \Omega \quad (2.12)$$

implies that $W_1(\epsilon)/W_2(\epsilon)$ tends to zero if $x = x_1$, and it tends to infinite if $x = x_2$, a contradiction. So, we get $\mu(J_1 \cap J_2) > 0$.

For all $x \in J_1 \cap J_2$, we have

$$0 < k := \lim_{\epsilon \rightarrow 0} \frac{W_1(\epsilon)}{W_2(\epsilon)} = \frac{w_2(x)}{w_1(x)}. \quad (2.13)$$

Next, we prove that $\mu(J_1 - J_2) = \mu(J_2 - J_1) = 0$. If $\mu(J_1 - J_2) > 0$ the above argument implies that $k = 0$, and if $\mu(J_2 - J_1) > 0$, we obtain $k = \infty$. In either case, it contradicts (2.13). Therefore, $\mu(J_1 \cap J_2) = \mu(J_1 \cup J_2) = \mu(A_1) = \mu(A_2)$ and

$$w_2(x) = kw_1(x), \quad \text{a.e. on } A_1 \cup A_2. \quad (2.14)$$

It immediately follows that (2.14) holds a.e. on Ω . $\square$

We also observe in the next example that (2.2) in Theorem 2.1 is essential.

Example 2.3. Let $a = 0, b = 1$, and $p = 2$. We consider $\mathcal{S}$ the space of constant functions,

$$f_m(x) = x, \quad g_m = \begin{cases} x + \frac{1}{m}, & \text{if } m \text{ is even,} \\ x + \frac{x}{m}, & \text{if } m \text{ is odd.} \end{cases} \quad (2.15)$$

Here, $m(g_m - f_m)$, is equal to 1 if m is even, and it is equal x in otherwise. Applying Theorem 2.1, we get $Q_{2m} \rightarrow 1/2$ and $Q_{2m-1} \rightarrow 2/3$, as $m \rightarrow \infty$.

3. The Band Shrinks Horizontally

Let $x_0 \in [a, b]$ and $I_\epsilon := [x_0 - \epsilon, x_0 + \epsilon] \subset [a, b]$, $\epsilon > 0$. In this section, we assume that f and g have continuous derivatives up to order n at x_0 . From now on, $\mathcal{S} = \Pi^n$ is the space of algebraic polynomials of degree at most n , and Q_ϵ denotes an element of Π^n that minimizes $\Phi_{I_\epsilon}(F, Q)$, $Q \in \Pi^n$.

We recall (see [10, 11]) the Newton divided difference formula for the interpolation polynomial of a function h of degree n at $x_1 \leq x_2 \leq \dots \leq x_{n+1}$,

$$P(x) = h(x_1) + (x - x_1)h[x_1, x_2] + \dots + (x - x_1) \cdots (x - x_n)h[x_1, \dots, x_{n+1}]. \quad (3.1)$$

Here, $h[x_1, \dots, x_{m+1}]$ denotes the m -th order Newton divided difference. If h has continuous derivatives up to order m , on an interval $[a, b]$ containing to $x_1, \dots, x_{m+1}$, then the m -th divided difference can be expressed as

$$h[x_1, \dots, x_{m+1}] = \frac{h^{(m)}(\xi)}{m!}, \quad (3.2)$$

for some ξ in the interval $[x_1, x_{m+1}]$. It is well known that the m -th divided difference is a continuous function as function of their arguments $x_1, \dots, x_{m+1}$.

For simplicity of notation, we write $T(f)$ and $T(g)$ the Taylor polynomials of f and g at x_0 of order n , respectively.

Lemma 3.1. *Let $q \geq 0$, and let $f_i : [a, b] \rightarrow \mathbb{R}$, $1 \leq i \leq 3$ be continuous functions with $f_1 \leq f_2$. If*

$$h(x) = \int_{f_1(x)}^{f_2(x)} |y - f_3(x)|^q \operatorname{sgn}(y - f_3(x)) dy, \quad a \leq x \leq b, \quad (3.3)$$

then h is a continuous function.

Proof. If $q > 0$, the continuity of h follows from the continuity of f_i , $1 \leq i \leq 3$ and the uniform continuity of $t^q \operatorname{sgn}(t)$ on compact sets. If $q = 0$, we have

$$h(x) = \begin{cases} f_1(x) - f_2(x), & \text{if } f_3(x) \geq f_2(x), \\ f_2(x) - f_1(x), & \text{if } f_3(x) \leq f_1(x), \\ f_1(x) + f_2(x) - 2f_3(x), & \text{if } f_1(x) \leq f_3(x) \leq f_2(x). \end{cases} \quad (3.4)$$

Thus, the continuity of h is immediate. □

Theorem 3.2. *If $p \geq 1$, then any net Q_ϵ converges to $(T(f) + T(g))/2$, as $\epsilon \rightarrow 0$.*

Proof. It is easy to see that Q_ϵ is characterized by

$$\int_{I_\epsilon} h(x) Q(x) dx = 0, \quad Q \in \Pi^n, \quad (3.5)$$

where

$$h(x) = \int_{f(x)}^{g(x)} |y - Q_\epsilon(x)|^{p-1} \operatorname{sgn}(y - Q_\epsilon(x)) dy. \quad (3.6)$$

By Lemma 3.1, h is continuous. Therefore, h interpolates to zero in at least $n + 1$ different points of the interval I_ϵ , say $x_1 < x_2 < \dots < x_{n+1}$. In fact, if h has m different zeros, $m \leq n$, we can find an element $Q \in \Pi^n$ such that $h(x)Q(x) > 0$ for all $x \neq x_i$, $1 \leq i \leq m$. It contradicts (3.5). So,

$$\int_{f(x_i)}^{g(x_i)} |y - Q_\epsilon(x_i)|^{p-1} \operatorname{sgn}(y - Q_\epsilon(x_i)) dy = 0, \quad 1 \leq i \leq n + 1. \quad (3.7)$$

It follows that $Q_\epsilon(x_i)$ is a best constant approximant to the identity function with norm $\|\cdot\|_{p,[f(x_i),g(x_i)]}$, $1 \leq i \leq n+1$. A straightforward computation shows that $Q_\epsilon(x_i) = (f(x_i)+g(x_i))/2$, $1 \leq i \leq n+1$; that is, Q_ϵ interpolates to the function $(f+g)/2$ at x_i , $1 \leq i \leq n+1$. From (3.1) and (3.2), it follows that

$$Q_\epsilon(x) = h(x_1) + (x - x_1)h^{(1)}(\xi_1) + \cdots + (x - x_1) \cdots (x - x_n) \frac{h^{(n)}(\xi_n)}{n!}, \quad (3.8)$$

where $h = (f+g)/2$, $\xi_i \in I_\epsilon$, $1 \leq i \leq n$. Taking limit for $\epsilon \rightarrow 0$ in (3.8) and using the continuity of the derivatives of the functions f and g , we get the theorem. $\square$

Corollary 3.3. *Let $p \geq 1$. Suppose that f and g have continuous derivatives up to order $n+1$ at x_0 and $f(x_0) < g(x_0)$. Then, for sufficiently small ϵ , Q_ϵ is the best l^{p+1} -simultaneous approximant in $L^{p+1}(I_\epsilon)$; that is, Q_ϵ minimizes*

$$\left(\|f - Q\|_{p+1,I_\epsilon}^{p+1} + \|g - Q\|_{p+1,I_\epsilon}^{p+1} \right)^{1/(p+1)}, \quad Q \in \Pi^n. \quad (3.9)$$

Proof. By hypothesis, there is $\epsilon_0 > 0$ such that $f < g$ on I_{ϵ_0} . Let P_ϵ be the best l^{p+1} -simultaneous approximant to f and g in $L^{p+1}(I_\epsilon)$. From Theorem 3.2, and [8, Theorem 3.4], there exists $0 < \epsilon_1 < \epsilon_0$ such that

$$Q_\epsilon(x), P_\epsilon(x) \in [f(x), g(x)], \quad x \in I_\epsilon, \quad \epsilon < \epsilon_1. \quad (3.10)$$

On the other hand, we have

$$\begin{aligned} (p+1)\Phi_{I_\epsilon}^p(F, Q) &= \int_{I_\epsilon} |f(x) - Q(x)|^{p+1} \operatorname{sgn}(Q(x) - f(x)) dx \\ &\quad + \int_{I_\epsilon} |g(x) - Q(x)|^{p+1} \operatorname{sgn}(g(x) - Q(x)) dx, \quad Q \in \Pi^n, \end{aligned} \quad (3.11)$$

From (3.10) and (3.11), we get

$$\|f - Q_\epsilon\|_{p+1,I_\epsilon}^{p+1} + \|g - Q_\epsilon\|_{p+1,I_\epsilon}^{p+1} = \|f - P_\epsilon\|_{p+1,I_\epsilon}^{p+1} + \|g - P_\epsilon\|_{p+1,I_\epsilon}^{p+1}, \quad (3.12)$$

for all $\epsilon < \epsilon_1$. So, $Q_\epsilon = P_\epsilon$ for all $\epsilon < \epsilon_1$. $\square$

Remark 3.4. If $\Phi_{[a,b]}(F, Q) = \sup_{x \in [a,b]} \sup_{y \in F(x)} |y - Q(x)|$, then

$$\Phi_{[a,b]}(F, Q) = \sup_{x \in [a,b]} \max\{|f(x) - Q(x)|, |g(x) - Q(x)|\} \quad (3.13)$$

implies that Q_ϵ is the best l^∞ -simultaneous approximant in $L^\infty([a, b])$, for all $\epsilon > 0$.

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References

- [1] B. Sendov, *Hausdorff Approximation, Mathematic and It Application*, Kluwer Academic, Dodrecht, The Netherlands, 1990.
- [2] J. B. Diaz and H. W. McLaughlin, "Simultaneous approximation of a set of bounded real functions," *Mathematics of Computation*, vol. 23, pp. 583–593, 1969.
- [3] P. D. Milman, "On best simultaneous approximation in normed linear spaces," *Journal of Approximation Theory*, vol. 20, no. 2, pp. 223–238, 1977.
- [4] S. Tanimoto, "On best simultaneous approximation," *Mathematica Japonica*, vol. 48, no. 2, pp. 275–279, 1998.
- [5] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer, New York, NY, USA, 1970.
- [6] C. K. Chui, H. Diamond, and L. A. Raphael, "On best data approximation," *Approximation Theory and Its Applications*, vol. 1, no. 1, pp. 37–56, 1984.
- [7] C. K. Chui, O. Shisha, and P. W. Smith, "Best local approximation," *Journal of Approximation Theory*, vol. 15, no. 4, pp. 371–381, 1975.
- [8] H. H. Cuenya and F. E. Levis, "Interpolation and best simultaneous approximation," *Journal of Approximation Theory*, vol. 162, no. 9, pp. 1577–1587, 2010.
- [9] R. Royden, *Real Analysis*, The Macmillan Company, London, UK, 1968.
- [10] S. D. Conte and C. De Boor, *Elementary Numerical Analysis*, McGraw-Hill, New York, NY, USA, 1972.
- [11] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, vol. 12 of *Texts in Applied Mathematics*, Springer, New York, NY, USA, 3rd edition, 2002.