International Scholarly Research Network
ISRN Mathematical Analysis
Volume 2011, Article ID 149830, 11 pages
doi:10.5402/2011/149830

Research Article

General Properties for Volterra-Type Operators in the Unit Disk

Rabha W. Ibrahim and Maslina Darus

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi, Selangor Darul Ehsan 43600, Malaysia

Correspondence should be addressed to Maslina Darus, maslina@ukm.my

Received 26 October 2010; Accepted 5 December 2010

Academic Editors: J. Colliander, B. Kaltenbacher, and A. L. Sasu

The object of this paper is to study general properties such as boundedness, compactness, and geometric properties for two integral operators of Volterra-Type in the unit disk.

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $U := \{z \in \mathbb{C} : |z| < 1\}$. Suppose that $g : U \to \mathbb{C}$ is a holomorphic map, $f \in \mathcal{H}$. The integral operator, called Volterra-type operator,

$$J_g f(z) = \int_0^z f\,dg = \int_0^1 f(tz)zg'(tz)dt = \int_0^z f(\xi)g'(\xi)d\xi, \quad z \in U, \tag{1.1}$$

was introduced by Pommerenke in [1]. Another natural integral operator is defined as follows:

$$I_g f(z) = \int_0^z f'(\xi)g(\xi)d\xi, \quad z \in U. \tag{1.2}$$

The importance of the operators J_g and I_g comes from the fact that

$$J_g f + I_g f = M_g f - f(0)g(0), \tag{1.3}$$

where the multiplication operator M_g is defined by

$$(M_g f)(z) = g(z)f(z), \quad f \in \mathcal{A},\ z \in U. \tag{1.4}$$

Furthermore, Volterra integral equations arise in many physical applications (see [2–4]).

In the past few years, many authors focused on the boundedness and compactness of Volterra-type integral operator between several spaces of holomorphic functions. In [1], Pommerenke showed that J_g is a bounded operator on the Hardy space H^2. The boundedness and compactness of $J_g f$ and $I_g f$ between some spaces of analytic functions, as well as their n-dimensional extensions, were investigated in [5–11].

For functions $f \in \mathcal{A}$, the integral operators $J_g f$ and $I_g f$ contain well-known integral operators in the analytic function theory and geometric function theory such as the generalized Bernardi-Libera-Livingston linear integral operator (cf. [12–14]) and the Srivastava-Owa fractional derivative operators (cf. [15, 16]). Recently, Breaz and Breaz introduced two integral operators of analytic functions taking the form (1.1) and (1.2) (see [17]). Further, the integral operators of Volterra-Type involving the integral operators were studied in [18–22]. Finally, these operators are involving the Cesáro integral operator (see [23–25]).

A function $f \in \mathcal{A}$ is called in the class Σ if and only if it has the norm (see [26])

$$\|f\| = \sup_{z\in U}\left(1-|z|^2\right)\left|\frac{f''(z)}{f'(z)}\right| < \infty, \quad (z \in U). \tag{1.5}$$

Note that the fraction $T_f := f''(z)/f'(z)$ is called pre-Schwarzian derivative which is usually used to discuss the univalency of analytic functions (see [27]). Moreover, the norm in (1.5) is a modification to one defined in [28].

The purpose of this paper is to study the boundedness, compactness, and some geometric properties of the integral operators $J_g f$ and $I_g f$ for the functions $f \in \Sigma$ and g is an analytic function on the open unit disk.

2. The Boundedness and Compactness

In this section, we consider the boundedness and compactness of the operators $J_g f$ and $I_g f$ on the classes Σ.

Consider the space $\mathcal{B}_{\log}$ of all functions $f \in \mathcal{A}$ which are satisfying

$$\|f\|_{\mathcal{B}_{\log}} = \sup_{z\in U}\left(1-|z|^2\right)\left|\frac{f'(z)}{f(z)}\right| \ln\frac{1}{\left(1-|z|^2\right)} < \infty, \quad (z \in U). \tag{2.1}$$

Theorem 2.1. *Assume that g is an analytic function on U. Then, for functions $f \in \mathcal{B}_{\log}$, J_g is bounded if and only if $g \in \Sigma$.*

Proof. Assume that J_g is bounded. Taking the function given by $f(z) = 1$, we see that $g \in \Sigma$. Conversely, assume that $g \in \Sigma$, we have

$$
\begin{aligned}
\left(1-|z|^2\right)\left|\frac{(J_g f)''(z)}{(J_g f)'(z)}\right| &= \left(1-|z|^2\right)\left|\frac{f(z)g''(z)+f'(z)g'(z)}{f(z)g'(z)}\right| \\
&= \left(1-|z|^2\right)\left[\left|\frac{g''(z)}{g'(z)}+\frac{f'(z)}{f(z)}\right|\right] \\
&\le \|g\| + \frac{\left(1-|z|^2\right)|f'(z)/f(z)|\ln\left[1/\left(1-|z|^2\right)\right]}{\ln\left[1/\left(1-|z|^2\right)\right]} \\
&\le \|g\| + \frac{\|f\|_{\mathcal{B}_{\log}}}{\ln\left[1/\left(1-|z|^2\right)\right]} \\
&\le \|g\| + \frac{\|f\|_{\mathcal{B}_{\log}}}{\ln\left[1/\left(1-|z|^2\right)\right]\left(1-|z|^2\right)}.
\end{aligned}
\tag{2.2}
$$

By taking the supremum for the last assertion over U and using the fact that the quantity

$$
\sup_{x\in(0,1]} x\left(\ln\frac{1}{x}\right)
\tag{2.3}
$$

is finite, the boundedness of the operator J_g follows. □

Theorem 2.2. *Assume that g is an analytic function on U. Then, $I_g : \Sigma \to \Sigma$ is bounded if and only if $g \in \mathcal{B}_{\log}$, where*

$$
\|g\|_{\mathcal{B}_{\log}} = \sup_{z\in U}\left(1-|z|^2\right)\left|\frac{g'(z)}{g(z)}\right|\ln\frac{1}{\left(1-|z|^2\right)}.
\tag{2.4}
$$

Proof. Assume that $g \in \mathcal{B}_{\log}$. Then, we obtain

$$
\begin{aligned}
\left(1-|z|^2\right)\left|\frac{(I_g f)''(z)}{(I_g f)'(z)}\right| &= \left(1-|z|^2\right)\left|\frac{f''(z)g(z)+f'(z)g'(z)}{f'(z)g(z)}\right| \\
&= \left(1-|z|^2\right)\left[\left|\frac{f''(z)}{f'(z)}+\frac{g'(z)}{g(z)}\right|\right] \\
&\le \|f\| + \frac{\left(1-|z|^2\right)|g'(z)/g(z)|\ln\left[1/\left(1-|z|^2\right)\right]}{\ln\left[1/\left(1-|z|^2\right)\right]}
\end{aligned}
$$

$$
\begin{aligned}
&\le \|f\| + \frac{\|g\|_{\mathcal{B}_{\log}}}{\ln\left[1/\left(1-|z|^2\right)\right]}, \quad (z \in U) \\
&\le \|f\| + \frac{\|g\|_{\mathcal{B}_{\log}}}{\ln\left[1/\left(1-|z|^2\right)\right]\left(1-|z|^2\right)}.
\end{aligned}
\tag{2.5}
$$

By taking the supremum for the last assertion over U, the boundedness of the operator I_g follows.

Conversely, assume that $I_g : \Sigma \to \Sigma$ is bounded, then there is a positive constant C such that

$$
\|I_g f\| \le C\|f\| \tag{2.6}
$$

for every $f \in \Sigma$. Set

$$
h_a(z) = \frac{(\overline{a}z-1)}{\overline{a}}\left[\left(1+\ln\frac{1}{1-\overline{a}z}\right)^2+1\right]\left[\ln\frac{1}{1-|a|^2}\right]^{-1}, \tag{2.7}
$$

for $a \in U$ such that $\sqrt{1-(1/e)} < |a| < 1$. Then, we have

$$
\begin{aligned}
h_a'(z) &= \left(\ln\frac{1}{1-\overline{a}z}\right)^2\left[\ln\frac{1}{1-|a|^2}\right]^{-1}, \\
h_a''(z) &= \frac{2\overline{a}}{1-\overline{a}z}\left(\ln\frac{1}{1-\overline{a}z}\right)\left[\ln\frac{1}{1-|a|^2}\right]^{-1}.
\end{aligned}
\tag{2.8}
$$

Thus,

$$
\frac{h_a''(z)}{h_a'(z)} = \frac{2\overline{a}}{1-\overline{a}z}\left[\ln\frac{1}{1-\overline{a}z}\right]^{-1}, \tag{2.9}
$$

and then

$$
\frac{h_a''(a)}{h_a'(a)} = \frac{2\overline{a}}{1-|a|^2}\left[\ln\frac{1}{1-|a|^2}\right]^{-1}. \tag{2.10}
$$

It is clear that the relation (2.9) is finite when $|z| < 1$, hence $\|h_a(z)\| < \infty$. Setting

$$
M := \sup_{\sqrt{1-(1/e)}<|a|<1} \|h_a(z)\| < \infty, \tag{2.11}
$$

therefore, we have

$$\begin{aligned}
\infty &> \|I_g\|\|h_a\| \geq \|I_g h_a\| \\
&\geq \sup_{z\in U}\left(1-|z|^2\right)\left|\frac{h_a''(z)}{h_a'(z)}+\frac{g'(z)}{g(z)}\right| \\
&\geq \left(1-|a|^2\right)\left|\frac{h_a''(a)}{h_a'(a)}+\frac{g'(a)}{g(a)}\right| \\
&\geq \left|\frac{2\overline{a}}{\ln\left(1/\left(1-|a|^2\right)\right)}+\left(1-|a|^2\right)\frac{g'(a)}{g(a)}\right| \\
&\geq \frac{-2|a|+\left(1-|a|^2\right)\left|g'(a)/g(a)\right|\ln\left(1/\left(1-|a|^2\right)\right)}{\ln\left(1/\left(1-|a|^2\right)\right)}.
\end{aligned} \tag{2.12}$$

Now letting

$$f_a(z) := 2\frac{(\overline{a}z-1)}{\overline{a}}\left[\left(1+\ln\frac{1}{1-\overline{a}z}\right)^2+1\right]\left[\ln\frac{1}{1-|a|^2}\right]^{-1}-\int_0^z \ln\frac{1}{1-\overline{a}x}dx \tag{2.13}$$

for $a\in U$ such that $\sqrt{1-(1/e)}<|a|<1$. Then, we obtain

$$\begin{aligned}
f_a'(z) &= 2\left(\ln\frac{1}{1-\overline{a}z}\right)^2\left[\ln\frac{1}{1-|a|^2}\right]^{-1}-\ln\frac{1}{1-\overline{a}z}, \\
f_a''(z) &= \frac{4\overline{a}}{1-\overline{a}z}\left(\ln\frac{1}{1-\overline{a}z}\right)\left[\ln\frac{1}{1-|a|^2}\right]^{-1}-\frac{\overline{a}}{1-\overline{a}z}.
\end{aligned} \tag{2.14}$$

Thus, we conclude that

$$\frac{f_a''(a)}{f_a'(a)} = \frac{3|a|/\left(1-|a|^2\right)}{\ln\left(1/\left(1-|a|^2\right)\right)}. \tag{2.15}$$

In the same manner of the previous case, we have

$$N := \sup_{\sqrt{1-(1/e)}<|a|<1}\|f_a\| < \infty. \tag{2.16}$$

Consequently, we have

$$\begin{aligned}
\infty &> \|I_g\|\|f_a\| \geq \|I_g f_a\| \\
&\geq \sup_{z\in U}\left(1-|z|^2\right)\left|\frac{f_a''(z)}{f_a'(z)}+\frac{g'(z)}{g(z)}\right| \\
&\geq \left(1-|a|^2\right)\left|\frac{f_a''(a)}{f_a'(a)}+\frac{g'(a)}{g(a)}\right| \\
&\geq \left(1-|a|^2\right)\left|\frac{3|a|/\left(1-|a|^2\right)}{\ln\left(1/\left(1-|a|^2\right)\right)}+\frac{g'(a)}{g(a)}\right| \\
&\geq \frac{-3|a|+\left(1-|a|^2\right)|g'(a)/g(a)|\ln\left(1/\left(1-|a|^2\right)\right)}{\ln\left(1/\left(1-|a|^2\right)\right)}.
\end{aligned} \tag{2.17}$$

From (2.12) and (2.17), we have

$$\left(1-|a|^2\right)\left|\frac{g'(a)}{g(a)}\right|\ln\frac{1}{\left(1-|a|^2\right)} < \infty \tag{2.18}$$

for all $\sqrt{1-(1/e)} < |a| < 1$. Also, we have

$$\sup_{|a|\leq\sqrt{1-(1/e)}}\left(1-|a|^2\right)\left|\frac{g'(a)}{g(a)}\right|\ln\frac{1}{\left(1-|a|^2\right)} \leq \sup_{\sqrt{1-(1/e)}\leq|a|<1}\left(1-|a|^2\right)\left|\frac{g'(a)}{g(a)}\right|\ln\frac{1}{\left(1-|a|^2\right)}. \tag{2.19}$$

From (2.18) and (2.19), we obtain $g \in \mathcal{B}_{\log}$, as desired.

In the following results, we study the compactness of the integral operators J_g and I_g in an open disc. □

Theorem 2.3. *Assume that g is an analytic function on U. Then, for functions $f \in \mathcal{B}_{\log}$, the integral operator J_g is compact if and only if $g \in \Sigma$.*

Proof. If J_g is compact, then it is bounded, and by Theorem 2.1 it follows that $g \in \Sigma$.

Now assume that $g \in \Sigma$, that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{B}_{\log}$, and $f_n \to 0$ uniformly on $\overline{U}$ as $n \to \infty$. Now for every $\varepsilon > 0$, there is $\delta \in (0,1)$ such that

$$\frac{1}{1-|z|^2} < \varepsilon, \tag{2.20}$$

where $\delta < |z| < 1$. Since δ is arbitrary, then we can chose $\ln(1/(1-|z|^2)) > 1$ for $\delta < |z| < 1$ and

$$\begin{aligned}
\|J_g f_n\| &= \sup_{z\in U}\left(1-|z|^2\right)\left|\frac{(J_g f_n)''(z)}{(J_g f_n)'(z)}\right| \\
&= \sup_{z\in U}\left(1-|z|^2\right)\left|\frac{f_n(z)g''(z)+f_n'(z)g'(z)}{f_n(z)g'(z)}\right| \\
&\le \sup_{z\in U}\left(1-|z|^2\right)\left|\frac{g''(z)}{g'(z)}\right| + \sup_{z\in U}\left(1-|z|^2\right)\left|\frac{f_n'(z)}{f_n(z)}\right|\left(\ln\frac{1}{1-|z|^2}\right) \\
&\le \frac{\|g\|}{1-|z|^2} + \|f_n\|_{\mathcal{B}_{\log}} \\
&< \varepsilon\|g\| + \|f_n\|_{\mathcal{B}_{\log}}.
\end{aligned} \tag{2.21}$$

Since for $f_n \to 0$ on $\overline{U}$ we have $\|f_n\|_{\mathcal{B}_{\log}} \to 0$, and that ε is an arbitrary positive number, by letting $n \to \infty$ in the last inequality, we obtain that $\lim_{n\to\infty}\|J_g f_n\| = 0$. Therefore, J_g is compact. □

Theorem 2.4. *Assume that g is an analytic function on U. Then, the integral operator $I_g : \Sigma \to \Sigma$ is compact if and only if g is a constant defer from zero.*

Proof. Assume that g is a constant without loss of generality and assume that $f(z) = z$. Then, it is clear that I_g is compact.

Conversely, assume that $I_g : \Sigma \to \Sigma$ is compact. Let $(z_n)_{n\in\mathbb{N}}$, be a sequence in U such that $|z_n| \to 1$ as $n \to \infty$. Our aim is to show that $g'(z_n) \to 0$ as $n \to \infty$, then by the maximum modulus theorem, we have g is a constant. In fact, setting

$$f_n(z) = 2\frac{(\overline{z}_n z - 1)}{\overline{z}_n}\left[\left(1+\ln\frac{1}{1-\overline{z}_n z}\right)^2 + 1\right]\left[\ln\frac{1}{1-|z|^2}\right]^{-1} - 4\int_0^z \ln\frac{1}{1-\overline{z}_n w}dw. \tag{2.22}$$

Then, we obtain

$$\begin{aligned}
f_n'(z) &= 2\left(\ln\frac{1}{1-\overline{z}_n z}\right)^2\left[\ln\frac{1}{1-|z|^2}\right]^{-1} - 4\left[\ln\frac{1}{1-\overline{z}_n z}\right], \\
f_n''(z) &= \frac{4\overline{z}_n}{1-\overline{z}_n z}\left(\ln\frac{1}{1-\overline{z}_n z}\right)\left[\ln\frac{1}{1-|z|^2}\right]^{-1} - \frac{4\overline{z}_n}{1-\overline{z}_n z}.
\end{aligned} \tag{2.23}$$

Consequently, we have

$$\frac{f_n''(z_n)}{f_n'(z_n)} = 0. \tag{2.24}$$

Similar to the proof of Theorem 2.2, we see that $f_n \to 0$ uniformly on $\overline{U}$. Since $I_g : \Sigma \to \Sigma$ is compact, then we get

$$\|I_g f_n\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.25}$$

Thus,

$$\begin{aligned} \left|\frac{g'(z_n)}{g(z_n)}\right| &\leq \sup_{z\in U}\left|\frac{g'(z)}{g(z)}\right| + \sup_{z\in U}\left|\frac{f_n''(z)}{f_n'(z)}\right| \\ &\leq \|I_g f_n\| \longrightarrow 0 \end{aligned} \tag{2.26}$$

Implies that $g_n'(z) \to 0$ and consequently g is a constant as desired. □

3. Some Geometric Properties

In this section, we introduce some geometric properties for analytic function $f \in \Sigma$. A function $f \in \mathcal{H}$ which normalized as $f(0) = f'(0) - 1 = 0$ denoted this class by $\mathcal{A}$. Recall that a function $f \in \mathcal{A}$ is said to be star-like of order $\mu \in [0,1)$ in U if it satisfies

$$f \in \mathcal{S}_\mu \Longleftrightarrow \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \mu, \quad (z \in U). \tag{3.1}$$

Also, a function $f \in \mathcal{A}$ is called convex in U if it satisfies

$$f \in \mathcal{K}_\mu \Longleftrightarrow \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \mu, \quad (z \in U). \tag{3.2}$$

It follows that

$$f \in \mathcal{K}_\mu \Longleftrightarrow zf' \in \mathcal{S}_\mu. \tag{3.3}$$

In the next result, we discuss the convexity of the integral operators J_g and I_g.

Theorem 3.1. *Assume that $f, g \in \mathcal{A}$. If $f \in \mathcal{S}_\mu$ and $g \in \mathcal{K}_\nu$ such that $0 \leq \mu + \nu < 1$, then the function $J_g f$ is convex of order $\mu + \nu$.*

Proof. Assume that $f, g \in \mathcal{A}$. Then, we obtain

$$\frac{z(J_g f)''(z)}{(J_g f)'(z)} = \frac{zg''(z)}{g'(z)} + \frac{zf'(z)}{f(z)}. \tag{3.4}$$

Consequently, we get

$$\Re\left\{1+\frac{z(J_g f)''(z)}{(J_g f)'(z)}\right\}=\Re\left\{\frac{zg''(z)}{g'(z)}+1\right\}+\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\mu+\nu. \tag{3.5}$$

Hence, $J_g \in \mathcal{K}_{\mu+\nu}$. □

Theorem 3.2. *Assume that $f, g \in \mathcal{A}$. If $f \in \mathcal{K}_\mu$ and $g \in \mathcal{S}_\nu$ such that $0 \leq \mu+\nu < 1$, then the function $I_g f$ is convex of order $\mu+\nu$.*

Proof. Assume that $f, g \in \mathcal{A}$. Then, we have

$$\frac{z(I_g f)''(z)}{(I_g f)'(z)}=\frac{zf''(z)}{f'(z)}+\frac{zg'(z)}{g(z)}. \tag{3.6}$$

Consequently, we get

$$\Re\left\{1+\frac{z(I_g f)''(z)}{(I_g f)'(z)}\right\}=\Re\left\{\frac{zf''(z)}{f'(z)}+1\right\}+\Re\left\{\frac{zg'(z)}{g(z)}\right\}>\mu+\nu. \tag{3.7}$$

Hence, $I_g \in \mathcal{K}_{\mu+\nu}$. □

Theorem 3.3. *Assume that $f, g \in \mathcal{A}$. If $f \in \mathcal{S}_\mu$ and $g \in \mathcal{S}_\nu$ such that $0 \leq \mu+\nu < 1$, then the multiplication operator $M_g f$ is star-like of order $\mu+\nu$.*

Proof. Assume that $f, g \in \mathcal{A}$. Then, we obtain

$$\Re\left\{\frac{z(M_g f)'(z)}{(M_g f)(z)}\right\}=\Re\left\{\frac{zf'(z)}{f(z)}\right\}+\Re\left\{\frac{zg'(z)}{g(z)}\right\}>\mu+\nu. \tag{3.8}$$

Hence, $M_g f \in \mathcal{S}_{\mu+\nu}$.

The next result comes directly from the definition of the class Σ and the fact that $\|T_f\| < \infty$ if and only if f is uniformly locally univalent (see [23]). □

Theorem 3.4. *Assume that g is an analytic function on U and $f \in \mathcal{A}$. Then, the functions $I_g f$ and $J_g f$ are in the class Σ if and only if f is locally univalent in U.*

Acknowledgment

The work presented here was supported by the MOHE Grant no. UKM-ST-06-FRGS0107-2009, Malaysia.

References

[1] Ch. Pommerenke, "Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation," *Commentarii Mathematici Helvetici*, vol. 52, no. 4, pp. 591–602, 1977.

[2] D. Rostami and K. Maleknejad, "Preconditioners for solving stochastic boundary integral equations with weakly singular kernels," *Computing*, vol. 63, no. 1, pp. 47–67, 1999.

[3] K. Maleknejad and N. Aghazadeh, "Numerical solution of Volterra integral equations of the second kind with convolution kernel by using Taylor-series expansion method," *Applied Mathematics and Computation*, vol. 161, no. 3, pp. 915–922, 2005.

[4] Y. Ren, B. Zhang, and H. Qiao, "A simple Taylor-series expansion method for a class of second kind integral equations," *Journal of Computational and Applied Mathematics*, vol. 110, no. 1, pp. 15–24, 1999.

[5] A. Aleman and J. A. Cima, "An integral operator on H^p and Hardy's inequality," *Journal d'Analyse Mathématique*, vol. 85, pp. 157–176, 2001.

[6] G. Benke and D.-C. Chang, "A note on weighted Bergman spaces and the Cesàro operator," *Nagoya Mathematical Journal*, vol. 159, pp. 25–43, 2000.

[7] D.-C. Chang, R. Gilbert, and J. Tie, "Bergman projection and weighted holomorphic functions," in *Reproducing Kernel Spaces and Applications*, vol. 143 of *Operator Theory: Advances and Applications*, pp. 147–169, Birkhäuser, Basel, Switzerland, 2003.

[8] D.-C. Chang and S. Stević, "The generalized Cesàro operator on the unit polydisk," *Taiwanese Journal of Mathematics*, vol. 7, no. 2, pp. 293–308, 2003.

[9] Z. Hu, "Extended Cesàro operators on mixed norm spaces," *Proceedings of the American Mathematical Society*, vol. 131, no. 7, pp. 2171–2179, 2003.

[10] A. G. Siskakis and R. Zhao, "A Volterra type operator on spaces of analytic functions," in *Function Spaces (Edwardsville, IL, 1998)*, vol. 232 of *Contemporary Mathematics*, pp. 299–311, American Mathematical Society, Providence, RI, USA, 1999.

[11] X. Zhu, "Volterra type operators from logarithmic Bloch spaces to Zygmund type spaces," *International Journal of Modern Mathematics*, vol. 3, no. 3, pp. 327–336, 2008.

[12] S. D. Bernardi, "Convex and starlike univalent functions," *Transactions of the American Mathematical Society*, vol. 135, pp. 429–446, 1969.

[13] R. J. Libera, "Some classes of regular univalent functions," *Proceedings of the American Mathematical Society*, vol. 16, pp. 755–758, 1965.

[14] A. E. Livingston, "On the radius of univalence of certain analytic functions," *Proceedings of the American Mathematical Society*, vol. 17, pp. 352–357, 1966.

[15] S. Owa, "On the distortion theorems. I," *Kyungpook Mathematical Journal*, vol. 18, no. 1, pp. 53–59, 1978.

[16] S. Owa and H. M. Srivastava, "Univalent and starlike generalized hypergeometric functions," *Canadian Journal of Mathematics*, vol. 39, no. 5, pp. 1057–1077, 1987.

[17] D. Breaz and N. Breaz, "Two integral operator," *Studia Universitatis Babes-Bolyai, Mathematica, Cluj-Napoca*, no. 3, pp. 13–21, 2002.

[18] J.-L. Liu, "Some applications of certain integral operator," *Kyungpook Mathematical Journal*, vol. 43, no. 2, pp. 211–219, 2003.

[19] J.-L. Liu, "Notes on Jung-Kim-Srivastava integral operator," *Journal of Mathematical Analysis and Applications*, vol. 294, no. 1, pp. 96–103, 2004.

[20] B. A. Uralegaddi and C. Somanatha, "Certain integral operators for starlike functions," *Journal of Mathematical Research and Exposition*, vol. 15, no. 1, pp. 14–16, 1995.

[21] J. L. Li, "Some properties of two integral operators," *Soochow Journal of Mathematics*, vol. 25, no. 1, pp. 91–96, 1999.

[22] S. Owa, "Properties of certain integral operators," *Georgian Mathematical Journal*, vol. 2, no. 5, pp. 535–545, 1995.

[23] A. G. Siskakis, "The Cesàro operator is bounded on H^1," *Proceedings of the American Mathematical Society*, vol. 110, no. 2, pp. 461–462, 1990.

[24] A. G. Siskakis, "Composition semigroups and the Cesàro operator on H^p," *Journal of the London Mathematical Society II*, vol. 36, no. 1, pp. 153–164, 1987.

[25] J. Miao, "The Cesàro operator is bounded on H^p for $0 < p < 1$," *Proceedings of the American Mathematical Society*, vol. 116, no. 4, pp. 1077–1079, 1992.

[26] Y. C. Kim, S. Ponnusamy, and T. Sugawa, "Mapping properties of nonlinear integral operators and pre-Schwarzian derivatives," *Journal of Mathematical Analysis and Applications*, vol. 299, no. 2, pp. 433–447, 2004.

[27] Y. C. Kim and T. Sugawa, "Norm estimates of the pre-Schwarzian derivatives for certain classes of univalent functions," *Proceedings of the Edinburgh Mathematical Society II*, vol. 49, no. 1, pp. 131–143, 2006.

[28] S. Li and S. Stević, "Volterra-type operators on Zygmund spaces," *Journal of Inequalities and Applications*, vol. 2007, Article ID 32124, 10 pages, 2007.