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*Research Article*

# Multiple Periodic Solutions to a Suspension Bridge Wave Equation with Damping

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This paper is concerned with the existence of multiple periodic solutions for a suspension bridge wave equation with damping. By using Leray-Schauder degree theory, the authors prove that the damped wave equation has multiple periodic solutions.

## 1. Introduction

In [1], also see [2–6], the author considered a horizontal cross-section of the center span of a suspension bridge and proposed a partial differential equation model for the torsional motion of the cross-section and treated the center span of the bridge as a beam of length $L$ and width $2l$ suspended by cables. Consider the horizontal cross-section of mass $m$ located at position $x$ along the length of the span. She treated this cross-section as a rod of length $2l$ and mass $m$ suspended by cables. Let $y(x,t)$ denote the downward distance of the center of gravity of the rod from the unloaded state and let $\theta(x,t)$ denote the angle of the rod from horizontal at time $t$. Assume that the cables do not resist compression, but resist elongation according to Hooke's law with spring constant $K$. Then the torsional and vertical motion of the span satisfy

$$\begin{gathered}
\theta_{tt} - \varepsilon_1 \theta_{xx} = \frac{3K}{ml} \cos\left[(y - l\sin\theta)^+\right] - \delta\theta_t + h_1(x,t), \\
y_{tt} + \varepsilon_2 y_{xxxx} = -\frac{K}{m}\left[(y - l\sin\theta)^+ + (y + l\sin\theta)^+\right] - \delta y_t + g + h_2(xt), \\
\theta(0,t) = \theta(L,t) = y(0,t) = y(L,t) = y_{xx}(0,t) = y_{xx}(L,t) = 0,
\end{gathered} \tag{1.1}$$

where $u^+ = \max\{u, 0\}$, $\varepsilon_1$, $\varepsilon_2$ are physical constants related to the flexibility of the beam, $\delta$ is the damping constant, $h_1$ and $h_2$ are external forcing terms, and $g$ is the acceleration due to gravity. The spatial derivatives describe the restoring force that the beam exerts, and the time derivatives $\theta_t$ and $y_t$ represent the force due to friction. The boundary conditions reflect the fact that the ends of the span are hinged.

Throughout the paper [1] the author assumes that the cables never lose tension; that is, it is assumed that $(y \pm \sin\theta) \geq 0$. In this case, we see that (1.1) becomes uncoupled, and the torsional and vertical motions satisfy, respectively,

$$\begin{gathered} \theta_{tt} - \varepsilon_1 \theta_{xx} = -\frac{6K}{m} \cos\theta \sin\theta - \delta\theta_t + h_1(x,t), \\ \theta(0,t) = \theta(L,t) = 0, \end{gathered} \tag{1.2}$$

$$\begin{gathered} y_{tt} + \varepsilon_2 y_{xxxx} = -\frac{2K}{m} y - \delta y_t + g + h_2(x,t), \\ y(0,t) = y(L,t) = y_{xx}(0,t) = y_{xx}(L,t) = 0. \end{gathered} \tag{1.3}$$

In paper [1], removing the damping term; that is, let $\delta = 0$, changing variables, and imposing boundary and periodicity conditions, the author rewrites (1.2) as

$$\begin{gathered} u_{tt} - u_{xx} + b \sin u = \varepsilon h(x,t), \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = u(x,\pi), \quad u_t(x,0) = u_t(x,\pi), \\ u(x,t) = u(\pi - x,t), \quad u(x,t) = u(x,\pi - t). \end{gathered} \tag{1.4}$$

And it proves that (1.4) has at least two solutions in the subspace $H$ of $L^2$. Where $H$ is defined as

$$H = \left\{ u \in L^2(\Omega) u(x,t) = u(\pi - x,t), u(x,t) = u(x,\pi - t),\ u \text{ is } \pi - \text{periodic in } t \right\}. \tag{1.5}$$

Notice that (1.4) is particular in no damping and the selection of $H$. Hence, in [1] the author left a problem which is relevant to this case.

*Problem 1.* "Under appropriate hypotheses on the forcing term, does a similar result hold for the damped equation?"

Motivated by this problem, in this paper, we suppose that the damping is present, that is, $\delta \neq 0$, and study the following problem:

$$\begin{gathered} u_{tt} - u_{xx} + \delta u_t + b \sin u = \varepsilon h(x,t), \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = u(x,\pi), \quad u_t(x,t) = u_t(x,\pi), \\ u(x,t) = u(\pi - x,t). \end{gathered} \tag{1.6}$$

## 2. Preliminaries

Let $N = \{0, 1, \ldots\}$ and $Z$ be the set of integers, $\Lambda = N \times N$. Let $\Omega = (0, \pi) \times (0, \pi)$ and $L^2(\Omega)$ be usual space of square integrable functions with usual inner product $(\cdot, \cdot)$ and corresponding norm $\|\cdot\|$. For the Sobolev space $H^1(\Omega)$, we denote the standard inner product by $(u, v)_1 = (u, v) + (u_x, v_x) + (u_t, v_t)$ and norm by $\|u\|_1$.

Define the operator $L_\delta u = u_{tt} - u_{xx} + \delta u_t : H \to H$ by

$$D(L_\delta) = \left\{ u \in H \mid u(x,t) = \sum_\Lambda u_{mn}\Phi_{mn}, \sum_\Lambda \left( \left( (2n+1)^2 - 4m^2 \right)^2 + 4m^2\delta^2 \right) |u_{mn}|^2 < \infty \right\},$$
$$L_\delta u = \sum_\Lambda \left( (2n+1)^2 - 4m^2 + 2m\delta \right) u_{mn}\Phi_{mn}, \quad \text{for all } u \in H. \tag{2.1}$$

We know that the eigenvalues and corresponding eigenfunctions of $L_\delta$ are

$$\begin{aligned} \lambda_{mn} &= (2n+1)^2 - 4m^2 + 2m\delta i, \quad (m,n) \in \Lambda, \\ \Phi_{mn}(x,t) &= e^{2mti} \sin(2n+1)x, \quad (m,n) \in \Lambda. \end{aligned} \tag{2.2}$$

In order to seek the solutions of (1.6), we first investigate the properties of operator $L_\delta$. We have the following Lemma.

**Lemma 2.1.** *$L_\delta^{-1}$ exists, $L_\delta^{-1} : H \to H$ is compact, and $\|L_\delta^{-1}\| = 1$.*

*Proof.* Because we are restricted to the subspace $H$ of $L^2$, and $\lambda_{mn} \neq 0$, we easily know $L_\delta^{-1}$ exists.

We prove $L_\delta^{-1} : H \to H$ is compact below. We find that

$$L_\delta^{-1} u = \sum_\Lambda \frac{1}{(2n+1)^2 - 4m^2 + 2m\delta i} u_{mn}\Phi_{mn}, \tag{2.3}$$

for all $u = \sum_\Lambda u_{mn}\Phi_{mn} \in H$. For any $(m,n) \in \Lambda$, we have

$$\left| (2n+1)^2 - 4m^2 + 2m\delta i \right|^2 \geq 1, \tag{2.4}$$

then

$$\begin{aligned} \left\| L_\delta^{-1} u \right\|^2 &= \sum_\Lambda \left| \frac{1}{(2n+1)^2 - 4m^2 + 2m\delta i} u_{mn} \right|^2 \\ &\leq \sum_\Lambda |u_{mn}|^2 = \|u\|^2. \end{aligned} \tag{2.5}$$

Hence,

$$\left\| L_\delta^{-1} u \right\| \leq \|u\|. \tag{2.6}$$

On the other hand,

$$\left\| L_\delta^{-1} u \right\|_1^2 = \sum_\Lambda \left| \frac{1 + (2n+1)^2 + 4m^2}{\left((2n+1)^2 - 4m^2\right)^2 + 4m^2\delta^2} u_{mn} \right|^2, \tag{2.7}$$

while

$$\begin{aligned} \frac{1 + (2n+1)^2 + 4m^2}{\left((2n+1)^2 - 4m^2\right)^2 + 4m^2\delta^2} &= \frac{1 + (2n+1)^2 + 4m^2}{(2n+2m+1)^2(2n-2m+1)^2 + 4m^2\delta^2} \\ &\leq \frac{1 + (2n+1)^2 + 4|m|^2}{(2n+2|m|+1)^2 + 4|m|^2\delta^2} \\ &\leq \frac{(2n+2|m|+1)^2 + 1}{(2n+2|m|+1)^2} \\ &\leq 2. \end{aligned} \tag{2.8}$$

Hence,

$$\left\| L_\delta^{-1} u \right\|_1^2 \leq 2 \sum_\Lambda |u|^2 = 2\|u\|. \tag{2.9}$$

By (2.6) and (2.9), we can find that the operator $L_\delta^{-1} : H \to H$ is compact since the embedding $H^1 \to L^2$ is compact.

Finally, we prove $\|L_\delta^{-1}\| = 1$. By (2.2) and

$$|\lambda_{mn}|^2 = \left|(2n+1)^2 - 4m^2 + 2m\delta i\right|^2 \geq 1. \tag{2.10}$$

Set $u = \Phi_{00}$, such that, $\|L_\delta^{-1} u\| / \|u\| = 1$. Therefore,

$$\left\| L_\delta^{-1} \right\| = 1. \tag{2.11}$$

Hence, we complete the proof of this lemma. □

*Definition 2.2.* One says that $u \in H$ is a solution to (1.6) if

$$u = L_{\delta}^{-1}(\varepsilon h - b \sin u). \tag{2.12}$$

To establish the existence of multiple periodic solutions to (1.6), we use Leray-Schauder degree theory to prove the existence of multiple zeros of a related operator $T_1$. To compute the degree of $T_1$, we continuously deform it to a linear operator $T_0$, the *Gâteaux* derivative of $T_1$, and compute its degree via a direct calculation.

It is not difficult to show that the homotopy property of Leray-Schauder degree ensures that the degree of an operator $T_1$ is preserved as $T_1$ is continuously deformed to its *Fréchet* derivative under appropriate hypotheses. However, the nonlinear term in (1.6), $f(u) = \sin u$, is not *Fréchet* differentiable in $L^2$ at $u = 0$.

There is a theorem in paper [1], in which, the author shows that, under certain conditions on the nonlinear term $f$ and the differential operator $L$, Leray-Schauder degree is indeed preserved under homotopy from the operator $T_1$ to its *Gâteaux* derivative $T_0$. This result can be used to establish multiplicity of solutions to equations of the form (1.6). The result follows.

**Lemma 2.3.** *Let $I_1$, $I_2$ be open, bounded intervals in $R$, and define $Q := I_1 \times I_2$. Let $B$ be a subspace of $L^p(Q)$, $p \geq 1$, and define $\|u\| := \|u\|_{L^p}$. Consider the problem*

$$Lu + f(u) = \varepsilon h(x,t), \tag{2.13}$$

*where $L$, $f$, and $h$ satisfy the following:*

(H1) *$L^{-1}$ is compact;*

(H2) *$\|L^{-1}\| \leq 1$;*

(H3) *$f(0) = 0$;*

(H4) *$f$ is Lipschitz with Lipschitz constant $M$;*

(H5) *$h \in B$ and $h \leq 1$;*

(H6) *the Gâteaux derivative $df(0,u)$ exists and satisfies $df(0,u) = \rho u$, where $\rho > 0$ and $-\rho$ is not an eigenvalue of $L$.*

*Define $T_0$: $B \to B$ by*

$$T_0(u) = u + \rho L^{-1}(u), \tag{2.14}$$

*and $T_1 : B \to B$ by*

$$T_1(u) = u - L^{-1}(\varepsilon h - f(u)). \tag{2.15}$$

*Then for $\varepsilon$ sufficiently small, there exists $\gamma > 0$ such that*

$$\deg(T_1, B_\gamma(0), 0) = \deg(T, B_\gamma(0), 0). \tag{2.16}$$

## 3. Result and Proof

The main result of this paper is as follows.

**Theorem 3.1.** *Let $h \in H$ with $\|h\| \leq 1$, and let $b \in (-\sqrt{25+4\delta^2}, -\sqrt{9+4\delta^2})$, $0 < \delta^2 < 14$. Then there exists $\varepsilon_0 > 0$ such that if $|\varepsilon| < \varepsilon_0$,* (1.6) *has at least two solutions in $H$.*

*Proof.* Let $L = L_\delta$ and $f(u) = b\sin u$, it is easy to know that $L$ and $f$ satisfy the conditions (H1–H5) in Lemma 2.3.

Reply Lemma 2.3, we define $T_0$: $H \to H$ by

$$T_0(u) = u + bL_\delta^{-1}(u), \tag{3.1}$$

and $T_1$: $H \to H$ by

$$T_1(u) = u - L_\delta^{-1}(\varepsilon h - b\sin(u)). \tag{3.2}$$

And note that zeros of $T_1$ correspond to solutions of (1.6). To prove the theorem, we will show the following:

(C1) there exists $R_0 > 0$ such that for $R > R_0$, $\deg(T_1, B_R(0), 0) = 1$;

(C2) there exists $\gamma \in (0, R_0)$ such that $\deg(T_1, B_\gamma(0), 0) = -1$.

Then, since $\deg(T_1, B_\gamma(0), 0) \neq 0$, there exists a zero of $T_1$ (i.e., a solution of (1.6)) in $B_\gamma(0)$. Moreover, by the additivity property of degree, $\deg(T_1, B_R(0) \setminus \overline{B_\gamma(0)}, 0) \neq 0$ and hence (1.6) has a second solution in the annulus $B_R(0) \setminus \overline{B_\gamma(0)}$.

To establish (C1), define

$$T_\beta u = u - \beta L_\delta^{-1}(\varepsilon h - b\sin(u)), \tag{3.3}$$

or $\beta \in [0,1]$, and note that this definition of $T_1$ is consistent with our previous definition. Note also that $T_0$ is simply the identity map; hence, for any $R > 0$ we have $\deg(T_0, B_R(0), 0) = 1$. The homotopy property of degree ensures that $\deg(T_\beta, B_R(0), 0)$ is constant provided that $0 \overline{\in} T_\beta(\partial B_R(0))$ for all $\beta \in [0,1]$.

Fix $\beta \in [0,1]$ and suppose $u \in H$ solves $T_\beta u = 0$. We will show that u is bounded above by some $R_0 > 0$ and that this bound is independent of $\beta$.

Since $T_\beta u = 0$, we have

$$\begin{aligned} \|u\| &= \beta\left\|L_\delta^{-1}(\varepsilon h - b\sin u)\right\| \leq \beta[\varepsilon_0 + b\|\sin u\|] \\ &\leq \left[\varepsilon_0 + bm(\Omega)^{1/2}\right] < \left[\varepsilon_0 + b\sqrt{2}\pi\right] < R_0, \end{aligned} \tag{3.4}$$

if we choose $R_0 > \varepsilon_0 + b\sqrt{2}\pi$.

Thus, for $R > R_0$, we have

$$\deg(T_1, B_R(0), 0) = \deg(T_0, B_R(0), 0) = 1, \tag{3.5}$$

and (C1) above holds.

To establish (C2), let $\varepsilon < \varepsilon_0$; we will determine the value of $\varepsilon_0$ later. For $\mu \in [0,1]$ define

$$T_\mu u = u + (1-\mu)L_\delta^{-1}(bu) - \mu L_\delta^{-1}(\varepsilon h - b\sin u), \tag{3.6}$$

and note again that this definition of $T_1$ is consistent with our previous definitions. We will again apply the homotopy property of degree (via Lemma 2.3) and a standard degree calculation to show that for some $\gamma > 0$

$$\deg(T_1, B_\gamma(0), 0) = \deg(T_0, B_\gamma(0), 0) = -1. \tag{3.7}$$

Observe that for $L = L_\delta$ and $f(u) := b\sin u$, hypotheses (H1)–(H5) of Lemma 2.3 are satisfied. To verify hypothesis (H6), we need to show that

$$df(0,u) = bu. \tag{3.8}$$

By definition of the *Gâteaux* derivative,

$$\begin{aligned} df(0,u) &= \frac{d}{dt} f(0+tu)|_{t=0} \\ &= \lim_{h\longrightarrow 0} \frac{f((t+h)u) - f(tu)}{h}|_{t=0} \\ &= \lim_{h\longrightarrow 0} \frac{b\sin(hu)}{h}. \end{aligned} \tag{3.9}$$

We will show that the limit above (in $H$) is $bu$.

Note first that in $R$ we have

$$\lim_{h\to 0} = \frac{\sin(hu)}{h} = \lim_{h\to 0} \frac{\sin(hu)}{h}\frac{u}{u} = u, \tag{3.10}$$

and hence

$$\left|\frac{\sin(hu)}{h} - u\right|^2 \longrightarrow 0, \tag{3.11}$$

as $h \to 0$. Invoking the convexity of $\omega^2$, we have

$$\left|\frac{\sin(hu)}{h} - u\right|^2 \le 4\left[\frac{1}{2}\left|\frac{\sin(hu)}{h}\right|^2 + \frac{1}{2}|u|^2\right] \le 4u^2. \tag{3.12}$$

Since $u \in L^2$, $|(\sin(hu)/h) - u|^2$ is dominated in $L^1$; thus by the dominated convergence theorem,

$$\left\| \frac{b\sin(hu)}{h} - bu \right\| \longrightarrow 0, \tag{3.13}$$

as $h \to 0$; therefore (3.8) holds. Moreover, by the form of eigenvalue of $L_\delta$ and our choice of $b$, $-b$ is not an eigenvalue of $L_\delta$; therefore hypothesis (H6) of Lemma 2.3 holds. Thus, by Lemma 2.3, for sufficiently small $\gamma, \varepsilon > 0$, we have

$$\deg(T_0, B_\gamma(0), 0) = \deg(T_1, B_\gamma(0), 0). \tag{3.14}$$

Finally, we will show that

$$\deg(T_0, B_\gamma(0), 0) = \deg\left(I + bL_\delta^{-1}, B_\gamma(0), 0\right) = -1. \tag{3.15}$$

Consider the finite dimensional subspace $M_N = \operatorname{span}\{\Phi_{mn}\}_0^N$ of $H$ and recall that, by compactness, $bL_\delta^{-1}$ can be approximated in operator norm by the operators $B_N : M_N \to M_N$ given by

$$B_N(u) = b\sum_{-N}^{N}\sum_{0}^{N}\frac{c_{mn}}{\lambda_{mn}}\Phi_{mn}. \tag{3.16}$$

By definition of Leray-Schauder degree, for $N$ sufficiently large,

$$\deg(T_0, B_\gamma(0), 0) = \deg\left(I + B_N, B_\gamma(0) \bigcap M_N, 0\right) = \sum_{u \in (I+B_N)^{-1}(0)} \operatorname{sgn} J_{I+B_N}(u), \tag{3.17}$$

where $J_\Phi(u)$ is the Jacobian determinant of $\Phi$ at $u$.

Since $I + B_N$ can be identified with an $(2N+1)^2 \times (2N+1)^2$ diagonal matrix whose entries are $1 + b/\lambda_{mn}$, we have

$$\deg\left(I + B_N, B_\gamma(0) \bigcap M_N, 0\right) = \operatorname{sgn} \prod_{m=-N}^{N}\prod_{n=0}^{N}\left(1 + \frac{b}{\lambda_{mn}}\right). \tag{3.18}$$

Now we consider the following two cases.

(D1) If $\lambda_{mn}$ contains imaginary part, suppose a pair of conjugate complex numbers are $a \pm ci$ $(c \neq 0)$, then,

$$\left(1 + \frac{b}{a+ci}\right)\left(1 + \frac{b}{a-ci}\right) = \frac{(a+b)^2 + c^2}{a^2 + c^2} > 0. \tag{3.19}$$

(D2) If $\lambda_{mn}$ is real, then $m = 0$, here $\lambda_{mn} = (2n+1)^2$.

Since $b \in (-\sqrt{25+4\delta^2}, -\sqrt{9+4\delta^2})$, and $0 < \delta^2 < 14$, the only negative value of $1 + b/(2n+1)^2$ occurs at $\lambda_{00}$. Therefore,

$$\deg\left(I + B_N, B_\gamma(0) \bigcap M_N, 0\right) = -1. \tag{3.20}$$

The proof of the theorem is complete. □

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