## Research Article

# Generalized Homogeneity of Means 

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We extend the classical notions of translativity and homogeneity of means to $F$-homogeneity, that is, invariance with respect to an operation $F: I \times I \rightarrow I$. We find the shape of $F$ for the arithmetic weighted mean and then the general form of $F$ for quasi-linear means. Also, we are interested in characterizations of means. It turns out that no quasi-arithmetic mean can be characterized by $F$-homogeneity with respect to a single operation $F$, one needs to take two of such operations in order to characterize a mean.

## 1. Introduction

Definition 1.1. Let $I \subset \mathbb{R}$ be an interval. A function $M: I \times I \rightarrow I$ such that

$$
\begin{equation*}
\min \{x, y\} \leq M(x, y) \leq \max \{x, y\} \quad x, y \in I \tag{1.1}
\end{equation*}
$$

is called a mean (on $I^{2}$ ).
Note that every mean is reflexive, that is

$$
\begin{equation*}
M(x, x)=x \quad \forall x \in I \tag{1.2}
\end{equation*}
$$

A mean $M$ is called symmetric, if

$$
\begin{equation*}
M(x, y)=M(y, x) \quad \forall x, y \in I \tag{1.3}
\end{equation*}
$$

In [1], Aczél and Dhombres distinguished two special types of means, defined in $I \times I$, where $I \subset \mathbb{R}_{+}$.
(i) Translative: if the function is translation invariant, that is for all $x, y \in I, t \in \mathbb{R}_{+}$

$$
\begin{equation*}
x+t, y+t \in I \Longrightarrow M(x+t, y+t)=M(x, y)+t \tag{1.4}
\end{equation*}
$$

(ii)Homogeneous: if the function is multiplication invariant, that is for all $x, y \in I, u \in \mathbb{R}_{+}$

$$
\begin{equation*}
x u, y u \in I \Longrightarrow M(x u, y u)=M(x, y) u \tag{1.5}
\end{equation*}
$$

For more information about means, see, for instance, [1-6].
In the present paper, we present three approaches to the question of translativity and homogeneity of means. First, we "discover" some functional equations which generalize properties (1.4) and (1.5). Let us adopt the following concept of generalized homogeneity which in many instances covers both translativity and homogeneity.

Definition 1.2. Let $I \subset \mathbb{R}$ be an interval and let $F: I \times I \rightarrow I$ be a function. A mean $M$ : $I \times I \rightarrow I$ such that

$$
\begin{equation*}
M(F(t, x), F(t, y))=F(t, M(x, y)) \tag{1.6}
\end{equation*}
$$

for every $x, y, t \in I$ is called $F$-homogeneous.
In the last section, we determine all the operations $F$ with respect to which quasi-linear means are $F$-homogeneous. Also, we characterize means as solutions to systems of functional equations, thus generalizing a result from [1] (cf. Proposition 9, page 249).

## 2. Pexider Equations for Means

We notice that the translative mean equation $M: I \times I \rightarrow I$ on an interval $I$ satisfying $I+I \subset I$, that is the equation

$$
\begin{equation*}
M(s+x, s+y)=s+M(x, y) \tag{2.1}
\end{equation*}
$$

and the homogeneous mean equation $M: I \times I \rightarrow I$ on an interval $I$ satisfying $I \cdot I \subset I$, that is the equation

$$
\begin{equation*}
M(s x, s y)=s M(x, y) \tag{2.2}
\end{equation*}
$$

may be treated as conditional forms of the following equations:

$$
\begin{gather*}
M(s+x, t+y)=\phi(s, t)+M(x, y)  \tag{2.3}\\
M(s x, t y)=\phi(s, t) M(x, y) \tag{2.4}
\end{gather*}
$$

respectively.

Here, the word "condition" means that (2.3) and (2.4) have to be satisfied for pairs ( $s, t$ ) of the set $W=\{(s, t) \in I \times I: s=t\}$.

To show that (2.1) actually is the conditional equation (2.3), we put $t=s, y=x$ into (2.3) and use reflexivity of $M$. Then,

$$
\begin{equation*}
s+x=M(s+x, s+x)=\phi(s, s)+M(x, x)=\phi(s, s)+x, \tag{2.5}
\end{equation*}
$$

whence $\phi(s, s)=s, s \in I$.
Putting $\underline{s}=(s, t)$ and $\underline{x}=(x, y)$, we can rewrite (2.3) in the form

$$
\begin{equation*}
M(\underline{s}+\underline{x})=\phi(\underline{s})+M(\underline{x}), \tag{2.6}
\end{equation*}
$$

which is satisfied for $\underline{s} \in I \times I$ and $\underline{x} \in I \times I$. This is Pexider conditional equation (in this case the condition refers to the fact that (2.6) holds for pairs $(\underline{s}, \underline{x}) \in(I \times I) \times(I \times I))$. To solve (2.6) let us note that fixing arbitrarily $x, y, s_{1}, s_{2}, t_{1}, t_{2} \in I$, and letting $\underline{x}=(x, y), \underline{s}=\left(s_{1}, s_{2}\right), \underline{t}=\left(t_{1}, t_{2}\right)$, we get, from (2.6)

$$
\begin{align*}
\phi(\underline{s}+\underline{t}) & =M(\underline{s}+\underline{t}+\underline{x})-M(\underline{x})=\phi(\underline{s})+M(\underline{t}+\underline{x})-M(\underline{x})  \tag{2.7}\\
& =\phi(\underline{s})+\phi(\underline{t})+M(\underline{x})-M(\underline{x})=\phi(\underline{s})+\phi(\underline{t}) .
\end{align*}
$$

Thus $\phi$ satisfies the Cauchy equation for $\underline{s}, \underline{t} \in I \times I$. Moreover, from (2.6) and properties of mean $M$, we obtain ( $x \in I$ is arbitrarily fixed)

$$
\begin{align*}
\min \left(s_{1}, s_{2}\right) & \leq M\left(s_{1}+x, s_{2}+x\right)-x=M(\underline{s}+(x, x))-M(x, x)  \tag{2.8}\\
& =\phi(\underline{s}) \leq \max \left(s_{1}, s_{2}\right),
\end{align*}
$$

so $\phi$ is majorized (and minorized) on $I \times I$ by a continuous function. The well known results (cf. for instance $[1,2,7]$ ) imply that $\phi$ is a linear mapping, that is there exist constants $\alpha, \omega \in \mathbb{R}$ such that

$$
\begin{equation*}
\phi\left(s_{1}, s_{2}\right)=\alpha s_{1}+\omega s_{2}, \quad\left(s_{1}, s_{2}\right) \in I \times I . \tag{2.9}
\end{equation*}
$$

From (2.8), we infer that $s=\phi(s, s)=(\alpha+\omega) s, s \in I$, so $\alpha=1-\omega$. Moreover, from (2.8), we have for $s_{1}<s_{2}, s_{1}, s_{2} \in I$

$$
\begin{equation*}
s_{1} \leq(1-\omega) s_{1}+\omega s_{2} \leq s_{2} \tag{2.10}
\end{equation*}
$$

that is, $\omega\left(s_{2}-s_{1}\right) \geq 0$ and $(1-\omega)\left(s_{2}-s_{1}\right) \geq 0$, so $\omega \in[0,1]$. In other words, we have proved that there exists an $\omega \in[0,1]$ such that

$$
\begin{equation*}
\phi\left(s_{1}, s_{2}\right)=(1-\omega) s_{1}+\omega s_{2}, \quad\left(s_{1}, s_{2}\right) \in I \times I . \tag{2.11}
\end{equation*}
$$

Now, put $s=y$ and $t=x$ into (2.3). Using the reflexivity of $M$ and (2.11), we obtain

$$
\begin{equation*}
x+y=M(y+x, x+y)=\phi(y, x)+M(x, y)=(1-\omega) y+\omega x+M(x, y) \tag{2.12}
\end{equation*}
$$

whence

$$
\begin{equation*}
M(x, y)=(1-\omega) x+\omega y, \quad(x, y) \in I \times I \tag{2.13}
\end{equation*}
$$

Thus we obtain the following.
Theorem 2.1. Let $I \subset \mathbb{R}$ be a non-degenerate interval such that $I+I \subset I$. Then, a mean $M: I \times I \rightarrow I$ and a function $\phi: I \times I \rightarrow \mathbb{R}$ satisfy (2.3) if and only if there exists a constant $\omega \in[0,1]$ such that

$$
\begin{equation*}
M(x, y)=\phi(x, y)=(1-\omega) x+\omega y, \quad(x, y) \in I \times I \tag{2.14}
\end{equation*}
$$

Now, we define in $\mathbb{R}^{2}$ an operation $\bullet$ by

$$
\begin{equation*}
\left(s_{1}, s_{2}\right) \bullet(x, y)=\left(s_{1} x, s_{2} y\right) \tag{2.15}
\end{equation*}
$$

Let $I \subset(0, \infty)$ be an interval such that $I \cdot I \subset I$. Suppose that $M: I \times I \rightarrow I$ is a mean and $\phi: I \times I \rightarrow \mathbb{R}$ is an arbitrary function. We consider the equation $\left(\underline{s}=\left(s_{1}, s_{2}\right), \underline{x}=(x, y)\right)$

$$
\begin{equation*}
M(\underline{s} \bullet \underline{x})=\phi(\underline{s}) M(\underline{x}) . \tag{2.16}
\end{equation*}
$$

From the assumption $I \subset(0, \infty)$, hence both $M$ and $\phi$ take on positive values. Putting $u_{1}=$ $\ln s_{1}, u_{2}=\ln s_{2}, v=\ln x, w=\ln y$ and defining $\widetilde{M}: \ln I \times \ln I \rightarrow \ln I, \tilde{\phi}: \ln I \times \ln I \rightarrow \mathbb{R}$ by

$$
\begin{align*}
& \widetilde{M}(v, w)=\ln M(\exp (v), \exp (w)),  \tag{2.17}\\
& \widetilde{\phi}\left(u_{1}, u_{2}\right)=\ln \phi\left(\exp \left(u_{1}\right), \exp \left(u_{2}\right)\right),
\end{align*}
$$

we see that

$$
\begin{equation*}
\widetilde{M}\left(u_{1}+v, u_{2}+w\right)=\widetilde{\phi}\left(u_{1}, u_{2}\right)+\widetilde{M}(v, w) \tag{2.18}
\end{equation*}
$$

for all $u_{1}, u_{2}, v, w \in \ln I$. It is easy to show that $\widetilde{M}$ is a mean, because so was $M$. By Theorem 2.1, we have, for some constant $\omega \in[0,1]$,

$$
\begin{equation*}
\widetilde{M}(v, w)=\widetilde{\phi}(v, w)=(1-\omega) v+\omega w, \quad(v, w) \in \ln I \times \ln I \tag{2.19}
\end{equation*}
$$

hence

$$
\begin{equation*}
M(x, y)=\phi(x, y)=x^{1-\omega} y^{\omega}, \quad(x, y) \in I \times I \tag{2.20}
\end{equation*}
$$

We obtain the following.

Theorem 2.2. Let $I \subset(0, \infty)$ be a non-degenerate interval such that $I \cdot I \subset I$. Then a mean $M$ : $I \times I \rightarrow I$ and a function $\phi: I \times I \rightarrow \mathbb{R}$ satisfy (2.16) if and only if there exists a constant $\omega \in[0,1]$ such that $M$ and $\phi$ are given by (2.20).

Remark 2.3. Let us note that two other Pexider equations,

$$
\begin{gather*}
M(\underline{s}+\underline{x})=\phi(\underline{s}) M(\underline{x}),  \tag{2.21}\\
M(\underline{s} \bullet \underline{x})=\phi(\underline{s})+M(\underline{x}),
\end{gather*}
$$

have no solutions $(M, \phi)$ such that $M$ is reflexive. In fact, putting $\underline{s}=(s, s)$ and $\underline{x}=(x, x)$, we get

$$
\begin{align*}
& s+x=\phi(s, s) x \\
& s x=\phi(s, s)+x \tag{2.22}
\end{align*}
$$

respectively. The equalities cannot be satisfied for all $s, x \in I$.

## 3. Further Generalizations

Now, we come back to (2.1). We notice that it is also a special case of the following equation:

$$
\begin{equation*}
M(s+x, \psi(s)+y)=M(s, \psi(s))+M(x, y) \tag{3.1}
\end{equation*}
$$

that is, the Cauchy equation for $M$, but with one of variables belonging to the graph of some fixed function $\psi: I \rightarrow I$. We will consider also the equation

$$
\begin{equation*}
M(s x, \psi(s) y)=M(s, \psi(s)) M(x, y) \tag{3.2}
\end{equation*}
$$

Note that if we take $\psi=\mathrm{id}_{I}$, we get (2.1) or (2.2).
We will prove results concerning the above equations. Let us start with the "additive" case.

Theorem 3.1. Let $I$ be a non-degenerate interval such that $I+I \subset I$. Let $\psi: I \rightarrow I$ be a function such that the mapping $\Psi: I \times I \rightarrow I \times I$ given by

$$
\begin{equation*}
\Psi\left(s_{1}, s_{2}\right)=\left(s_{1}+\psi\left(s_{2}\right), s_{2}+\psi\left(s_{1}\right)\right) \tag{3.3}
\end{equation*}
$$

is a surjection. Then, a symmetric and reflexive mapping $M: I \times I \rightarrow I$ satisfies (3.1) if and only if $M$ is the arithmetic mean.

Proof. Fix a $s \in I$ and put $x=\psi(s), y=s$ into (3.1). Using the reflexivity and the symmetry of $M$, we obtain

$$
\begin{align*}
s+\psi(s) & =M(s+\psi(s), \psi(s)+s)=M(s, \psi(s))+M(\psi(s), s)  \tag{3.4}\\
& =2 M(s, \psi(s))
\end{align*}
$$

whence

$$
\begin{equation*}
M(s, \psi(s))=\frac{s+\psi(s)}{2}, \quad s \in I \tag{3.5}
\end{equation*}
$$

Now, fix a $(x, y) \in I \times I$. Let $\left(s_{1}, s_{2}\right) \in I \times I$ be chosen so that $\Psi\left(s_{1}, s_{2}\right)=(x, y)$. From (3.1) and (3.5), we get

$$
\begin{align*}
M(x, y) & =M\left(s_{1}+\psi\left(s_{2}\right), \psi\left(s_{1}\right)+s_{2}\right)=M\left(s_{1}, \psi\left(s_{1}\right)\right)+M\left(\psi\left(s_{2}\right), s_{2}\right) \\
& =\frac{s_{1}+\psi\left(s_{1}\right)+\psi\left(s_{2}\right)+s_{2}}{2}=\frac{x+y}{2} . \tag{3.6}
\end{align*}
$$

This completes the proof.
Analogously as in the proof of Theorem 2.2, we obtain the following.
Theorem 3.2. Let $I \subset(0, \infty)$ be an interval such that $I \cdot I \subset I$. Let $\psi: I \rightarrow I$ be a function such that $\Psi: I \times I \rightarrow I \times I$ defined by

$$
\begin{equation*}
\Psi\left(s_{1}, s_{2}\right)=\left(s_{1} \psi\left(s_{2}\right), s_{2} \psi\left(s_{1}\right)\right) \tag{3.7}
\end{equation*}
$$

is a bijection. Then, a symmetric and reflexive mapping $M: I \times I \rightarrow I$ is a solution of (3.2) if and only if $M$ is the geometric mean.

Example 3.3. The assumption of a surjectivity of $\Psi$ is essential. This is shown by example of the function $\psi=$ id for which we have $\Psi\left(s_{1}, s_{2}\right)=\left(s_{1}+s_{2}, s_{1}+s_{2}\right)\left(\Psi\left(s_{1}, s_{2}\right)=\left(s_{1} s_{2}, s_{1} s_{2}\right)\right)$, equations (3.1) and (3.2) take forms (2.1) and (2.2), respectively, which have other solutions, even in the class of quasi-arithmetic means (cf. [1]).

We have the following corollaries of theorems.
Corollary 3.4. Let $c \in(0, \infty) \backslash\{1\}$ be fixed. The arithmetic mean is the only symmetric mean satisfying the equation

$$
\begin{equation*}
M(s+x, c s+y)=M(s, c s)+M(x, y) \tag{3.8}
\end{equation*}
$$

for all $s, x, y \in(0, \infty)$.
Proof. The function

$$
\begin{equation*}
(0, \infty)^{2} \ni\left(s_{1}, s_{2}\right) \longrightarrow\left(s_{1}+c s_{2}, s_{2}+c s_{1}\right) \tag{3.9}
\end{equation*}
$$

is a bijection of $(0, \infty)^{2}$ onto itself; it is enough to use Theorem 3.1.
Similarly, using Theorem 3.2, we obtain the following.

Corollary 3.5. Let a $c \in \mathbb{R} \backslash\{-1,1\}$ be fixed. The geometric mean is the only symmetric mean satisfying the equation

$$
\begin{equation*}
M\left(s x, s^{c} y\right)=M\left(s, s^{c}\right) M(x, y) \tag{3.10}
\end{equation*}
$$

for all $s, x, y \in(0, \infty)$.

## 4. F-Homogeneity of Quasi-Linear Means

We will now consider the following problem. What are operations $F$ with respect to which quasi-linear means are $F$-homogeneous?

Let us begin with the weighted arithmetic mean defined on an interval $I$. Let $p \in[0,1]$ be fixed and let $F: I \times I \rightarrow I$ be an operation satisfying the equation

$$
\begin{equation*}
(1-p) F(s, x)+p F(s, y)=F(s,(1-p) x+p y) \tag{4.1}
\end{equation*}
$$

for all $s, x, y \in I$.
First, we will suppose that $F: I \times I \rightarrow I$ is a function continuous in second variable.
Let us define $f_{s}:=F(s, \cdot)$ for every $s \in I$. Then, for every $s$, the function $f_{s}$ solves the equation

$$
\begin{equation*}
f_{s}((1-p) x+p y)=(1-p) f_{s}(x)+p f_{s}(y) \tag{4.2}
\end{equation*}
$$

In view of the equality (cf. [8])

$$
\begin{equation*}
\frac{u+v}{2}=(1-p)\left[(1-p) \frac{u+v}{2}+p u\right]+p\left[(1-p) v+p \frac{u+v}{2}\right] \tag{4.3}
\end{equation*}
$$

which holds for all $u, v \in \mathbb{R}$ and $p \in[0,1]$, it follows that

$$
\begin{equation*}
f_{s}\left(\frac{x+y}{2}\right)=\frac{f_{s}(x)+f_{s}(y)}{2} \tag{4.4}
\end{equation*}
$$

The assumption of continuity in the second variable implies (cf. e.g., [2] or [7]) the existence of functions $A, B: I \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
f_{s}(x)=F(s, x)=A(s) x+B(s) \tag{4.5}
\end{equation*}
$$

for all $s, x \in I$.
Now, let us additionally assume that $F$ is associative, that is, for all $s, t, x \in I$ we have

$$
\begin{equation*}
F(F(s, t), x)=F(s, F(t, x)) \tag{4.6}
\end{equation*}
$$

In view of (4.5), we obtain the equivalent equality

$$
\begin{equation*}
A(A(s) t+B(s)) x+B(A(s) t+B(s))=A(s) A(t) x+A(s) B(t)+B(s) \tag{4.7}
\end{equation*}
$$

for all $s, t, x \in I$. Thus we see that the operation given by (4.5) is associative if and only if the following system of equations

$$
\begin{gather*}
A(A(s) t+B(s))=A(s) A(t)  \tag{4.8}\\
B(A(s) t+B(s))=A(s) B(t)+B(s) \tag{4.9}
\end{gather*}
$$

holds for all $s, t \in I$. Assume now that $F$ is symmetric (hence it is also continuous with respect to the first variable). This means that, by (4.9),

$$
\begin{equation*}
A(s) B(t)+B(s)=A(t) B(s)+B(t) \tag{4.10}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
(A(s)-1) B(t)=(A(t)-1) B(s) \tag{4.11}
\end{equation*}
$$

for all $s, t \in I$.
We consider the following two cases.
(a) $A=1$. Then $F(s, t)=t+B(s)$, and the symmetry of $F$ yields the equality

$$
\begin{equation*}
t+B(s)=s+B(t) \tag{4.12}
\end{equation*}
$$

that is

$$
\begin{equation*}
B(s)-s=B(t)-t \tag{4.13}
\end{equation*}
$$

for all $s, t \in I$. So, there exists a constant $\alpha_{0} \in \mathbb{R}$ such that

$$
\begin{equation*}
B(s)=s+\alpha_{0}, \quad s \in I \tag{4.14}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
F(s, t)=t+s+\alpha_{0}=\gamma_{\alpha_{0}}^{-1}\left(\gamma_{\alpha_{0}}(s)+\gamma_{\alpha_{0}}(t)\right), \quad t, s \in I \tag{4.15}
\end{equation*}
$$

where $\gamma_{\alpha_{0}}: I \rightarrow I$ is defined by

$$
\begin{equation*}
\gamma_{\alpha_{0}}(s)=s+\alpha_{0} . \tag{4.16}
\end{equation*}
$$

Note that the condition $\gamma_{\alpha_{0}}(I) \subset I$ is a restriction imposed both on $I$ and $\alpha_{0}$.
(b) $A \neq 1$. Let $A\left(s_{0}\right) \neq 1$ for some $s_{0} \in I$. Putting $s=s_{0}$ into (4.11), we calculate

$$
\begin{equation*}
B(t)=d(A(t)-1) \tag{4.17}
\end{equation*}
$$

where $d:=B\left(s_{0}\right) /\left(A\left(s_{0}\right)-1\right)$. Using the symmetry of $F$, we get

$$
\begin{equation*}
A(s) t+d(A(s)-1)=F(s, t)=F(t, s)=A(t) s+d(A(t)-1) \tag{4.18}
\end{equation*}
$$

We have

$$
\begin{equation*}
A(s)(t+d)=A(t)(s+d), \quad s, t \in I \tag{4.19}
\end{equation*}
$$

whence

$$
\begin{equation*}
\frac{A(s)}{s+d}=\frac{A(t)}{t+d}, \quad s, t \in I \backslash\{-d\} \tag{4.20}
\end{equation*}
$$

In other words,

$$
\begin{equation*}
A(s)=\alpha_{2} s+\alpha_{2} d=\alpha_{2} s+\alpha_{1} \tag{4.21}
\end{equation*}
$$

for all $s \neq-d$, where $\alpha_{2}$ is a constant and $\alpha_{1}=\alpha_{2} d$. From (4.19), it results that $s=-d$ implies $A(s)=0$, so (4.21) holds for all $s \in I$. Hence,

$$
\begin{equation*}
B(t)=d\left(\alpha_{2} t+\alpha_{1}-1\right)=\alpha_{2} d t+d\left(\alpha_{1}-1\right)=\alpha_{1} t+d\left(\alpha_{1}-1\right), \quad t \in I \tag{4.22}
\end{equation*}
$$

and, finally, denoting $d\left(\alpha_{1}-1\right)$ by $\alpha_{0}$, we get

$$
\begin{equation*}
F(s, t)=\alpha_{2} s t+\alpha_{1}(s+t)+\alpha_{0}, \quad s, t \in I . \tag{4.23}
\end{equation*}
$$

Now, it is easy to check that associativity of $F$ is equivalent to

$$
\begin{equation*}
\alpha_{0} \alpha_{2}=\alpha_{1}^{2}-\alpha_{1} \tag{4.24}
\end{equation*}
$$

We have the following possibilities.
(i) $\alpha_{2}=0$ and $\alpha_{1}=0$. In this case, we have

$$
\begin{equation*}
F(s, t)=\alpha_{0} \tag{4.25}
\end{equation*}
$$

and it is enough to assume that $\alpha_{0} \in I$.
(ii) $\alpha_{2}=0$ and $\alpha_{1}=1$. Then, similarly as in case (a),

$$
\begin{equation*}
F(s, t)=s+t+\alpha_{0}=\gamma_{\alpha_{0}}^{-1}\left(\gamma_{\alpha_{0}}(s)+\gamma_{\alpha_{0}}(t)\right) \tag{4.26}
\end{equation*}
$$

where $\gamma_{\alpha_{0}}: I \rightarrow I$ is defined by (4.16).
(iii) $\alpha_{2} \neq 0$. Now, we have

$$
\begin{align*}
F(s, t) & =\frac{1}{\alpha_{2}}\left(\left(\alpha_{2} s+\alpha_{1}\right)\left(\alpha_{2} t+\alpha_{1}\right)-\alpha_{1}\right)  \tag{4.27}\\
& =\delta_{\alpha_{1}, \alpha_{2}}^{-1}\left(\delta_{\alpha_{1}, \alpha_{2}}(s) \delta_{\alpha_{1}, \alpha_{2}}(t)\right)
\end{align*}
$$

where $\delta_{\alpha_{1}, \alpha_{2}}: I \rightarrow I$ is given by

$$
\begin{equation*}
\delta_{\alpha_{1}, \alpha_{2}}(s)=\alpha_{1}+\alpha_{2} s \tag{4.28}
\end{equation*}
$$

Note that $\delta_{\alpha_{1}, \alpha_{2}}(I) \subset I$ is a condition imposed on the interval $I$ and constants $\alpha_{1}, \alpha_{2}$ as well.
Thus, we have proved.
Proposition 4.1. Let $I$ be a non-degenerate interval, let $F: I \times I \rightarrow I$ be a function which is associative, symmetric, and continuous in each variable. Let $p \in[0,1]$ be fixed. Then, the weighted arithmetic mean is $F$-homogeneous (that is (4.1) holds) if and only if there exist constants $\alpha_{0}, \alpha_{1}$, and $\alpha_{2}$ such that $F$ is given by (4.23), the condition (4.24) is satisfied and
(i) $\left(\alpha_{2}=\alpha_{1}=0 \Rightarrow \alpha_{0} \in I\right)$,
(ii) $\left(\left(\alpha_{2}=0 \wedge \alpha_{1}=1\right) \Rightarrow\left(\gamma_{\alpha_{0}}(I)+\gamma_{\alpha_{0}}(I) \subset \gamma_{\alpha_{0}}(I) \subset I\right)\right)$ where $\gamma_{\alpha_{0}}$ is given by (4.16),
(iii) $\left(\alpha_{2} \neq 0 \Rightarrow\left(\delta_{\alpha_{1}, \alpha_{2}}(I) \cdot \delta_{\alpha_{1}, \alpha_{2}}(I) \subset \delta_{\alpha_{1}, \alpha_{2}}(I) \subset I\right)\right)$ where $\delta_{\alpha_{1}, \alpha_{2}}$ is given by (4.28).

Remark 4.2. We see that among the operations listed in Proposition 4.1 are the following,
(i) $F(s, t)=s+t\left(\alpha_{0}=\alpha_{2}=0, \alpha_{1}=1\right), I \in\{\mathbb{R},(-\infty, a),(-\infty, a],(b, \infty),[b, \infty): a \leq 0, b \geq$ $0\}$,
(ii) $F(s, t)=s \cdot t\left(\alpha_{2}=1, \alpha_{1}=\alpha_{0}=0\right),(I, \cdot)$ is a subsemigroup of $(\mathbb{R}, \cdot)$,
(iii) $F(s, t)=2 s t-(s+t)+1=1 / 2((2 s-1)(2 t-1)+1)\left(\alpha_{2}=2, \alpha_{1}=-1, \alpha_{0}=1\right)$, in this case $I \in\{\mathbb{R},(a, \infty),[a, \infty), a \geq 1\}$.

Now, let us generalize Proposition 4.1 to the case of arbitrary quasi-linear means. We admit the following definition.

Definition 4.3. Let $I$ and $J$ be non-degenerate intervals, let $F: I \times I \rightarrow I$ be an arbitrary function, let $f: I \rightarrow J$ be a bijection. Then, we define the function $F_{f}: J \times J \rightarrow J$ by

$$
\begin{equation*}
F_{f}(u, v)=f\left(F\left(f^{-1}(u), f^{-1}(v)\right)\right) \tag{4.29}
\end{equation*}
$$

Remark 4.4. We can easily see that $F$ is associative and symmetric if and only if $F_{f}$ has the same properties. Moreover, if $f$ is continuous, then $F$ is continuous in each variable if and only if $F_{f}$ is continuous in each variable.

We obviously have the following.
Lemma 4.5. Let I and J be non-degenerate intervals. Let $f: I \rightarrow J$ be a continuous bijection, let $F: I \times I \rightarrow I$ be a function, and let $p \in[0,1]$ be fixed. Then the quasi-linear mean $M_{f}: I \times I \rightarrow I$ given by

$$
\begin{equation*}
M_{f}(x, y)=f^{-1}((1-p) f(x)+p f(y)) \tag{4.30}
\end{equation*}
$$

is F-homogeneous if and only if the weighted arithmetic mean on the interval $J$ is $F_{f}$-homogeneous.
From Proposition 4.1 and Lemma 4.5, we obtain the following.
Theorem 4.6. Let I and $J$ be non-degenerate intervals and let $f: I \rightarrow J$ be a continuous bijection, let $F: I \times I \rightarrow I$ be a function which is associative, symmetric, and continuous in each variable. Let $p \in[0,1]$ be fixed. Then, the quasi-linear mean $M_{f}: I \times I \rightarrow I$ is $F$-homogeneous if and only if there exist constants $\alpha_{0}, \alpha_{1}$, and $\alpha_{2}$ such that $F$ is given by

$$
\begin{equation*}
F(s, t)=f^{-1}\left(\alpha_{2} f(s) f(t)+\alpha_{1}(f(s)+f(t))+\alpha_{0}\right), \quad s, t \in I \tag{4.31}
\end{equation*}
$$

the condition (4.24) is satisfied, and
(i) $\left(\alpha_{2}=\alpha_{1}=0 \Rightarrow \alpha_{0} \in I\right)$,
(ii) $\left(\left(\alpha_{2}=0 \wedge \alpha_{1}=1\right) \Rightarrow\left(\gamma_{\alpha_{0}}(J)+\gamma_{\alpha_{0}}(J) \subset \gamma_{\alpha_{0}}(J) \subset J\right)\right)$ where $\gamma_{\alpha_{0}}: J \rightarrow J$ is given by

$$
\begin{equation*}
\gamma_{\alpha_{0}}(u)=u+\alpha_{0} \tag{4.32}
\end{equation*}
$$

(iii) $\left(\alpha_{2} \neq 0 \Rightarrow\left(\delta_{\alpha_{1}, \alpha_{2}}(J) \cdot \delta_{\alpha_{1}, \alpha_{2}}(J) \subset \delta_{\alpha_{1}, \alpha_{2}}(J) \subset J\right)\right)$ where $\delta_{\alpha_{1}, \alpha_{2}}: J \rightarrow J$ is given by

$$
\begin{equation*}
\delta_{\alpha_{1}, \alpha_{2}}(u)=\alpha_{1}+\alpha_{2} u \tag{4.33}
\end{equation*}
$$

Remark 4.7. Among operations $F$, for which $M_{f}$ is homogeneous, are the following

$$
\begin{align*}
(s, t) & \longrightarrow f^{-1}(f(s) f(t))  \tag{4.34}\\
(s, t) & \longrightarrow f^{-1}(f(s)+f(t)) \tag{4.35}
\end{align*}
$$

It is possible to see that $M_{f}$ is the unique quasi-linear mean which is homogeneous with respect to (4.34) and (4.35) (cf. [1], Theorem 15.8).

Remark 4.8. We see that the inverse to Theorem 4.6 does not hold. More exactly, Fhomogeneity alone does not characterize the mean $M_{f}$. In order to see it, define $F: \mathbb{R}_{+} \times \mathbb{R}_{+} \rightarrow$ $\mathbb{R}_{+}$by

$$
\begin{align*}
F(s, t) & =s t+2(s+t)+2=(s+2)(t+2)-2 \\
& =\delta_{2,1}^{-1}\left(\delta_{2,1}(s) \delta_{2,1}(t)\right) \tag{4.36}
\end{align*}
$$

We show that there exist many $F$-homogeneous quasi-linear means, even when we restrict our attention to symmetric ones. In fact, suppose that $M$ is an $F$-homogeneous quasiarithmetic mean, that is

$$
\begin{equation*}
M(F(s, t), F(s, u))=F(s, M(t, u)), \quad s, t, u \in \mathbb{R}_{+} \tag{4.37}
\end{equation*}
$$

or

$$
\begin{equation*}
M\left(\delta_{2,1}^{-1}\left(\delta_{2,1}(s) \delta_{2,1}(t)\right), \delta_{2,1}^{-1}\left(\delta_{2,1}(s) \delta_{2,1}(u)\right)\right)=\delta_{2,1}^{-1}\left(\delta_{2,1}(s) \delta_{2,1}(M(t, u))\right) . \tag{4.38}
\end{equation*}
$$

Substituting $x=\delta_{2,1}(s), y=\delta_{2,1}(t)$, and $z=\delta_{2,1}(u)$, and defining $\widetilde{M}:[2, \infty) \times[2, \infty) \rightarrow[2, \infty)$ by

$$
\begin{equation*}
\widetilde{M}(x, y)=\delta_{2,1}\left(M\left(\delta_{2,1}^{-1}(x), \delta_{2,1}^{-1}(y)\right)\right) \tag{4.39}
\end{equation*}
$$

we see that $\widetilde{M}$ is homogeneous on $[2, \infty)$. It follows (cf. for instance [1]) that either

$$
\begin{equation*}
\widetilde{M}(x, y)=\sqrt{x \cdot y} \tag{4.40}
\end{equation*}
$$

or there is a $k \neq 0$ such that

$$
\begin{equation*}
\widetilde{M}(x, y)=\left(\frac{x^{k}+y^{k}}{2}\right)^{1 / k} \tag{4.41}
\end{equation*}
$$

whence either

$$
\begin{equation*}
M(s, t)=\sqrt{(s+2)(t+2)}-2 \tag{4.42}
\end{equation*}
$$

or

$$
\begin{equation*}
M(s, t)=\left(\frac{(s+2)^{k}+(t+2)^{k}}{2}\right)^{1 / k}-2 \tag{4.43}
\end{equation*}
$$

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