

Research Article

A New Proof of the Existence of Free Lie Algebras and an Application

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The existence of free Lie algebras is usually derived as a consequence of the Poincaré-Birkhoff-Witt theorem. Moreover, in order to prove that (given a set X and a field \mathbb{K} of characteristic zero) the Lie algebra $\mathcal{L}(\mathbb{K}\langle X \rangle)$ of the Lie polynomials in the letters of X (over the field \mathbb{K}) is a free Lie algebra generated by X , all available proofs use the embedding of a Lie algebra \mathfrak{g} into its enveloping algebra $\mathcal{U}(\mathfrak{g})$. The aim of this paper is to give a much simpler proof of the latter fact without the aid of the cited embedding nor of the Poincaré-Birkhoff-Witt theorem. As an application of our result and of a theorem due to Cartier (1956), we show the relationships existing between the theorem of Poincaré-Birkhoff-Witt, the theorem of Campbell-Baker-Hausdorff, and the existence of free Lie algebras.

1. Introduction

We begin with the main definition we will be concerned with in this paper. In the sequel, \mathbb{K} will denote a fixed field, and all linear objects (linear maps, algebras, spans, etc.) will be tacitly meant with respect to \mathbb{K} .

Definition 1.1. Let X be any nonempty set. We say that L is a free Lie algebra generated by X if L is a Lie algebra, and there exists a map $\phi : X \rightarrow L$ satisfying the following property. For every Lie algebra \mathfrak{g} and for every map $f : X \rightarrow \mathfrak{g}$, there exists exactly one Lie algebra morphism f^ϕ such that the diagram below commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathfrak{g} \\ \phi \downarrow & \nearrow f^\phi & \\ L & & \end{array} \quad (1.1)$$

When $X \subset L$ set-theoretically and the above property is satisfied with the inclusion map $\iota : X \hookrightarrow L$ replacing ϕ , L will be called a free Lie algebra *over* X .

It is easy to see that a free Lie algebra generated by X —if it exists—is unique up to (Lie algebra) isomorphism. The existence of a free Lie algebra generated by a set can be proved in a standard way (see [1, Chapitre II, Section 2, n.1] and [2, Section 0.2]). As for the existence of a free Lie algebra *over* a set, it is natural to expect that (given a field \mathbb{K} of characteristic zero) the Lie \mathbb{K} -algebra of the Lie polynomials in the letters of X is a free Lie algebra over X . To prove this fact, however, more profound results are required, as we will explain below.

In the sequel, if V is a vector space, we denote by $\mathcal{T}(V)$ its tensor algebra and by $\mathcal{L}(V)$ the smallest Lie-subalgebra of $\mathcal{T}(V)$ containing V . The aim of this paper is to prove the following result.

Theorem 1.2. *Let \mathbb{K} be a field of characteristic zero, and let $\mathbb{K}\langle X \rangle$ be the free vector space generated by X , then $\mathcal{L}(\mathbb{K}\langle X \rangle)$ is a free Lie algebra over X .*

Classically, the above theorem is derived from the theorem of Poincaré, Birkhoff, and Witt (henceforth referred to as PBW), which we here recall for the reading convenience and to fix some notation.

Theorem of Poincaré-Birkhoff-Witt

Let \mathbb{K} be a field of characteristic zero, and let \mathfrak{g} be a Lie algebra. We denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} and by 1 the unit element of $\mathcal{U}(\mathfrak{g})$. Let π denote the natural projection $\pi : \mathcal{T}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$, and let us set $\mu := \pi|_{\mathfrak{g}}$. Suppose that \mathfrak{g} is endowed with an indexed (linear) basis $\{x_i\}_{i \in I}$, where I is totally ordered by the relation \preccurlyeq , then the following elements form a linear basis for $\mathcal{U}(\mathfrak{g})$:

$$1, \quad \mu(x_{i_1}) \cdots \mu(x_{i_n}), \quad \text{where } n \in \mathbb{N}, \quad i_1, \dots, i_n \in I, \quad i_1 \preccurlyeq \cdots \preccurlyeq i_n. \quad (1.2)$$

As a by-product of this theorem, we have the following crucial result.

Corollary 1.3. *The function $\mu : \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g})$ is injective.*

The above corollary is the essential tool one needs in order to prove Theorem 1.2. To the best of our knowledge, all the books using free Lie algebras prove Theorem 1.2 by making use of Theorem PBW (or, precisely, of Corollary 1.3); see Bourbaki [1, Chapitre II, Section 3, n.1, Théorème 1] (where it employed [3, Chapitre I, Section 2, n.7, Corollaire 3 du Théorème 1] which is the PBW Theorem); see Reutenauer [2, Theorem 0.5] (where the injectivity of $\mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$ is used); see also Hochschild [4, Chapter X, Section 2], Humphreys [5, Chapter V, Section 17.5], Jacobson [6, Chapter V, Section 4], and Varadarajan [7, Section 3.2].

For the sake of completeness, let us briefly recall the argument which uses PBW.

Lemma 1.4. *Let V be a vector space, then $\mathcal{L}(V)$ satisfies the following property: for every Lie algebra \mathfrak{g} and for every linear map $F : V \rightarrow \mathfrak{g}$, there exists exactly one Lie algebra morphism $\bar{F} : \mathcal{L}(V) \rightarrow \mathfrak{g}$ prolonging F .*

Proof. The universal property of $\mathcal{T}(V)$ allows us to prove the existence of exactly one morphism of associative algebras \tilde{F} such that the following diagram commutes:

$$\begin{array}{ccccc}
 V & \xrightarrow{F} & \mathfrak{g} & \xrightarrow{\mu} & \mathcal{U}(\mathfrak{g}) \\
 \downarrow & & & \nearrow \tilde{F} & \\
 \mathcal{T}(V) & & & &
 \end{array} \tag{1.3}$$

It is easily seen that $\tilde{F}(t) \in \mu(\mathfrak{g})$ for every $t \in \mathcal{L}(V)$, so that we can compose the restriction of \tilde{F} to $\mathcal{L}(V)$ with $\mu^{-1} : \mu(\mathfrak{g}) \rightarrow \mathfrak{g}$ (the function μ^{-1} is well posed on $\mu(\mathfrak{g})$ thanks to Corollary 1.3). The map

$$\bar{F} : \mathcal{L}(V) \longrightarrow \mathfrak{g}, \quad \bar{F} = \mu^{-1} \circ (\tilde{F}|_{\mathcal{L}(V)}) \tag{1.4}$$

satisfies all the requirements in the statement of the lemma. \square

Let us now prove Theorem 1.2 with the aid of Lemma 1.4 above. Let X be any set and let \mathfrak{g} , f be as in Definition 1.1. Let us denote by ϕ the inclusion map $X \hookrightarrow \mathcal{L}(\mathbb{K}\langle X \rangle)$. We apply Lemma 1.4 with $V = \mathbb{K}\langle X \rangle$ and F obtained by prolonging the function f by linearity on V , then the Lie algebra morphism $\bar{F} : \mathcal{L}(\mathbb{K}\langle X \rangle) \rightarrow \mathfrak{g}$ is easily seen to satisfy the requirements of f^ϕ in Definition 1.1, and Theorem 1.2 is proved.

The aim of this paper is to provide a new proof of Theorem 1.2, independently of PBW and of Corollary 1.3. This will be done in Sections 2 and 3.

The remainder of the paper (Section 4) provides a motivation for the first part. The occasion to search for a proof of Theorem 1.2 which is alternative to the usual one arose when—in our studies in monograph [8]—we came across a nonstandard demonstration of Theorem PBW due to Cartier [9]. Cartier’s proof uses the Theorem of Campbell, Baker, and Hausdorff (CBH, shortly) in order to prove PBW. In its turn, CBH is usually proved by means of Theorem PBW (see, e.g., Bourbaki [1, Chapitre II, Section 3, n.1, Corollaire 2], Hochschild [4, Proposition 2.1], Jacobson [6, Theorem 9, Chapter V], and Serre [10, 11, Theorems 4.2, 7.1]). So Cartier shows that, beside the usual argument

$$\text{PBW} \implies \text{CBH}, \tag{1.5}$$

surprisingly, the reverse path can be followed too.

In order to make this reverse path “CBH \implies PBW” fully consistent, it is necessary to provide a proof of CBH which is independent of PBW. (Seemingly, apart from Hausdorff’s original argument [12], an algebraic proof of CBH independent of PBW was not available at the time of Cartier’s paper [9], dated 1956. Twelve years later, Eichler [13] gave another proof, using only free Lie algebras, and then seven years after Eichler, Djoković [14] provided another one, using formal power series computations. More recently, Reutenauer [2, Section 3.4] has proved CBH with a rigorous algebraic formalization of the early ideas by Pascal, Baker, Campbell, and Hausdorff. See [8, Chapters 1, 4] for more details on all these topics.) Cartier provides this proof in [9], making use of free Lie algebras generated by

a finite set. As long as the existence of free Lie algebras is again a consequence of PBW, clearly it cannot be exploited to prove Theorem PBW itself. Hence, in [9] a proof of the existence of free Lie algebras generated by finite sets is given, independently of PBW, and relying only on the ideas contained in the classical proof of the theorem of Dynkin, Specht, and Wever.

By making use of these ideas, we here provide a proof, with no prerequisites, of the existence of free Lie algebras over *arbitrary* sets, thus generalizing the result in [9]. We hope that a new proof, which is alternative to all those presented in books, is welcome, especially since it makes no use of such a deep result as the Theorem of Poincaré, Birkhoff, and Witt. As an application, we are able to highlight the full interdependence of PBW and CBH and the existence of free Lie algebras (see Section 4), which—to the best of our knowledge—has never been pointed out in the specialized literature so far (see Theorem 4.1 and Corollary 4.3).

2. The Free Lie Algebra Generated by a Set

In the present short section, included for the sake of completeness and to fix the notation used throughout, we recall an argument, which dates back to Bourbaki [1, Chapitre II, Section 2, n.1] (see also Reutenauer [2, Section 0.2]) proving directly the existence of free Lie algebras *generated* by a set.

For $n \in \mathbb{N}$, $n \geq 1$, consider the set $M_n(X)$ (also denoted by M_n , for short) of all (roughly speaking) the noncommutative, nonassociative words of length n on the elements of X . The M_n s can be defined inductively by means of disjoint unions (denoted by \coprod) of Cartesian products in the following way:

$$\begin{aligned} M_1(X) &:= X, & M_2(X) &:= X \times X, & M_3(X) &:= (X \times X) \times X \coprod X \times (X \times X), \\ M_n(X) &:= \coprod_{i+j=n} M_i(X) \times M_j(X), & n &\geq 2. \end{aligned} \tag{2.1}$$

Let $M(X) := \coprod_{n \geq 1} M_n(X)$. We can define a (noncommutative, nonassociative) binary operation on $M(X)$ as follows. For any $w, w' \in M(X)$, say, $w \in M_n$ and $w' \in M_{n'}$, we denote by $w \cdot w'$ the unique element of $M_{n+n'}$ corresponding to (w, w') in the canonical injections $M_n \times M_{n'} \subset M_{n+n'} \subset M(X)$. This binary operation endows $M(X)$ with the structure of a magma, called the *free magma generated by X* . Let us now set $\text{Lib}(X) := \mathbb{K}\langle M(X) \rangle$, the free vector space generated by the free magma $M(X)$. (The free vector space generated by a set M is here thought of as the \mathbb{K} -vector space of the functions from M to \mathbb{K} vanishing outside a finite set; equivalently, it will be treated as the set of formal (finite) linear combinations of elements of M , where M is a basis for $\mathbb{K}\langle M \rangle$.) The canonical map $M(X) \hookrightarrow \text{Lib}(X)$ (sending $w \in M(X)$ to the characteristic function of $\{w\}$ in $M(X)$) will be simply denoted by set inclusion. The operation on $M(X)$ extends by linearity to an operation on $\text{Lib}(X)$ turning it into a nonassociative algebra, called the *free (nonassociative) algebra generated by X* . We have the following result, whose proof is standard and hence omitted.

Lemma 2.1. *Let X be a set, then, for every \mathbb{K} -algebra A and every function $f : X \rightarrow A$, there exists a unique algebra morphism $\bar{f} : \text{Lib}(X) \rightarrow A$ prolonging f .*

Given an (not necessarily associative) algebra $(M, *)$, we say that $S \subseteq M$ is a (two-sided) magma ideal in M if S is a vector subspace of M such that $s * m$ and $m * s$ belong to S ,

for every $s \in S$ and every $m \in M$. Moreover, if $H \subseteq M$ is any set, the smallest (two-sided) magma ideal in M containing H will be said to be the magma ideal generated by H in M .

Let us now consider the algebra $\text{Lib}(X)$ and

$$A := \{Q(a) = a \cdot a, J(a, b, c) = a \cdot (b \cdot c) + b \cdot (c \cdot a) + c \cdot (a \cdot b) \mid a, b, c \in \text{Lib}(X)\}. \quad (2.2)$$

We henceforth denote by \mathfrak{a} the magma ideal generated by A in $\text{Lib}(X)$. We next consider the quotient vector space $\text{Lie}(X) := \text{Lib}(X)/\mathfrak{a}$ and the natural projection $\pi : \text{Lib}(X) \rightarrow \text{Lie}(X)$. Then the map

$$\begin{aligned} \text{Lie}(X) \times \text{Lie}(X) &\longrightarrow \text{Lie}(X), \\ (\pi(a), \pi(b)) &\longmapsto [\pi(a), \pi(b)] := \pi(a \cdot b), \quad a, b \in \text{Lib}(X) \end{aligned} \quad (2.3)$$

is well posed, and it endows $\text{Lie}(X)$ with a \mathbb{K} -algebra structure, which turns out to be a Lie algebra over \mathbb{K} (see the very definition of A).

Proposition 2.2. *Let X be any set and, with the above notation, let us set $\varphi := \pi|_X : X \rightarrow \text{Lie}(X)$ then*

- (1) *$\text{Lie}(X)$, together with the map φ , is a free Lie algebra generated by X (according to Definition 1.1);*
- (2) *the set $\{\varphi(x)\}_{x \in X}$ is independent in $\text{Lie}(X)$, whence φ is injective;*
- (3) *the set $\varphi(X)$ Lie-generates $\text{Lie}(X)$, that is, the smallest Lie subalgebra of $\text{Lie}(X)$ containing $\varphi(X)$, coincides with $\text{Lie}(X)$.*

The proof of this proposition is derived by collecting together various results appearing in [1, Chapitre II, Section 2], with the additional care of transposing them to a nonassociative setting (see, e.g., [8, Theorem 2.54] for all the details).

3. The Isomorphism $\text{Lie}(X) \simeq \mathcal{L}(\mathbb{K}\langle X \rangle)$

From now on, we turn to prove Theorem 1.2 without using neither Theorem PBW nor Corollary 1.3. The arguments are more delicate than those in the preceding section.

We fix throughout a field \mathbb{K} of characteristic zero. We denote by $\mathcal{L}(\mathbb{K}\langle X \rangle)$ the smallest Lie subalgebra of $\mathcal{T}(\mathbb{K}\langle X \rangle)$ (the tensor algebra of the vector space $\mathbb{K}\langle X \rangle$) containing X . Let φ be as in Proposition 2.2. Being $\text{Lie}(X)$ a free Lie algebra generated by X , there exists a unique Lie algebra morphism

$$\Phi : \text{Lie}(X) \longrightarrow \mathcal{L}(\mathbb{K}\langle X \rangle) \text{ such that } \Phi(\varphi(x)) = x, \quad \text{for every } x \in X. \quad (3.1)$$

Our main task is to show that Φ is an isomorphism, *without using PBW theorem*. This will immediately prove that $\mathcal{L}(\mathbb{K}\langle X \rangle)$ is a free Lie algebra *over* X , according to Definition 1.1. We will do this by means of some auxiliary functions.

Lemma 3.1. *With the above notation, one has the grading $\text{Lie}(X) = \bigoplus_{n \geq 1} B_n$, where $B_1 = \text{span}(\varphi(X))$, and, for any $n \geq 2$,*

$$B_n = [B_1, B_{n-1}] = \text{span}\{[\varphi(x), y] : x \in X, y \in B_{n-1}\}. \quad (3.2)$$

Proof. It is easy to prove the grading $\text{Lib}(X) = \bigoplus_n \text{Lib}_n(X)$, where $\text{Lib}_n(X)$ is the span of $M_n(X)$. On the other hand, a simple argument shows that \mathfrak{a} is also the magma ideal generated by the elements

$$\begin{aligned} w \cdot w, \quad (w + w') \cdot (w + w') - w \cdot w - w' \cdot w', \\ w \cdot (w' \cdot w'') + w' \cdot (w'' \cdot w) + w'' \cdot (w \cdot w'), \end{aligned} \quad (3.3)$$

with $w, w', w'' \in M(X)$. As a consequence, we have an analogous grading $\mathfrak{a} = \bigoplus_n \mathfrak{a}_n$, with $\mathfrak{a}_n \subseteq \text{Lib}_n(X)$, for every $n \in \mathbb{N}$. This gives $\text{Lie}(X) = \bigoplus_n C_n$, where $C_n = \text{Lib}_n(X)/\mathfrak{a}$ is spanned by the n -degree commutators of the elements of $\varphi(X)$ (bracketing is taken in arbitrary order). In its turn, we obviously have $C_n = B_n$, where B_n is the span of the “nested” brackets

$$[\varphi(x_1) \cdots [\varphi(x_{n-1}), \varphi(x_n)] \cdots], \quad \text{for } x_1, \dots, x_n \in X. \quad (3.4)$$

Finally, it is a simple proof to check that $[B_n, B_m] \subseteq B_{n+m}$ for every n, m . \square

Thanks to Lemma 3.1, the following auxiliary map is well posed:

$$\delta : \text{Lie}(X) \longrightarrow \text{Lie}(X), \quad \delta\left(\sum_n b_n\right) := \sum_n n b_n \quad (b_n \in B_n, \forall n \in \mathbb{N}). \quad (3.5)$$

In the remainder of the section, for any vector space V , we denote by $\text{End}(V)$ the set of the endomorphisms of V . By the universal properties of the free vector space $\mathbb{K}\langle X \rangle$ and of tensor algebras, there exists a unique morphism of unital associative algebras $\theta : \mathcal{T}(\mathbb{K}\langle X \rangle) \rightarrow \text{End}(\text{Lie}(X))$, such that $\theta(x) = \text{ad } \varphi(x)$, for every $x \in X$.

Finally, there exists a unique linear map $g : \mathcal{T}(\mathbb{K}\langle X \rangle) \rightarrow \text{Lie}(X)$ such that, for every $x_1, \dots, x_k \in X$ and every $k \geq 2$, it holds $g(1_{\mathbb{K}}) = 0$, $g(x_1) = \varphi(x_1)$, and $g(x_1 \otimes \cdots \otimes x_k) = [\varphi(x_1) \cdots [\varphi(x_{k-1}), \varphi(x_k)]]$.

The diagram below gives an idea of the setting we are working in

$$\begin{array}{ccccc} & & & \text{End}(\text{Lie}(X)) & \\ & & \nearrow \text{ad} & \uparrow \theta & \\ X & \xrightarrow{\varphi} & \text{Lie}(X) & \xrightarrow{\Phi} & \mathcal{L}(\mathbb{K}\langle X \rangle) \hookrightarrow \mathcal{T}(\mathbb{K}\langle X \rangle) \\ & & \searrow \delta & \downarrow g & \\ & & & \text{Lie}(X) & \end{array} \quad (3.6)$$

We are now ready for the following result (see also Cartier [9, Lemma 1, page 242], where finite X s are considered); part (c) of the lemma is the core of a result due to Dynkin [15].

In the following statement, we denote by $\mathcal{T}_+(\mathbb{K}\langle X \rangle)$ the ideal of $\mathcal{T}(\mathbb{K}\langle X \rangle)$ whose elements have vanishing zero-degree component.

Lemma 3.2. *With all the above notation, one has the following:*

- (a) δ is a derivation of the Lie algebra $\text{Lie}(X)$ and $\delta \circ \varphi \equiv \varphi$ on X ,
- (b) $g(xy) = \theta(x)(g(y))$, for every $x \in \mathcal{T}(\mathbb{K}\langle X \rangle)$, $y \in \mathcal{T}_+(\mathbb{K}\langle X \rangle)$,
- (c) $\theta \circ \Phi \equiv \text{ad}$ on $\text{Lie}(X)$, that is, $\theta(\Phi(\xi))(\eta) = [\xi, \eta]$, for every $\xi, \eta \in \text{Lie}(X)$,
- (d) $g \circ \Phi \equiv \delta$ on $\text{Lie}(X)$.

Proof. (a) Take elements $t, t' \in \text{Lie}(X) = \bigoplus_{n \geq 1} B_n$ (see Lemma 3.1), $t = \sum_n t_n$, $t' = \sum_n t'_n$, with $t_n, t'_n \in B_n$ for every $n \geq 1$, and t_n, t'_n are definitively equal to 0, then we have

$$\begin{aligned}
 \delta([t, t']) &= \delta \sum_{n \geq 1} \sum_{i+j=n} [t_i, t'_j] = \sum_{n \geq 1} n \sum_{i+j=n} [t_i, t'_j] \\
 &= \sum_{n \geq 1} \sum_{i+j=n} (i+j) [t_i, t'_j] = \sum_{n \geq 1} \sum_{i+j=n} [i t_i, t'_j] + \sum_{n \geq 1} \sum_{i+j=n} [t_i, j t'_j] \\
 &= \left[\sum_i i t_i, \sum_j t'_j \right] + \left[\sum_i t_i, \sum_j j t'_j \right] = [\delta(t), t'] + [t, \delta(t')].
 \end{aligned} \tag{3.7}$$

Moreover, from the definition of the B_n s in Lemma 3.1, we have $\varphi(X) \subset B_1$, so

$$\delta(\varphi(x)) = \varphi(x), \quad \text{for every } x \in X. \tag{3.8}$$

(b) An inductive argument: if $x = k \in \mathbb{K} = \mathcal{T}_0(V)$, (b) is trivially true, $g(k y) = k g(y)$ (since g is linear) and

$$\theta(k)(g(y)) = k \text{Id}_{\text{Lie}(X)}(g(y)) = k g(y). \tag{3.9}$$

Thus, we are left to prove (b) when both x, y belong to $\mathcal{T}_+(\mathbb{K}\langle X \rangle)$; moreover, by linearity, we can assume without loss of generality that $x = v_1 \otimes \cdots \otimes v_k$ and $y = w_1 \otimes \cdots \otimes w_h$ with $h, k \geq 1$, and the v s and w s are elements of X ,

$$\begin{aligned}
 g(xy) &= g(v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_h) \\
 &= \text{ad}(\varphi(v_1)) \circ \cdots \circ \text{ad}(\varphi(v_k))([\varphi(w_1), \dots, [\varphi(w_{h-1}), \varphi(w_h)] \dots]) \\
 &= \theta(v_1 \otimes \cdots \otimes v_k)(g(w_1 \otimes \cdots \otimes w_h)) = \theta(x)(g(y)).
 \end{aligned} \tag{3.10}$$

(c) Let $\xi \in \text{Lie}(X)$, then (c) follows if we show that $\theta(\Phi(\xi))$ and $\text{ad}(\xi)$ coincide (note that they are both endomorphisms of $\text{Lie}(X)$). In its turn, this is equivalent to the identity of $\theta \circ \Phi$ with the map

$$\text{ad}: \text{Lie}(X) \longrightarrow \text{End}(\text{Lie}(X)), \quad \xi \longmapsto \text{ad}(\xi). \quad (3.11)$$

Now observe that both ad and $\theta \circ \Phi$ are Lie algebra morphisms (indeed, Φ is a Lie algebra morphism by construction, and θ is a Lie algebra morphism since it is a unital associative algebra morphism). Hence, the equality of ad and $\theta \circ \Phi$ follows if we prove that they are equal on a system of Lie generators for $\text{Lie}(X)$, namely, on $\varphi(X)$ (recall Proposition 2.2-(3.8)), for every $x \in X$, we indeed have

$$(\theta \circ \Phi)(\varphi(x)) = \theta(x) = \text{ad}(\varphi(x)). \quad (3.12)$$

(d) It suffices to show that $g \circ \Phi$ is a derivation of $\text{Lie}(X)$; we have

$$\begin{aligned} (g \circ \Phi)[\xi, \eta] &= g([\Phi(\xi), \Phi(\eta)]) = g(\Phi(\xi)\Phi(\eta) - \Phi(\eta)\Phi(\xi)) \\ &= \theta(\Phi(\xi))g(\Phi(\eta)) - \theta(\Phi(\eta))g(\Phi(\xi)) \\ &= [\xi, g(\Phi(\eta))] + [g(\Phi(\xi)), \eta]. \end{aligned} \quad (3.13)$$

□

In the third and fourth equalities, we used (b) and (c), respectively. Since $g \circ \Phi$ and δ are derivations of $\text{Lie}(X)$ coinciding on $\varphi(X)$, (d) follows.

Theorem 3.3. *Let \mathbb{K} be field of characteristic zero.*

If Φ is as in (3.1), Φ is an isomorphism of Lie algebras.

Proof. From the very definition of δ and the fact that \mathbb{K} has characteristic zero, it follows that δ is injective.

From the identity $g \circ \Phi \equiv \delta$ (see (d) in Lemma 3.2) and the injectivity of δ , we immediately infer the injectivity of Φ . Since Φ is clearly surjective (indeed, $\text{Lie}(X)$ and $\mathcal{L}(\mathbb{K}\langle X \rangle)$ are Lie generated by $\varphi(X)$ and X , respectively), the theorem is proved. □

In order to explicitly show the relationship of the arguments we employed to prove Theorem 1.2 with the so-called Theorem of Dynkin, Specht, and Wever, we seize the opportunity to show that the latter is implicitly contained in Theorem 3.3 and Lemma 3.2. (See e.g., ([4], [18], Proposition 2.2), ([6], [20], Chapter V, Section 4, Theorem 8), and ([2], [26], Theorem 1.4), ([10, 11], [29], Chapter IV, Section 8, LA 4.15). For the original proofs, see [15–17].) Despite its well-known importance, we state it as a corollary of the former results.

Corollary 3.4 (Dynkin, Specht, and Wever). *Let \mathbb{K} be a field of characteristic zero. Consider the linear map $P : \mathcal{T}(\mathbb{K}\langle X \rangle) \rightarrow \mathcal{L}(\mathbb{K}\langle X \rangle)$ such that*

$$\begin{aligned} P(1_{\mathbb{K}}) &= 0, & P(x_1) &= x_1, \\ P(x_1 \otimes \cdots \otimes x_k) &= k^{-1} [x_1, \dots, [x_{k-1}, x_k] \dots], & \forall k \geq 2, \end{aligned} \quad (3.14)$$

for any $x_1, \dots, x_k \in X$. Then P is a projection, that is, P is surjective and it is the identity on $\mathcal{L}(\mathbb{K}\langle X \rangle)$.

Consequently, one has the following characterization of Lie elements:

$$\mathcal{L}(\mathbb{K}\langle X \rangle) = \{t \in \mathcal{T}(\mathbb{K}\langle X \rangle) \mid P(t) = t\}. \quad (3.15)$$

Proof. The well-posedness and surjectivity of P are obvious. To prove that P is the identity on $\mathcal{L}(\mathbb{K}\langle X \rangle)$, it suffices to test it on a homogeneous bracket, say $t = [x_1, \dots, [x_{k-1}, x_k] \dots]$, with $k \in \mathbb{N}$ and $x_1, \dots, x_k \in X$. It is clear that it holds $P(t) = k^{-1}(\Phi \circ g)(t)$. As a consequence, being Φ invertible (by Theorem 3.3),

$$P(t) = k^{-1}(\Phi \circ g \circ \Phi \circ \Phi^{-1})(t) = k^{-1}(\Phi \circ \delta \circ \Phi^{-1})(t) = k^{-1}\Phi(k \Phi^{-1}(t)) = t. \quad (3.16)$$

In the second equality, we used (d) of Lemma 3.2; in the third, we used the definition of δ and the fact that $\Phi^{-1}(t) = [\varphi(x_1), \dots, [\varphi(x_{k-1}), \varphi(x_k)] \dots] \in B_k$. \square

4. The Relationship between the Theorems of PBW and of CBH

The deep intertwinement between PBW and CBH and the existence of free Lie algebras can be visualized in the following chain of equivalent statements.

Theorem 4.1. *Consider the following six statements (all linear structures are understood to be over a field of characteristic zero):*

- (a) *the set $\{\mu(x_i)\}_{i \in I}$ is independent in $\mathcal{U}(\mathfrak{g})$;*
- (b) *any Lie algebra \mathfrak{g} can be embedded in its enveloping algebra $\mathcal{U}(\mathfrak{g})$;*
- (c) *for every set X , there exists a free Lie algebra over X ;*
- (d) *for every finite set X , there exists a free Lie algebra over X ;*
- (e) *free Lie algebras over finite sets do exist, and the Campbell-Baker-Hausdorff theorem holds;*
- (f) *the Poincaré-Birkhoff-Witt theorem holds.*

Then these results can be proved each by the aid of the other in the following circular sequence:

$$(a) \implies (b) \implies (c) \implies (d) \implies (e) \implies (f) \implies (a). \quad (4.1)$$

Remark 4.2. Notice that statement (c) was proved in Section 3 above, without any prerequisite nor the aid of any of the other statements in Theorem 4.1. In particular, the isomorphism of $\text{Lie}(X)$ (the free Lie algebra generated by X) with $\mathcal{L}(\mathbb{K}\langle X \rangle)$ (the Lie algebra of the Lie polynomials in the letters of X) can be proved independently of (b) and (f), as we announced.

Proof. (a) \Rightarrow (b): this is obvious by the definition $\mu(x_i) = \pi(x_i)$,

(b) \Rightarrow (c): this is the usual approach to the existence of free Lie algebras over a set recalled in the Introduction,

(c) \Rightarrow (d): this is obvious,

(d) \Rightarrow (e): the derivation of the Campbell-Baker-Hausdorff theorem from the existence of free Lie algebras over finite sets appears in Eichler's proof [13],

(e) \Rightarrow (f): the proof of PBW from CBH (with the joint use of free Lie algebras over finite sets) is contained in Cartier's paper [9],

(f) \Rightarrow (a): this is obvious. \square

As by-products of the previous theorem, we highlight the following corollaries, containing some probably unexpected facts.

Corollary 4.3. *The following are consequences of Theorem 4.1. Again, all linear structures are on a field of characteristic zero.*

- (a) *The linear independence of the $\mu(x_i)$ s is sufficient (besides being necessary) for the system in (1.2) to form a basis of $\mathcal{U}(\mathfrak{g})$.*
- (b) *The sole embedding $\mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$ proves the Poincaré-Birkhoff-Witt theorem.*
- (c) *The existence of free Lie algebras over finite sets is sufficient for the existence of all free Lie algebras over arbitrary sets.*
- (d) *The existence of free Lie algebras proves the Poincaré-Birkhoff-Witt theorem (not only the reverse fact is true).*

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