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Research Article

Some New Identities on the q-Genocchi Numbers and Polynomials with Weight α

Seog-Hoon Rim and Joohee Jeong

Department of Mathematics Education, Kyungpook National University, Daegu 702-701, Republic of Korea

Correspondence should be addressed to Joohee Jeong, jhjeong@knu.ac.kr

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We construct a new type of q-Genocchi numbers and polynomials with weight α. From these q-Genocchi numbers and polynomials with weight α, we establish some interesting identities and relations.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N}\bigcup\{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number q. In this paper, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. As a definition of q-numbers, we use the notation of q-number of

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q} \tag{1.1}$$

(cf. [1–11]). Note that $\lim_{q\to 1}[x]_q = x$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim [1, 3],

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{1.2}$$

From (1.2), we have the well-known integral equation

$$q^n I_{-1}(f_n) + (-1)^{n-1} I_q(f) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l f(l) \tag{1.3}$$

(see [1, 3]), where $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$.

For $\alpha \in \mathbb{N}$, in [11], the q-Genocchi polynomials with weight α are introduced by

$$t \int_{\mathbb{Z}_p} e^{[x+y]_{q^\alpha} t} \, d\mu_{-q}(y) = \sum_{n=0}^{\infty} \widetilde{G}_{n,q}^{(\alpha)}(x) \frac{t^n}{n!}. \tag{1.4}$$

By comparing the coefficients of both sides of (1.4), we have

$$\begin{gathered} \widetilde{G}_{0,q}^{(\alpha)}(x) = 0, \\ \frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{(n+1)} = \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n \, d\mu_{-q}(y), \qquad \text{for } n \in \mathbb{N}. \end{gathered} \tag{1.5}$$

In the special case, $x = 0$, $\widetilde{G}_{n,q}^{(\alpha)}(0) = \widetilde{G}_{n,q}^{(\alpha)}$ are called the nth q-Genocchi numbers with weight α.

2. q-Genocchi Numbers and Polynomials with Weight α

In this section, we show some new identities on the q-Genocchi numbers and polynomials with weight α. And we establish the distribution relation for q-Genocchi polynomials with weight α.

From (1.5), we can easily see that

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{n+1} = \frac{[2]_q}{[\alpha]_q^n (1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{1+q^{\alpha l+1}}. \tag{2.1}$$

From (1.5) and (2.1), we note that

$$\begin{aligned} \frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{n+1} &= \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n \, d\mu_{-1}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \int_{\mathbb{Z}_p} [y]_{q^\alpha}^l \, d\mu_{-q}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \frac{\widetilde{G}_{l+1,q}^{(\alpha)}}{l+1}. \end{aligned} \tag{2.2}$$

Note that $(1/(l+1))\binom{n}{l} = (1/(n+1))\binom{n+1}{l+1}$ and from (2.2), we have the relation of polynomials and numbers,

$$\begin{aligned}\frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{n+1} &= \frac{1}{(n+1)}\sum_{l=0}^{n}\binom{n+1}{l+1}[x]_{q^\alpha}^{n+1-l}q^{\alpha l x}\widetilde{G}_{l+1,q}^{(\alpha)} \\ &= \frac{1}{(n+1)q^{\alpha x}}\sum_{l=0}^{n+1}\binom{n+1}{l+1}[x]_{q^\alpha}^{n+1-l}q^{\alpha(l+1)x}\widetilde{G}_{l+1,q}^{(\alpha)} \\ &= \frac{1}{(n+1)q^{\alpha x}}\left([x]_{q^\alpha}+q^{\alpha l x}\widetilde{G}_{q}^{(\alpha)}\right)^{n+1},\end{aligned} \tag{2.3}$$

with the usual convention of replacing $(\widetilde{G}_q^{(\alpha)})^n$ by $(\widetilde{G}_{n,q}^{(\alpha)})$.

Thus, by (2.3), we have a theorem.

Theorem 2.1. *For $\alpha \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned}q^{\alpha x}\widetilde{G}_{n+1,q}^{(\alpha)}(x) &= (n+1)q^{\alpha x}\left([x]_{q^\alpha}+q^{\alpha x}\widetilde{G}_{q}^{(\alpha)}\right)^{n+1} \\ &= \sum_{l=0}^{n+1}\binom{n+1}{l}[x]_{q^\alpha}^{n+1-l}q^{\alpha x}\widetilde{G}_{l,q}^{(\alpha)}.\end{aligned} \tag{2.4}$$

In (1.3), *if we take $n = 1$,*

$$qI_{-1}(f_1) + I_{-1}(f) = [2]_q. \tag{2.5}$$

We apply $f(x) = e^{[x]_{q^\alpha}t}$ with (1.5), *and we have the following:*

$$\begin{aligned}[2]_q &= \sum_{n=0}^{\infty}\left(q\int_{\mathbb{Z}_p}[x+1]_{q^\alpha}^{n}d\mu_{-q}(x)+\int_{\mathbb{Z}_p}[x]_{q^\alpha}^{n}d\mu_{-q}(x)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(q\frac{\widetilde{G}_{n+1,q}^{(\alpha)}(1)}{n+1}+\frac{\widetilde{G}_{n+1,q}^{(\alpha)}}{n+1}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

By comparing the coefficients on both the sides in (2.6), *we get*

$$q\frac{\widetilde{G}_{n+1,q}^{(\alpha)}(1)}{n+1}+\frac{\widetilde{G}_{n+1,q}^{(\alpha)}}{n+1} = \begin{cases}[2]_q & \text{if } n=0, \\ 0 & \text{if } n>0.\end{cases} \tag{2.7}$$

From (2.2) *and* (2.7), *we can derive the following:*

$$\begin{gathered}\widetilde{G}_{1,q}^{(\alpha)}(1)=1,\\ q^{1-\alpha}\left(q^{\alpha}\widetilde{G}_{q}^{(\alpha)}+1\right)^{n}+\widetilde{G}_{n,q}^{(\alpha)}=0 \quad \text{if } n\in\mathbb{N},\end{gathered} \tag{2.8}$$

with the usual convention of replacing $(\widetilde{G}_{q}^{(\alpha)})^{n}$ *by* $\widetilde{G}_{n,q}^{(\alpha)}$.

For a fixed odd positive integer d *with* $(p,d)=1$, *we set*

$$\begin{gathered}X=X_{d}=\frac{\lim_{\overline{N}}\mathbb{Z}}{dp^{N}\mathbb{Z}}, \quad X_{1}=\mathbb{Z}_{p},\\ X^{*}=\bigcup_{\substack{0<a<dp,\\ (a,p)=1}}(a+dp\mathbb{Z}_{p}),\\ a+dp^{N}\mathbb{Z}_{p}=\left\{x\in X\mid x\equiv a\ \left(\operatorname{mod}\ dp^{N}\right)\right\},\end{gathered} \tag{2.9}$$

where $a\in\mathbb{Z}$ *satisfies the condition* $0\le a<dp^{N}$. *For the distribution relation for the* q*-Genocchi polynomials with weight* α, *we consider the following:*

$$\begin{aligned}\int_{\mathbb{Z}_p}[x+y]_{q^{\alpha}}^{n}\,d\mu_{-q}(y)&=\int_{\mathbb{Z}_p}[n+y]_{q^{\alpha x}}^{n}\,d\mu_{-q}(y)\\ &=\frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^{a}q^{a}\int_{\mathbb{Z}_p}\left[\frac{x+a}{d}+y\right]_{q^{\alpha d}}^{n}d\mu_{-q}(y).\end{aligned} \tag{2.10}$$

By (1.5) *and* (2.10), *we get a theorem.*

Theorem 2.2. *For* $\alpha\in\mathbb{N}$ *and* $n\in\mathbb{Z}_{+}$, $d\in\mathbb{N}$ *with* $d\equiv 1(\operatorname{mod}\ 2)$, *one has*

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha)}(x)}{n+1}=\frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q}}\sum_{a=0}^{d-1}(-1)^{a}q^{a}\widetilde{G}_{n+1,q^{d}}^{(\alpha)}\left(\frac{x+a}{a}\right). \tag{2.11}$$

3. Higher-Order q-Genocchi Numbers and Polynomials with Weight α

In this section, we define higher-order q-Genocchi polynomials $\widetilde{G}_{n+1,q}^{(\alpha)}(h,k\mid x)$ and numbers $\widetilde{G}_{n+1,q}^{(\alpha)}(h,k)$ with weight α. We find an integral equation for higher-order q-Genocchi numbers with weight α. And we establish a combination property.

Let $\alpha \in \mathbb{Z}$ and $h, k \in \mathbb{Z}_+$, for $n \in \mathbb{Z}_+$, then we define higher-order q-Genocchi polynomials with weight α as follows:

$$\begin{aligned}
\frac{\widetilde{G}_{n+1,q}^{(\alpha)}(h,k \mid x)}{(n+1)} &= \underbrace{\int_{Z_p} \cdots \int_{Z_p}}_{k\text{-times}} [x_1 + x_2 + \cdots + x_k + x]_{q^\alpha}^n q^{(h-1)x_1 + \cdots + (h-k)x_k} \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\
&= \frac{[2]_q^k}{[\alpha]_q^n (1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{(1+q^{\alpha l + h}) \cdots (1+q^{\alpha l + h - k + 1})} \\
&= \frac{[2]_q^k}{[\alpha]_q^n (1-q)^m} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{(-q^{\alpha l + h} : q^{-1})_k},
\end{aligned} \tag{3.1}$$

where $(x : q)_h = \prod_{i=0}^{h-1} (1 - xq^i)$.

In the special case, $x = 0$, $\widetilde{G}_{n+1,q}^{(\alpha)}(h,k|0) = \widetilde{G}_{n+1,q}^{(\alpha)}(h,k)$ are called the $(n+1)$th (h,k)-Genocchi numbers with weight α.

In (3.1), apply the following identity:

$$[x_1 + x_2 + \cdots + x_k]_{q^\alpha} (1 - q^\alpha) + q^\alpha (x_1 + x_2 + \cdots + x_k) = 1, \tag{3.2}$$

and we have a theorem.

Theorem 3.1. *For $\alpha \in \mathbb{N}$ and $h, k \in \mathbb{Z}_+$, one has*

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha)}(h,k)}{n+1} = (1 - q^\alpha) \frac{\widetilde{G}_{n+2,q}^{(\alpha)}(h,k)}{n+2} + \frac{\widetilde{G}_{n+1,q}^{(\alpha)}(h+\alpha,k)}{n+1}. \tag{3.3}$$

We consider, for $\alpha \in \mathbb{N}$ and $h, k \in \mathbb{Z}_+$,

$$\begin{aligned}
&\sum_{j=0}^{i} \binom{i}{j} (q^\alpha - 1)^j \frac{\widetilde{G}_{n+j-i+1}^{(\alpha)}(h-\alpha,k)}{n+j-i+1} \\
&\quad = \sum_{j=0}^{i} \binom{i}{j} (q^\alpha - 1)^j \underbrace{\int_{Z_p} \cdots \int_{Z_p}}_{k\text{-times}} \left[\sum_{\ell=1}^{k} x_\ell \right]_{q^\alpha}^{n-i-j} q^{\sum_{\ell=1}^{k} (h-\alpha-\ell) x_\ell} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\
&\quad = \sum_{j=0}^{i} \binom{i}{j} (q^\alpha - 1)^j \underbrace{\int_{Z_p} \cdots \int_{Z_p}}_{k\text{-times}} \left[\sum_{\ell=1}^{k} x_\ell \right]_{q^\alpha}^{n-i+j} q^{\sum_{\ell=1}^{k} (h-\ell) x_\ell} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\
&\quad = \sum_{j=0}^{i} \binom{i}{j} (q^\alpha - 1)^j \frac{\widetilde{G}_{n+j-i+1}^{(\alpha)}(h,k)}{n+j-i+1}.
\end{aligned} \tag{3.4}$$

Therefore, we obtain the following combinatorial property.

Theorem 3.2. *For $\alpha \in \mathbb{N}$ and $h, k \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^{i} \binom{i}{j} (q^{\alpha}-1)^{j} \frac{\widetilde{G}_{n+j-i+1}^{(\alpha)}(h-\alpha,k)}{n+j-i+1} = \sum_{j=0}^{i-1} \binom{i-1}{j} (q^{\alpha}-1)^{j} \frac{\widetilde{G}_{n+j-i+1}^{(\alpha)}(h,k)}{n+j-i+1}. \tag{3.5}$$

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