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Research Article

Application of Spectral Methods to Boundary Value Problems for Differential Equations

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We try to generalize the concept of a spectrum in the nonlinear case starting from its splitting into several subspectra, not necessarily disjoint, following the classical decomposition of the spectrum. To obtain an extension of spectrum with rich properties, we replace the identity map by a nonlinear operator J acting between two Banach spaces X and Y, which takes into account the analytical and topological properties of a given operator F, although the original definitions have been given only in the case $X = Y$ and $J = I$. The FMV spectrum reflects only asymptotic properties of F, while the Feng's spectrum takes into account the global behaviour of F and gives applications to boundary value problems for ordinary differential equations or for the second-order differential equations, which are referred to as three-point boundary value problems with the classical or the periodic boundary conditions.

1. Introduction

Let us first recall the concept of a spectrum for linear operators acting in a complex Banach space X. We denote by $L(X)$ the algebra of all bounded linear operators on X and the resolvent set of L is defined by

$$\rho(L) = \left\{\lambda \in \not\subset / (\lambda I - L)^{-1} \in L(X)\right\} \tag{1.1}$$

and the spectrum of L by

$$\sigma(L) = \not\subset / \rho(L). \tag{1.2}$$

This spectrum consists of all complex scalars such that $\lambda I - L$ is not invertible because the inverse does not exists as a bounded operator. This can happen when $\lambda I - L$ is not one to one, which means that there exists $x \neq 0$ such that $Lx = \lambda x$. Such a value of λ is called an eigenvalue of L and the set of all eigenvalues is called the point spectrum.

For a linear compact operator L, the spectrum $\sigma(L)$ has some remarkable properties: it is compact, nonempty and it is countable, so has an empty interior. Moreover, it is bounded by the spectral radius given by Gelfand's formula:

$$r(L) = \lim_{n \to \infty} \sqrt[n]{\|L^n\|} \tag{1.3}$$

and it commutes with any polynomial p, that is,

$$\sigma(p(L)) = p(\sigma(L)) \tag{1.4}$$

by the spectral mapping theorem.

Finally, the map $\rho(L) \ni \lambda \to (\lambda I - L)^{-1} \in L(X)$ is analytic and the multivalued map $L(X) \ni L \to \sigma(L) \in 2^{\not{C}}$ is upper semicontinuous.

It is imposible to have a theory for nonlinear operators which collect all the useful properties of a spectrum that are satisfied by linear maps; see [1]. The spectrum of a nonlinear map contains rather little information about the map itself and may be empty. There are various notions of a spectrum for different classes of nonlinear maps that are useful in the study of nonlinear equations; see [2].

So, given a continuous nonlinear operator $F : X \to X$, one should try to define a spectrum $\sigma(F)$ such that $\sigma(F)$ has the usual properties like nonemptiness, compactness, and so forth, as in the linear case; $\sigma(F)$ contains the point spectrum $\sigma_p(F)$ of F as in the linear case (where $\sigma_p(F) = \{\lambda \in \not{C} / F(u) = \lambda u\}$, for some $u \neq 0$); $\sigma(F)$ has reasonable applications, for instance, in existence and uniqueness problems, to boundary value problems, bifurcation problems; see [3].

We try to generalize the concept of a spectrum in the nonlinear case starting from its splitting into several subspectra, not necessarily disjoint. We follow the classical decomposition of the spectrum:

$$\sigma(L) = \sigma_p(L) \cup \sigma_d(L) \cup \sigma_{\mathrm{co}}(L), \tag{1.5}$$

where $\sigma_p(L)$ is the point spectrum of $L(\lambda I - L$ is not $1 - 1)$; $\sigma_d(L)$ is the defect spectrum of $L(\lambda I - L$ is not onto); $\sigma_{\mathrm{co}}(L)$ is the compression spectrum of $L(\lambda I - L$ is not proper).

One way of defining an apropiate spectrum is to restrict attention to specific classes of maps and to replace the algebra $L(X)$ in (1.1) by other classes of continuous nonlinear operators. More general, let X be a Banach space over a field $K(R$ or $\not{C})$ and $M(X)$ denote a class of continuous maps which contains the identity operator I. This leads to the Rhodius resolvent spectrum [4]. One of the first such definition was given by Neuberger (1969), who took $M(X) = C^1(X)$, the Fréchet differentiable maps on X. The corresponding spectrum is allways nonempty but may not be closed. Another possible choice is that of Lipschitz continuous maps $F : X \to X$, which leads to the Kachurovskij spectrum [5], which is closed but may be empty.

All these spectra have "bad" properties, they do not satisfy the above minimal requirements (see [3, 6–9], for a comparison between these spectra).

Later, generalizations of spectrum are based on the Kuratowski measure of noncompactness $\alpha(M)$ of a bounded set $M \subset X$. It is defined as infimum of all $\varepsilon > 0$ such that M may be covered by finitely sets which have at most diameter ε. The name is motivated by the fact that $\alpha(M) = 0$ if and only if M has a compact closure.

A nonlinear operator $F : X \to Y$ satisfy two conditions:

$$\begin{gathered}\alpha(F(M)) \leq k\alpha(M), \\ \alpha(F(M)) \geq K\alpha(M), \quad (\forall)X \supset M\text{bounded}.\end{gathered} \tag{1.6}$$

The smallest constant k denoted by $[F]_A$ and the largest constant K denoted by $[F]_a$ are the first (metric) characteristics. So $[F]_A = 0$ if and only if F is compact and $[F]_a > 0$ implies that F is proper on closed bounded sets.

A major contribution was made in 1978 by Furi et al. see [10]. They employed the concept of stably solvable maps together with the asymptotic characteristics:

$$[F]_Q = \limsup_{\|x\| \to \infty} \frac{\|F(x)\|}{\|x\|}, \qquad [F]_q = \liminf_{\|x\| \to \infty} \frac{\|F(x)\|}{\|x\|}, \tag{1.7}$$

the upper and lower quasinorm, respectively.

A continuous function $F : X \to Y$, is stably solvable if, for any compact map $G : X \to Y$ with $[G]_Q = 0$, the coincidence equation $F(u) = G(u)$ has a solution $u \in X$. Taking $G(u) = v$ for a fixed $v \in Y$ it is clear that the stable solvability is equivalent to surjectivity (only if F is linear); see [11].

To obtain an extension of spectrum with rich properties, we replace the identity map by a nonlinear operator $J : X \to Y$ which takes into account the analytical and topological properties of the given operator F. We can define spectra for pair of operators (F, J) between two Banach spaces X and Y, although the original definitions have been given only the case $X = Y$ and $J = I$, see [1].

The Furi-Martelli-Vignoli spectrum of the pair $(F, J) : X \to Y$ is defined by the union:

$$\sigma_{\text{FMV}}(F, J) = \sigma_{SS}(F, J) \cup \sigma_q(F, J) \cup \sigma_a(F, J), \tag{1.8}$$

where $\lambda \in \sigma_{SS}(F, J)$ *if* $\lambda J - F$ is not stably solvable; $\lambda \in \sigma_q(F, J)$ *if* $[\lambda J - F]_q = 0$; $\lambda \in \sigma_a(F, J)$ *if* $[\lambda J - F]_a = 0$.

Relating to the previous decomposition for $L \in L(x)$, we get the relations:

$$\sigma_{SS}(L, I) = \sigma_d(L), \qquad \sigma_q(L, I) \supseteq \sigma_p(L), \qquad \sigma_a(L, I) \subseteq \sigma_{\text{co}}(L). \tag{1.9}$$

The FMV spectrum is closed and upper semicontinuous, as we can see in [10], but has one defect: did not contain the point spectrum, in the sense that it did not contain the eigenvalue λ such that $\lambda x - F(x) = 0$ from some nonzero x.

The FMV theory was so successful for many developments until 1997, when Feng, see [12], introduced a spectrum defined in a similar way, but with other concepts of solvability and characteristics which contains the classical point spectrum.

The Feng spectrum takes into account the global behaviour of F, while the FMV spectrum reflects only asymptotic properties of F.

Feng developed an attractive theory and was able to use the theory to give applications to boundary value problems.

2. Application for Ordinary Differential Equations Involving Spectral Methods

We consider the problem:

$$\begin{gathered} \dot{x}(t) - A(t)x(t) = \varepsilon g(t, x(t)), \quad 0 \leq t \leq T \\ Lx = \theta, \end{gathered} \tag{2.1}$$

where $A : [0, T] \to R^{n\times n}$ is continuous matrix valued function; $g : [0, T] \times R^n \to R^n$ is a Carathéodory function; $L : C([0, T], R^n) \to R^n$ is a bounded linear operator which associates to each continuous function $x : [0, T] \to R^n$ a vector $Lx \in R^n$ and $\varepsilon \neq 0$ is a scalar parameter.

Putting

$$Dx(t) = \frac{dx}{dt} - A(t)x, \tag{2.2}$$

$$G(x)(t) = g(t, x(t)), \tag{2.3}$$

we may write (2.1) as an operator equation:

$$Dx = \varepsilon G(x) \tag{2.4}$$

in the Banach space $X = \{x \in (C([0, 1], R^n)/Lx = \theta\}$.

By $U(t, s)$ we denote the Cauchy function of the operator family $A(t)$ which means the unique solution of the linear Volterra integral equation:

$$U(t, s) = I + \int_s^t A(\pi)U(\pi, s)d\pi, \quad 0 \leq t, s \leq T \tag{2.5}$$

and by

$$Ez(t) = \int_0^t U(t, s)z(s)ds, \quad 0 \leq t \leq T \tag{2.6}$$

the associated evolution operator.

It is clear the fact that $DE = I$, that is, the operator (2.6) is the right inverse to the differential operator (2.2).

Assume that the composition $L_U = LU_0$ of the boundary operator L in (2.1) and the operator $U_0 : R^n \to C([0,T], R^n)$ defined by $(U_0x)(t) = U(t,0)x, x \in R^n$ is an isomorphism in R^n.

The nonlinear operator F defined by

$$F(x) = \left(I - U_0 L_U^{-1} L\right) EG(x) \tag{2.7}$$

maps the Banach space X into itself.

For any $x \in X$, we have

$$LF(x) = LEG(x) - LU_0 L_U^{-1} LEG(x) = LEG(x) - LEG(x) = \theta \Longrightarrow F(x) \in X. \tag{2.8}$$

We put

$$M = \sup_{0 \le t,s \le T} \|U(t,s)\| \tag{2.9}$$

and we denote by

$$\mu_G(r) = \sup_{\|x\| \le r} \|G(x)\| \tag{2.10}$$

the growth function of the Nemytskij operator (2.3).

Proposition 2.1. *Suppose that the nonlinearity $g : [0,T] \times R^n \to R^n$ satisfies a growth condition:*

$$|g(t,u)| \le a(t) + b(t)|u|, \quad 0 \le t \le T, \ u \in R^n \tag{2.11}$$

for some $a, b \in L_1([0,T])$. Define a scalar function $\varphi : (0,\infty) \to (0,\infty)$ by

$$\varphi(r) = M^2 \left\| L_U^{-1} \right\| \|L\| \mu_G(r), \quad r > 0 \tag{2.12}$$

with M given by (2.9) *and $\mu_G(r)$ given by* (2.10)*. Then the following linear problem:*

$$\lambda x - Lx = y, \quad y \in X \tag{2.13}$$

admits a solution $x \in X$ if and only if $1/\varepsilon$ belongs to the point spectrum of the operator (2.7)*. Moreover, the asymptotic point spectrum of this operator satisfies the inclusion*

$$\sigma_q(F) \subseteq \left\{ \lambda \in \frac{R}{\lambda} \exp\left(-\frac{M\|b\|_1}{\lambda} \right) \le \lim_{r \to \infty} \frac{\varphi(r)}{r} \right\}. \tag{2.14}$$

Proof. We put $\lambda = 1/\varepsilon$. It is well known the fact that every solution of the boundary value problem (2.1) solves the eigenvalue equation $F(x) = \lambda x$ and vice versa. We only have to prove (2.14).

Let $x \in X$ be a solution of nonlinear equation, see [2]:

$$\lambda x - F(x) = y, \quad y \in X \tag{2.15}$$

for some $\lambda > 0$. Then

$$\begin{aligned} |\lambda||x(t)| &\leq \left|y(t)\right| + |EG(x)(t)| + \left|U_0 L_U^{-1} LEG(x)(t)\right| \\ &\leq \left|y(t)\right| + M\|a\|_1 + M\int_0^t b(s)|x(s)|ds + M^2\left\|L_U^{-1}\right\|\|L\|\mathrm{g}|(t,x(t))| \end{aligned} \tag{2.16}$$

so $|x(t)| \leq c_r + (M/\lambda)\int_0^t b(s)|x(s)|ds$, $\|x\|_\infty \leq r$, where

$$c_r = \frac{1}{\lambda}\left[\|y\|_\infty + M\|a\|_1 + M^2\left\|L_U^{-1}\right\|\|L\|\mu_G(r)\right]. \tag{2.17}$$

Applying Gronwall's lemma to (2.16) we have

$$|x(t)| \leq c_r \exp\left(\frac{M\|b\|_1}{\lambda}\right) \tag{2.18}$$

hence

$$\frac{\lambda\exp(-M\|b\|_1)}{\lambda} \leq \frac{c_r\lambda}{\|x\|_\infty} + \frac{\|\lambda x - F(x)\|}{\|x\|_\infty} + \frac{M\|a\|_1}{\|x\|_\infty} + \frac{\varphi(r)}{\|x\|_\infty}, \quad \text{for } 0 < \|x\|_\infty \leq r. \tag{2.19}$$

Passing to the limit $r \to \infty$ we conclude that $[\lambda I - F]_q > 0$ for any λ such that

$$\lim_{r\to\infty}\frac{\varphi(r)}{r} < \lambda\exp\left(-\frac{M\|b\|_1}{\lambda}\right) \tag{2.20}$$

(this means that the nonlinear operator F maps the Banach space X into itself); see [13].

The last hypotesis is easily checked by asuming that $\mathrm{g}(t,u) = a(t) + b(t)u$ with $a, b \in L_1[0,T]$. The growth function in this case satisfies the trivial estimate:

$$\mu_G(r) \leq \|a\|_1 + \|b\|_1 r \tag{2.21}$$

so, the condition (2.20) becomes

$$M^2\left\|L_U^{-1}\right\|\|L\|\|b\|_1 < \frac{1}{\varepsilon}\exp(-M\varepsilon\|b\|_1). \tag{2.22}$$

Putting $M\|b\|_1\varepsilon = \eta$ and $\omega(\eta) = 1/\eta e^\eta$ we can rewrite (2.22) as $\omega(\eta) > M\|L_U^{-1}\|\|L\|$.

But the function $\omega : (0,\infty) \to (0,\infty)$ is strictly decreasing, hence invertible, with $\lim_{\eta\to 0_+}\omega(\eta) = \infty$, $\lim_{\eta\to\infty}\omega(\eta) = 0$ and (2.22) is true for $0 < \varepsilon < \omega^{-1}(M\|L_U^{-1}\|\|L\|)/M\|b\|_1$.

Passing to $\lambda = 1/\varepsilon$, this gives an explicit bound of the type (2.14) for an asymptotic point spectrum $\sigma_q(F)$. □

3. Another Type of Boundary Value Problems for the Second-Order Differential Equation

We consider the second-order differential equation

$$\ddot{x}(t) + g(t) f(x(t)) = 0 \tag{3.1}$$

with the condition

$$x(0) = 0, \qquad x(1) = \alpha x(\eta) \tag{3.2}$$

or

$$\dot{x}(0) = 0, \qquad x(1) = \alpha x(\eta), \tag{3.3}$$

where $\eta \in (0,1)$ is fixed.

This kind of problems are referred to as three-point boundary value problems. Many existence results have been obtained and is known that, when $\alpha\eta \neq 1$ in (3.2) or $\alpha \neq 1$ in (3.3), these boundary value problems may be transformed equivalently in a Hammerstein integral equation:

$$x(s) = \int_0^1 k(s,t) g(t) f(x(t)) dt, \tag{3.4}$$

where the Kernel function K depends on the boundary condition (3.2) or (3.3).

In case of the boundary condition (3.2), the Kernel K (Green's function) from (3.4) is given by

$$K(s,t) = \frac{s(1-t)}{1-\alpha\eta} - l(s,t,\alpha,\eta), \tag{3.5}$$

where

$$l(s,t,\alpha,\eta) = \begin{cases} \dfrac{\alpha s(\eta - t)}{1-\alpha\eta} + s - t, & t \leq \min\{\eta, s\}, \\ \dfrac{\alpha s(\eta - t)}{1-\alpha\eta}, & s < t < \eta, \\ \mathrm{s} - t, & \eta < t \leq s, \\ 0, & t > \max\{\eta, \mathrm{s}\}. \end{cases} \tag{3.6}$$

In case of the boundary condition (3.3), the Kernel is given by

$$k(s,t) = \frac{1-t}{1-\alpha} - m(s,t,\alpha,\eta), \tag{3.7}$$

where

$$m(s,t,\alpha,\eta) = \begin{cases} \dfrac{\alpha(\eta-t)}{1-\alpha} + s - t, & t \le \min\{\eta, s\}, \\ \dfrac{\alpha(\eta-t)}{1-\alpha}, & s < t \le \eta, \\ s-t, & \eta < t \le s, \\ 0, & t > \max\{\jmath, \mathrm{s}\}. \end{cases} \tag{3.8}$$

We define the scalar function k by

$$k(t) = \max_{0\le s\le 1} |k(s,t)|, \quad 0 \le t \le 1. \tag{3.9}$$

The function f in (3.1) is continuous and positive and satisfy the growth condition:

$$\left|f(t,u)\right| \le a(t) + b(t)|u|, \quad 0 \le t \le 1,\ u \in R, \tag{3.10}$$

where we may suppose that the functions a and b are constant and we can rewrite (3.10) as

$$\left|f(u)\right| \le a + b|u|. \tag{3.11}$$

Solving the three-point boundary value problems (3.1) with condition (3.2), or (3.1) with condition (3.3) can be reduced to solving a Hammerstein integral equation of the from:

$$\lambda x(s) - \int_0^1 k(s,t) f(t,x(t))dt = y(s), \quad 0 \le s \le 1 \tag{3.12}$$

with $\lambda = 1$ and $y(s) = 0$ that is (3.4).

We have the following four propositions (for proofs, see [Nonlinear Spectral Theory [13, pages 355–358]]).

Proposition 3.1. *Suppose that* $\alpha\eta \neq 1$ *and* $f : R \to R$ *is a Carathéodory function which satisfies the growth condition* (3.11). *Then the boundary value problem* (3.1) *with condition* (3.2) *has at least one solution provided that*

$$\|bg\|_1 < \begin{cases} \dfrac{4(1-\alpha\eta)}{1-\alpha}, & \alpha\eta \leq 0, \\ \dfrac{4(1-\alpha\eta)}{\max\{\alpha,1\}}, & 0 < \alpha\eta < 1, \\ \dfrac{4(\alpha\eta-1)}{\alpha}, & \alpha\eta > 1, \end{cases} \tag{3.13}$$

or

$$\|bg\|_2 < \begin{cases} \dfrac{\sqrt{30}(1-\alpha\eta)}{1-\alpha}, & \alpha\eta \leq 0, \\ \dfrac{\sqrt{30}(1-\alpha\eta)}{\max\{\alpha,1\}}, & 0 < \alpha\eta < 1, \\ \dfrac{\sqrt{30}(\alpha\eta-1)}{\alpha}, & \alpha\eta > 1. \end{cases} \tag{3.14}$$

Proposition 3.2. *Suppose that* $\alpha \neq 1$ *and* $f : R \to R$ *is a Carathéodory function which satisfies the growth condition* (3.11). *Then the boundary value problem* (3.1) *with condition* (3.3) *has at least one solution provided that*

$$\|bg\|_1 < \begin{cases} 1, & \alpha \leq 0, \\ 1-\alpha, & 0 < \alpha < 1, \\ \dfrac{\alpha-1}{\alpha}, & \alpha > 1, \end{cases} \tag{3.15}$$

or

$$\|bg\|_2 < \begin{cases} \sqrt{3}, & \alpha \leq 0, \\ \sqrt{3}(1-\alpha), & 0 < \alpha < 1, \\ \dfrac{\sqrt{3}(\alpha-1)}{\alpha}, & \alpha \geq 1. \end{cases} \tag{3.16}$$

Proposition 3.3. *Let* $\alpha\eta \neq 1$. *Then the following alternative holds.*

(i) *The quasilinear there-point boundary value problem:*

$$\begin{aligned} \ddot{x}\ (t) &= \mu f(t, x(t), \dot{x}(t)) + y(t), \\ x(0) &= 0, \\ x(1) &= \alpha x(\eta), \end{aligned} \tag{3.17}$$

where $y \in C[0,1]$ *is given and* $\mu \neq 0$ *has a solution for* $\mu = 1$ *and any function* $y \in L_1[0,1]$.

(ii) *There exist some* $\mu \leq 1$ *such that the boundary value problem* (3.17) *has a nontrivial solution for* $y(t) = 0$.

(We are interested in solutions x *of* (3.17) *in the Sobolev space* $W_1^2[0,1]$ *of all absolutely continuous functions* x *such that* $\dot{x}$ *is also absolutely continuous and* $\ddot{x} \in L_1[0,1]$*).*

Proposition 3.4. *Let* $\alpha\eta \neq 1$. *Then the bounduary value problem* (3.17) *has a solution for any function* $y \in L_1[0,1]$ *provided that*

$$|\mu|\left(\|p\|_1 + \|q\|_1\right) < \frac{1}{c(\alpha\eta)}, \tag{3.18}$$

where

$$c(\alpha,\eta) = 1 + \frac{\alpha\eta + 1}{|1 - \alpha\eta|} \tag{3.19}$$

or

$$c(\alpha,\eta) = \begin{cases} \dfrac{2}{1-\alpha\eta}, & \alpha\eta < 1, \\ \dfrac{2\alpha\eta}{\alpha\eta - 1}, & \alpha\eta > 1. \end{cases} \tag{3.20}$$

We can foccus on the equation

$$\ddot{x}\ (t) = \mu f(t, x(t), \dot{x}(t)) + y(t), \tag{3.21}$$

where $f : [0,1] \times R^n \times R^n \rightarrow R^n$ *and* $y : [0,1] \rightarrow R^n$ *are supposed to be continuous vector functions and* $\mu \neq 0$.

We also can consider (3.21) *together with the classical boundary condition*

$$x(0) = x(1) = 0 \tag{3.22}$$

or with the periodic boundary conditions

$$\begin{gathered} x(0) = x(1), \\ \dot{x}(0) = \dot{x}(1). \end{gathered} \tag{3.23}$$

Now we have two different problems. The boundary value problem (3.21) with condition (3.22) may be studied by means of the Feng or FMV spectrum, while the second boundary value problem given by (3.21) with condition (3.23) requires the semilinear spectra. For the first problem we put

$$\begin{gathered} X = \left\{x \in C^2[0,1] / x(0) = x(1) = 0\right\}, \\ Y = C[0,1] \end{gathered} \tag{3.24}$$

and we define the operators $L, F : X \to Y$ by

$$\begin{aligned} Lx(t) &= \ddot{x}(t), \\ F(x)(t) &= f(t, x(t), \dot{x}(t)). \end{aligned} \tag{3.25}$$

Then L is invertible on Y with inverse

$$L^{-1}y(s) = \int_0^1 k(s,t)y(t)dt, \tag{3.26}$$

where

$$k(s,t) = \begin{cases} s(t-1), & 0 \le s \le t \le 1, \\ t(s-1), & 0 \le t \le s \le 1 \end{cases} \tag{3.27}$$

is the classical Green's function of L.

The solvability of semilinear equation, see [14],

$$\lambda Lx - F(x) = y, \quad y \in Y \tag{3.28}$$

reduces to the solvability of the classical eigenvalue equation

$$\lambda x - L^{-1}F(x) = z, \quad z \in X. \tag{3.29}$$

For the second problem, see [2, 14], we put

$$\begin{gathered} X = \left\{ x \in C^2[0,1] / x(0) = x(1),\ \dot{x}(0) = \dot{x}(1) \right\}, \\ Y = C[0,1]. \end{gathered} \tag{3.30}$$

In this case, the operator L is not invertible and again we define the operators $L, F : X \to Y$ by (3.25).

We have

$$\begin{gathered} N(L) = \{x \in X / x(t) = \text{const}\} \cong R^n, \\ R(L) = \left\{ y \in Y / Qy = \theta \right\} \cong \frac{Y}{R^n}, \end{gathered} \tag{3.31}$$

where

$$Qy = \int_0^1 y(t)dt. \tag{3.32}$$

For the projection $P : X \to R$ we may choose $Px = x(0)$, so we have the decompositions

$$X = R^n \oplus X_0 \text{ and } Y = R^n \oplus Y_0. \tag{3.33}$$

So $\dim N(L) = co\dim R(L) = n$ which shows that L is a Fredholm operator of index zero.

The restriction of (3.26) to the range of L is the operator given by

$$L_p^{-1} = (L/_{X_0})^{-1} : R(L) \longrightarrow X_0. \tag{3.34}$$

The linear operator (the natural quotient map) $\Pi : Y \to Y/R(L)$ and $\Lambda : Y/R(L) \to N(L)$ (the natural isomorphism induced by L) are given by

$$\begin{gathered} \Pi y = [y] = \{\tilde{y} \in Y/Q\tilde{y} = Qy\}, \\ \Lambda[y] = Qy. \end{gathered} \tag{3.35}$$

As a canonical homeomorphism $h : Y/R(L) \to Y_0$ we may choose $h[y] = Qy$. So the linear isomorphism $L + h\Lambda^{-1}P : X \to Y$ is given by

$$\left(L + h\Lambda^{-1}P\right)x(t) = \ddot{x}(t) + x(0) \tag{3.36}$$

and its inverse $\Lambda\Pi + K_{PQ} = \Lambda\Pi + L_P^{-1}(I - Q) : Y \to X$, is

$$(\Lambda\Pi + K_{PQ})y(s) = \int_0^1 k(s,t)y(t)dt + \left(1 - \int_0^1 (k(s,t)dt) \int_0^1 y(t)dt\right). \tag{3.37}$$

If L is a bijection between X and Y we have $X_0 = X$, $Y_0 = \theta$, $Px = Qy = \theta$ and $K_{PQ} = L^{-1}$.

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