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Research Article

Conharmonic Curvature Tensor on $N(K)$-Contact Metric Manifolds

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The object of the present paper is to characterize $N(k)$-contact metric manifolds satisfying certain curvature conditions on the conharmonic curvature tensor. In this paper we study conharmonically symmetric, ξ-conharmonically flat, and ϕ-conharmonically flat $N(k)$-contact metric manifolds.

1. Introduction

Let M and $\overline{M}$ be two Riemannian manifolds with g and $\overline{g}$ being their respective metric tensors related through

$$\overline{g}(X,Y) = e^{2\sigma} g(X,Y), \tag{1.1}$$

where σ is a real function. Then M and $\overline{M}$ are called conformally related manifolds and the correspondence between M and $\overline{M}$ is known as conformal transformation [1].

It is known that a harmonic function is defined as a function whose Laplacian vanishes. A harmonic function is not invariant, in general. The condition under which a harmonic function remains invariant have been studied by Ishii [2] who introduced the conharmonic transformation as a subgroup of the conformal transformation (1.1) satisfying the condition

$$\sigma^i_{,i} + \sigma_{,i}\sigma^i_{,} = 0, \tag{1.2}$$

where *comma* denotes the covariant differentiation with respect to the metric g.

A rank-four tensor $\widetilde{C}$ that remains invariant under conharmonic transformation for a $(2n+1)$-dimensional Riemannian manifold M is given by

$$\begin{aligned}\widetilde{C}(X,Y,Z,W) &= \widetilde{R}(X,Y,Z,W)\\ &\quad -\frac{1}{2n-1}\big[g(Y,Z)S(X,W)-g(X,Z)S(Y,W)\\ &\qquad +S(Y,Z)g(X,W)-S(X,Z)g(Y,W)\big],\end{aligned} \tag{1.3}$$

where $\widetilde{R}$ denotes the Riemannian curvature tensor of type $(0,4)$ defined by

$$\widetilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W), \tag{1.4}$$

where R is the Riemannian curvature tensor of type $(1,3)$ and S denotes Ricci tensor of type $(0,2)$, respectively.

The curvature tensor defined by (1.3) is known as conharmonic curvature tensor. A manifold whose conharmonic curvature vanishes at every point of the manifold is called conharmonically flat manifold. Thus this tensor represents the deviation of the manifold from conharmonic flatness. It satisfies all the symmetric properties of the Riemannian curvature tensor $\widetilde{R}$. There are many physical applications of the tensor $\widetilde{C}$. For example, in [3], Abdussattar showed that sufficient condition for a space-time to be conharmonic to a flat space-time is that the tensor $\widetilde{C}$ vanishes identically. A conharmonically flat space-time is either empty in which case it is flat or filled with a distribution represented by energy momentum tensor T possessing the algebraic structure of an electromagnetic field and conformal to a flat space-time [3]. Also he described the gravitational field due to a distribution of pure radiation in presence of disordered radiation by means of spherically symmetric conharmonically flat space-time. Conharmonic curvature tensor have been studied by Siddiqui and Ahsan [4], Özgür [5], and many others.

Let M be an almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η,g). At each point $p\in M$, decompose the tangent space T_pM into direct sum $T_pM=\phi(T_pM)\oplus\{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of T_pM generated by $\{\xi_p\}$. Thus the conformal curvature tensor C is a map

$$C: T_pM\times T_pM\times T_pM \longrightarrow \phi(T_pM)\oplus\{\xi_p\},\quad p\in M. \tag{1.5}$$

It may be natural to consider the following particular cases:

(1) $C: T_p(M)\times T_p(M)\times T_p(M)\ \rightarrow\ L(\xi_p)$, that is, the projection of the image of C in $\phi(T_p(M))$ is zero;

(2) $C: T_p(M)\times T_p(M)\times T_p(M)\ \rightarrow\ \phi(T_p(M))$, that is, the projection of the image of C in $L(\xi_p)$ is zero;

(3) $C:\ \phi(T_p(M))\times\phi(T_p(M))\times\phi(T_p(M))\ \rightarrow\ L(\xi_p)$, that is, when C is restricted to $\phi(T_p(M))\times\phi(T_p(M))\times\phi(T_p(M))$, the projection of the image of C in $\phi(T_p(M))$ is zero. This condition is equivalent to

$$\phi^2C(\phi X,\phi Y,\phi Z)=0. \tag{1.6}$$

Here cases 1, 2, and 3 are synonymous to conformally symmetric, ξ-conformally flat, and ϕ-conformally flat.

In [6], it is proved that a conformally symmetric K-contact manifold is locally isometric to the unit sphere. In [7], it is proved that a K-contact manifold is ξ-conformally flat if and only if it is an η-Einstein Sasakian manifold. In [8], some necessary conditions for a K-contact manifold to be ϕ-conformally flat are proved. In [9], a necessary and sufficient condition for a Sasakian manifold to be ϕ-conformally flat is obtained. In [10], projective curvature tensor in K-contact and Sasakian manifolds has been studied. Moreover, the author [11] considered some conditions on conharmonic curvature tensor $\widetilde{C}$, which has many applications in physics and mathematics, on a hypersurface in the semi-Euclidean space E_s^{n+1}. He proved that every conharmonically Ricci-symmetric hypersurface M satisfying the condition $\widetilde{C} \cdot R = 0$ is pseudosymmetric. He also considered the condition $\widetilde{C} \cdot \widetilde{C} = L_{\widetilde{C}} Q(g, \widetilde{C})$ on hypersurfaces of the semi-Euclidean space E_s^{n+1}.

Motivated by the studies of conformal curvature tensor in (see [6–9]) and the studies of projective curvature tensor in K-contact and Sasakian manifolds in [10] and Lorentzian para-Sasakian manifolds in [5], in this paper we study conharmonic curvature tensor in $N(k)$-contact metric manifolds.

Analogous to the considerations of conformal curvature tensor, we give following definitions.

Definition 1.1. A $(2n+1)$-dimensional $N(k)$-contact metric manifold is said to be conharmonically symmetric if $(\nabla_W \widetilde{C})(X, Y)Z = 0$, where $X, Y, Z, W \in TM$.

Definition 1.2. A $(2n+1)$-dimensional $N(k)$-contact metric manifold is said to be ξ-conharmonically flat if $\widetilde{C}(X, Y)\xi = 0$ for $X, Y \in TM$.

Definition 1.3. A $(2n+1)$-dimensional $N(k)$-contact metric manifold is said to be ϕ-conharmonically flat if $\widetilde{C}(\phi X, \phi Y, \phi Z, \phi W) = 0$, where $X, Y, Z, W \in TM$.

The paper is organized as follows. After preliminaries in Section 2, in Section 3 we consider conharmonically symmetric $N(k)$-contact metric manifolds. In this section we prove that if an n-dimensional $N(k)$-contact metric manifold is conharmonically symmetric, then it is locally isometric to the product $E^{(n+1)}(0) \times S^n(4)$. Section 4 deals with ξ-conharmonically flat $N(k)$-contact metric manifolds and we prove that an n-dimensional $N(k)$-contact metric manifold is ξ-conharmonically flat if and only if it is an η-Einstein manifold. Besides these some important corollaries are given in this section. Finally, in Section 5, we prove that a ϕ-conharmonically flat $N(k)$-contact metric manifold is a Sasakian manifold with vanishing scalar curvature.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying (see [12, 13])

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \tag{2.1}$$

An almost contact metric structure is said to be normal if the almost induced complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right) \tag{2.2}$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$, and f is a smooth function on $M \times \mathbb{R}$. Let g be the compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.3}$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2.1) it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \qquad g(X, \xi) = \eta(X), \tag{2.4}$$

for any vector fields X, Y on the manifold. An almost contact metric structure becomes a contact metric structure if $g(X, \phi Y) = d\eta(X, Y)$, for all vector fields X, Y.

A contact metric manifold is said to be Einstein if $S(X, Y) = \lambda g(X, Y)$, where λ is a constant and η-Einstein if $S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y)$, where α and β are smooth functions.

A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.5}$$

$X, Y \in TM$, where ∇ is the Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a Killing vector field is said to be a K-contact metric manifold. A Sasakian manifold is K-contact but not conversely. However a 3-dimensional K-contact manifold is Sasakian [14].

It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ [15]. Again on a Sasakian manifold [16] we have

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.6}$$

As a generalization of both $R(X, Y)\xi = 0$ and the Sasakian case, Blair et al. [17] introduced the (k, μ)-nullity distribution on a contact metric manifold and gave several reasons for studying it. The (k, μ)-nullity distribution $N(k, \mu)$ [17] of a contact metric manifold M is defined by

$$\begin{aligned} N(k, \mu) : p &\longrightarrow N_p(k, \mu) \\ &= \{W \in T_pM : R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\}, \end{aligned} \tag{2.7}$$

for all $X, Y \in TM$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold M with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold. If $\mu = 0$, the (k, μ)-nullity distribution reduces to k-nullity distribution [18]. The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by [18]

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\}, \tag{2.8}$$

with k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold as $N(k)$-contact metric manifold [19]. If $k = 1$, then the manifold is Sasakian, and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [15].

Given a non-Sasakian (k, μ)-contact manifold M, Boeckx [20] introduced an invariant

$$I_M = \frac{1 - \mu/2}{\sqrt{1 - k}} \tag{2.9}$$

and showed that, for two non-Sasakian (k, μ)-manifolds M_1 and M_2, we have $I_{M_1} = I_{M_2}$ if and only if, up to a D-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds.

Thus we see that from all non-Sasakian (k, μ)-manifolds of dimension $(2n + 1)$ and for every possible value of the invariant I, one (k, μ)-manifold M can be obtained with $I_M = 1$. For $I > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c, where we have $I = (1+c)/|1-c|$. Boeckx also gives a Lie algebra construction for any odd dimension and value of $I < -1$.

Using this invariant, Blair et al. [19] constructed an example of a $(2n + 1)$-dimensional $N(1 - 1/n)$-contact metric manifold, $n > 1$. The example is given in the following.

Since the Boeckx invariant for a $(1 - 1/n, 0)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature c so choosing that the resulting D-homothetic deformation will be a $(1 - 1/n, 0)$-manifold. That is, for $k = c(2 - c)$ and $\mu = -2c$ we solve

$$1 - \frac{1}{n} = \frac{k + a^2 - 1}{a^2}, \qquad 0 = \frac{\mu + 2a - 2}{a} \tag{2.10}$$

for a and c. The result is

$$c = \frac{\sqrt{n} \pm 1}{n - 1}, \qquad a = 1 + c, \tag{2.11}$$

and taking c and a to be these values we obtain $N(1 - 1/n)$-contact metric manifold.

However, for a $N(k)$-contact metric manifold M of dimension $(2n + 1)$, we have [19]

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.12}$$

where $h = (1/2)\pounds_\xi \phi$,

$$h^2 = (k-1)\phi^2, \tag{2.13}$$

$$R(X,Y)\xi = k\left[\eta(Y)X - \eta(X)Y\right], \tag{2.14}$$

$$\begin{aligned} S(X,Y) &= 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \\ &\quad + [2nk - 2(n-1)]\eta(X)\eta(Y), \quad n \geq 1, \end{aligned} \tag{2.15}$$

$$S(Y,\xi) = 2nk\eta(X), \tag{2.16}$$

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y), \tag{2.17}$$

$$(\nabla_X h)(Y) = \left\{(1-k)g(X,\phi Y) + g(X,h\phi Y)\right\}\xi + \eta(Y)\left[h(\phi X + \phi hX)\right], \tag{2.18}$$

In a $(2n+1)$-dimensional almost contact metric manifold, if $\{e_1,\ldots,e_{2n},\xi\}$ is a local orthonormal basis of the tangent space of the manifold, then $\{\phi e_1,\ldots,\phi e_{2n},\xi\}$ is also a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i,e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n, \tag{2.19}$$

$$\sum_{i=1}^{2n} S(e_i,e_i) = \sum_{i=1}^{2n} S(\phi e_i, \phi e_i) = r - 2nk, \tag{2.20}$$

$$\sum_{i=1}^{2n} g(e_i,Z)S(Y,e_i) = \sum_{i=1}^{2n} g(\phi e_i,Z)S(Y,\phi e_i) = S(Y,Z) - 2nk\eta(Z), \tag{2.21}$$

for $Y, Z \in T(M)$. In particular in view of $\eta \circ \phi = 0$, we get

$$\sum_{i=1}^{2n} g(e_i,\phi Z)S(Y,e_i) = \sum_{i=1}^{2n} g(\phi e_i,\phi Z)S(Y,\phi e_i) = S(Y,\phi Z). \tag{2.22}$$

Here we state a lemma due to Baikoussis and Koufogiorgos [21] which will be used in this paper.

Lemma 2.1. *Let M^{2n+1} be an η-Einstein manifold of dimension $(2n+1)(n \geq 1)$. If ξ belongs to the k-nullity distribution, then $k = 1$ and the structure is Sasakian.*

3. Conharmonically Symmetric $N(k)$-Contact Metric Manifolds

In this section we study conharmonically symmetric $N(k)$-contact metric manifolds. Differentiating (1.3) covariantly with respect to W, we obtain

$$\begin{aligned}\left(\nabla_W \widetilde{C}\right)(X,Y)Z &= (\nabla_W R)(X,Y)Z \\ &\quad - \frac{1}{2n-1}\left[g(Y,Z)(\nabla_W Q)X - g(X,Z)(\nabla_W Q)Y + (\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y\right].\end{aligned} \tag{3.1}$$

Therefore for conharmonically symmetric $N(k)$-contact metric manifolds we have

$$\begin{aligned}(\nabla_W R)(X,Y)Z = \frac{1}{2n-1}\big[&g(Y,Z)(\nabla_W Q)X - g(X,Z)(\nabla_W Q)Y + (\nabla_W S)(Y,Z)X \\ &-(\nabla_W S)(X,Z)Y\big].\end{aligned} \tag{3.2}$$

Differentiating (2.12) covariantly with respect to W and using (2.15) we obtain

$$(\nabla_W R)(X,Y)\xi = k\left[g(W,\phi Y)X + g(hW,\phi Y)X - g(W,\phi X)Y - g(hW,\phi X)Y\right]. \tag{3.3}$$

Again, differentiating (2.14) covariantly with respect to W and using (2.16) and (2.17) we have

$$\begin{aligned}(\nabla_W S)(Y,Z) = 2(n-1)\big[&(1-k)g(W,\phi Y)\eta(Z) + g(W,h\phi Y)\eta(Z) + g(h\phi W,Z)\eta(Y) \\ &+g(h\phi hW,Z)\eta(Y)\big] \\ &+\{2(1-n)+2nk\}\big[g(W,\phi Y)\eta(Z) + g(hW,\phi Y)\eta(Z) + g(W,\phi Z)\eta(Y) \\ &+g(hW,\phi Z)\eta(Y)\big].\end{aligned} \tag{3.4}$$

Therefore we have

$$(\nabla_W Q)(Y) = 2k\left[g(W,\phi Y)\xi - (\phi W)\eta(Y)\right] + 2nk\left[g(W,h\phi Y) + (h\phi W)\eta(Y)\right]. \tag{3.5}$$

Putting $Z = \xi$ in (3.2) and using (3.3), (3.4), and (3.5) we obtain

$$\begin{aligned}&(2n-1)k\left[g(W,\phi Y)X + g(hW,\phi Y)X - g(W,\phi X)Y - g(hW,\phi X)Y\right] \\ &\quad = 2k\left[g(W,\phi X)\phi^2 Y - g(W,\phi Y)\phi^2 X\right] \\ &\qquad + 2nk\left[g(W,h\phi X)\phi^2 Y - g(W,h\phi Y)\phi^2 X\right].\end{aligned} \tag{3.6}$$

Taking inner product of (3.6) with ξ and using (2.1) we obtain

$$(2n-1)k\left[g(W,\phi Y)\eta(X)+g(hW,\phi Y)\eta(X)-g(W,\phi X)\eta(Y)-g(hW,\phi X)\eta(Y)\right]=0. \quad (3.7)$$

From (3.7) we get, either $k=0$ or

$$g(W,\phi Y)\eta(X)+g(hW,\phi Y)\eta(X)-g(W,\phi X)\eta(Y)-g(hW,\phi X)\eta(Y)\big]=0. \quad (3.8)$$

Putting hY instead of Y in (3.8) and using (2.12) we obtain

$$g(W,\phi hY)\eta(X)=(k-1)g(W,\phi Y)\eta(X). \quad (3.9)$$

Using (3.9) in (3.7) yields

$$k\left[g(W,\phi Y)\eta(X)-g(W,\phi X)\eta(Y)\right]=0. \quad (3.10)$$

The relation (3.10) gives $k=0$, since $g(W,\phi Y)\eta(X)-g(W,\phi X)\eta(Y)=0$ gives $g(W,\phi Y)=0$ (by putting $X=\xi$), which is not the case for a $N(k)$-contact metric manifold, in general.

Therefore in either case we obtain $k=0$.

Hence we have the following.

Theorem 3.1. *A conharmonically symmetric n-dimensional $N(k)$-contact metric manifold is locally isometric to the product $E^{(n+1)}(0)\times S^n(4)$.*

Remark 3.2. The converse of the above theorem is not true in general. However if $k=0$, then we get $R(X,Y)\xi=0$, and hence from the definition of the conharmonic curvature tensor we obtain $\widetilde{C}(X,Y)\xi=0$, that is, the manifold under consideration is ξ-conharmonically flat. Thus if an $N(k)$-contact manifold is locally isometric to $E^{(n+1)}(0)\times S^n(4)$, then the manifold is ξ-conharmonically flat.

4. ξ-Conharmonically Flat $N(k)$-Contact Metric Manifolds

In this section we consider a $(2n+1)$-dimensional ξ-conharmonically flat $N(k)$-contact metric manifolds. Then from (1.3) we obtain

$$R(X,Y)\xi=\frac{1}{2n-1}\left[g(Y,\xi)QX-g(X,\xi)QY+S(Y,\xi)X-S(X,\xi)Y\right]. \quad (4.1)$$

Using (2.1), (2.13), and (2.15) in (4.1) we obtain

$$\left[\eta(Y)QX-\eta(X)QY\right]+k\left[\eta(Y)X-\eta(X)Y\right]=0. \quad (4.2)$$

Putting $Y=\xi$ in (4.2) and using (2.1) and (2.15) we get

$$QX=-kX+(2n+1)k\eta(X)\xi. \quad (4.3)$$

Taking inner product with W of (4.3) yields

$$S(X,W) = -kg(X,W) + (2n+1)k\eta(X)\eta(W). \tag{4.4}$$

From relation (4.4), we conclude that the manifold is an η-Einstein manifold.

Conversely, we assume that a $(2n+1)$-dimensional $N(k)$-contact manifold satisfies the relation (4.4). Then we easily obtain from (1.3) that $\widetilde{C}(X,Y)\xi = 0$.

In view of the above discussions we state the following.

Theorem 4.1. *A $(2n+1)$-dimensional $N(k)$-contact metric manifold is ξ-conharmonically flat if and only if it is an η-Einstein manifold.*

Hence in view of Lemma 2.1 we state the following.

Corollary 4.2. *Let M be a $(2n+1)$-dimensional ξ-conharmonically flat $N(k)$-contact metric manifold, then $k = 1$ and the structure is Sasakian.*

Let $\{e_1, e_2, \ldots, e_n, e_{n+1}, \ldots e_{2n}, e_{2n+1} = \xi\}$ be a local orthonormal basis of the tangent space of the manifold. Putting $X = W = e_i$ in (4.4) and summing up from 1 to $2n+1$ we obtain in view of (2.18) and (2.19) that

$$r = 0. \tag{4.5}$$

Therefore we have the following corollary.

Corollary 4.3. *In a $(2n+1)$-dimensional ξ-conharmonically flat $N(k)$-contact metric manifold, the scalar curvature r vanishes.*

5. ϕ-Conharmonically Flat $N(k)$-Contact Metric Manifolds

This section deals with a $(2n+1)$-dimensional ϕ-conharmonically flat $N(k)$-contact metric manifold. Then we have from (1.3) that

$$\begin{aligned}
&\widetilde{R}(\phi X, \phi Y, \phi Z, \phi W) \\
&\quad = \frac{1}{2n-1}\big[g(\phi Y, \phi Z)S(\phi X, \phi W) - g(\phi X, \phi Z)S(\phi Y, \phi W) + S(\phi Y, \phi Z)g(\phi X, \phi W) \\
&\qquad\qquad - S(\phi X, \phi Z)g(\phi Y, \phi W)\big].
\end{aligned} \tag{5.1}$$

Let $\{e_1, e_2, \ldots, e_{2n}, \xi\}$ be a local orthonormal basis of the tangent space of the manifold. Then $\{\phi e_1, \phi e_2, \ldots, \phi e_{2n}, \xi\}$ is also a local orthonormal basis of the tangent space. Putting $X = W = e_i$ in (5.1) and summing up from 1 to $2n$ we have

$$\begin{aligned}
&\sum_{i=1}^{2n} \widetilde{R}(\phi e_i, \phi Y, \phi Z, \phi e_i) \\
&= \frac{1}{2n-1} \sum_{i=1}^{2n} \big[g(\phi Y, \phi Z) S(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z) S(\phi Y, \phi e_i) + S(\phi Y, \phi Z) g(\phi e_i, \phi e_i) \\
&\qquad - S(\phi e_i, \phi Z) g(\phi Y, \phi e_i) \big].
\end{aligned} \tag{5.2}$$

Using (2.18), (2.19), (2.20), and (2.21) in (5.2) we obtain

$$S(\phi Y, \phi Z) = (r - k) g(\phi Y, \phi Z). \tag{5.3}$$

Replacing Y and Z by ϕY and ϕZ in (5.3) and using (2.1) we have

$$S(Y, Z) = (r - k) g(Y, Z) + [(2n+1)k - r] \eta(Y) \eta(Z). \tag{5.4}$$

Putting $Y = Z = e_i$ in (5.4) and taking summation over $i = 1$ to $2n + 1$ we get by using (2.18) and (2.19) that

$$r = 0. \tag{5.5}$$

In view of the above discussions we have the following.

Proposition 5.1. *A $(2n+1)$-dimensional ϕ-conharmonically flat $N(k)$-contact metric manifold is an η-Einstein manifold with vanishing scalar curvature.*

Therefore in view of the Lemma 2.1 we state the following theorem.

Theorem 5.2. *A $(2n+1)$-dimensional ϕ-conharmonically flat $N(k)$-contact metric manifold is a Sasakian manifold with vanishing scalar curvature.*

Definition 5.3. In a $(2n+1)$-dimensional $N(k)$-contact metric manifold, if the Ricci tensor S satisfies $(\nabla_X S)(\phi Y, \phi Z) = 0$, then the Ricci tensor is said to be η-parallel.

The notion of η-parallel Ricci tensor for Sasakian manifold was introduced by Kon [22].

Putting $r = 0$ in (5.4) we have

$$S(Y, Z) = -k g(Y, Z) + (2n+1) k \eta(Y) \eta(Z). \tag{5.6}$$

Replacing Y and Z by ϕY and ϕZ in (5.6) and using (2.1) we obtain

$$S(\phi Y, \phi Z) = -k g(\phi Y, \phi Z). \tag{5.7}$$

Relation (5.7) yields

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{5.8}$$

since k is a constant. Therefore we have the following corollary.

Corollary 5.4. *A $(2n+1)$-dimensional ϕ-conharmonically flat $N(k)$-contact metric manifold satisfies η-parallel Ricci tensor.*

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