International Scholarly Research Network
ISRN Discrete Mathematics
Volume 2011, Article ID 430396, 15 pages
doi:10.5402/2011/430396

*Research Article*

# Chromatic Classes of 2-Connected $(n, n+4)$-Graphs with Exactly Three Triangles and at Least Two Induced 4-Cycles

**G. C. Lau[1] and Y. H. Peng[2]**

[1] *Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, Johor Campus, Segamat, Malaysia*
[2] *Department of Mathematics, Universiti Putra Malaysia, 43400 Serdang, Malaysia*

Correspondence should be addressed to G. C. Lau, drlaugc@gmail.com

Received 7 July 2011; Accepted 15 August 2011

Academic Editor: G. Isaak



For a graph $G$, let $P(G,\lambda)$ be its chromatic polynomial. Two graphs $G$ and $H$ are chromatically equivalent, denoted $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. A graph $G$ is chromatically unique if $P(H,\lambda) = P(G,\lambda)$ implies that $H \cong G$. In this paper, we determine all chromatic equivalence classes of 2-connected $(n, n+4)$-graphs with exactly three triangles and at least two induced 4-cycles. As a byproduct of these, we obtain various new families of $\chi$-equivalent graphs and $\chi$-unique graphs.

## 1. Introduction

Let $P(G)$, or simply $P(G)$, denote the chromatic polynomial of a simple graph $G$. Two graphs $G$ and $H$ are chromatically equivalent (simply $\chi$-equivalent), denoted $G \sim H$, if $P(G) = P(H)$. A graph $G$ is chromatically unique (simply $\chi$-unique) if $P(H) = P(G)$ implies that $H \cong G$. Let $\langle G\rangle$ denote the equivalence class determined by the graph $G$ under $\sim$. Clearly, $G$ is $\chi$-unique if and only if $\langle G\rangle = \{G\}$. A graph $H$ is called a *relative* of $G$ if there is a sequence of graphs $G = H_1, H_2, \ldots, H_k = H$ such that each $H_i$ is a $K_{r_i}$-gluing of some graphs (say $X_i$ and $Y_i$) and that $H_{i+1}$ is obtained from $H_i$ by forming another $K_{r_i}$-gluing of $X_i$ and $Y_i$ for $1 \le i \le k-1$. We say $H$ is a graph of *type* $G$ if $H$ is a relative of $G$ or $H \cong G$. A family $\mathcal{S}$ of graphs is said to be *relative-closed* (simply $\chi_r$-closed) if

(i) no two graphs in $\mathcal{S}$ are relatives of each other,

(ii) for any graph $G \in \mathcal{S}$, $P(H,\lambda) = P(G,\lambda)$ implies that $H \in \mathcal{S}$ or $H$ is a relative of a graph in $S$.

If $\mathcal{S}$ is a $\chi_r$-closed family, then the chromatic equivalence class of each graph in $\mathcal{S}$ can be determined by studying the chromaticity of each graph in $\mathcal{S}$.

If $G$ is a graph of order $n$ and size $m$, we say $G$ is an $(n,m)$-graph. The chromatic equivalence classes of 2-connected $(n,n+i)$-graph have been fully determined for $i=0,1$ in [1, 2] and partially determined for $i=2,3$ in [3–5]. Peng and Lau have also characterized and classified certain chromatic equivalence classes of 2-connected $(n,n+4)$-graph in [6, 7]. In [8], by using the idea of cyclomatic number, the authors obtained the $\chi_r$-closed family of 2-connected $(n,n+4)$-graphs with exactly three triangles.

In this paper, all the chromatic equivalence classes of 2-connected $(n,n+4)$-graphs with exactly three triangles and at least two induced $C_4$s are determined. As a byproduct of these, we obtain various new families of $\chi$-equivalent graphs and $\chi$-unique graphs. The readers may refer to [9] for terms and notation used but not defined here.

## 2. Notation and Basic Results

Let $C_n$ (or $n$-cycle) be the cycle of order $n$. An induced 4-cycle is the cycle $C_4$ without chord. The following are some useful known results and techniques for determining the chromatic polynomial of a graph. Throughout this paper, all graphs are assumed to be connected unless otherwise stated.

**Lemma 2.1** (Fundamental Reduction Theorem (Whitney [10])). *Let $G$ be a graph and $e$ an edge of $G$. Then*

$$P(G)=P(G-e)-P(G\cdot e), \tag{2.1}$$

*where $G-e$ is the graph obtained from $G$ by deleting $e$, and $G\cdot e$ is the graph obtained from $G$ by identifying the end vertices of $e$.*

Let $G_1$ and $G_2$ be graphs, each containing a complete subgraph $K_p$ with $p$ vertices. If $G$ is a graph obtained from $G_1$ and $G_2$ by identifying the two subgraphs $K_p$, then $G$ is called a $K_p$-gluing of $G_1$ and $G_2$. Note that a $K_1$-gluing and a $K_2$-gluing are also called a vertex-gluing and an edge-gluing, respectively.

**Lemma 2.2** (Zykov [11]). *Let $G$ be a $K_r$-gluing of $G_1$ and $G_2$. Then*

$$P(G)=\frac{P(G_1)P(G_2)}{P(K_r)}. \tag{2.2}$$

*Lemma 2.2 implies that all $K_r$-gluings of $G_1$ and $G_2$ are $\chi$-equivalent. It follows from Lemma 2.2 that if $H$ is a relative of $G$, then $H\sim G$.*

The following conditions for two graphs $G$ and $H$ to be $\chi$-equivalence are well known (see, e.g., [4]).

**Lemma 2.3.** *Let $G$ and $H$ be two $\chi$-equivalent graphs. Then $G$ and $H$ have, respectively, the same number of vertices, edges, and triangles. If both $G$ and $H$ do not contain $K_4$, then they have the same number of induced $C_4$s.*

A generalized $\theta$-graph is a 2-connected graph consisting of three edge-disjoint paths between two vertices of degree 3. All other vertices have degree two. These paths have lengths $x$, $y$ and $z$, respectively, where $x \geq y \geq z$. The graph is of order $x+y+z-1$ and size $x+y+z$ (see [2]). We will denote $K_2$ as $C_2$ for convenience.

**Lemma 2.4.**

$$\text{(i)} \quad P(C_n) = (\lambda-1)^n + (-1)^n(\lambda-1), \qquad n \geq 2, \tag{2.3}$$

$$\text{(ii)} \quad P(\theta_{x,y,z}) = \begin{cases} \dfrac{P(C_{x+1})P(C_{y+1})P(C_{z+1})}{\lambda^2(\lambda-1)^2} + \dfrac{P(C_x)P(C_y)P(C_z)}{\lambda^2}, & \text{if } z \neq 1, \\ \dfrac{P(C_{x+1})P(C_{y+1})}{\lambda(\lambda-1)} & \text{if } z = 1. \end{cases} \tag{2.4}$$

Lemma 2.4(i) can be proved by induction while Lemma 2.4(ii) follows from Lemmas 2.1 and 2.2. For integers $x$, $y$, $z$, $n$, and $\lambda$, let us write

$$\begin{gathered} Q_n(\lambda) = \sum_{i=0}^{n-2}(-1)^i(\lambda-1)^{n-2-i}, \\ M_{x,y,z}(\lambda) = Q_{x+1}(\lambda)Q_{y+1}(\lambda)Q_{z+1}(\lambda) + (\lambda-1)^2 Q_x(\lambda)Q_y(\lambda)Q_z(\lambda). \end{gathered} \tag{2.5}$$

Note that when $\lambda = 1$, we have $Q_n(1) = (-1)^n$ and $M_{x,y,z}(1) = (-1)^{x+y+z+1}$. Lemma 2.4 can then be written as the following lemma.

**Lemma 2.5** (see [4]). *(i)* $P(C_n) = \lambda(\lambda-1)Q_n(\lambda)$ *and (ii)* $P(\theta_{x,y,z}) = \lambda(\lambda-1)M_{x,y,z}(\lambda)$.

We also need the following lemma.

**Lemma 2.6** (Whitehead and Zhao [12]). *A graph $G$ contains a cut-vertex if and only if* $(\lambda-1)^2 \mid P(G)$.

Lemma 2.6 also implies that if $H \sim G$, then $H$ is 2-connected if and only if $G$ is so.

## 3. Classification of Graphs

Let $\mathcal{F}$ be the $\chi_r$-closed family of 2-connected $(n, n+4)$-graphs with three triangles and at least two induced $C_4$s. In [8], we classified all the 31 types of graph $F \in \mathcal{F}$ as shown in Figure 1. Since the approach used to classify all the graphs $F$ is rather long and repetitive, we will not discuss it here. The reader may refer to Theorems 1 and 3 in [8] for a detail derivation of the graphs.

We are now ready to determine the chromaticity of all 31 types of $\chi_r$-closed family of 2-connected $(n, n+4)$-graphs having exactly 3 triangles and at least two induced $C_4$s as shown in Figure 1. We first note that if $H \sim F_i (1 \leq i \leq 31)$ in Figure 1, then $H$ must be of type $F_j (1 \leq j \leq 31)$ in Figure 1 as well. For convenience, we will say that the graph $F_i$, or any of its relatives, is of type $(i)$.

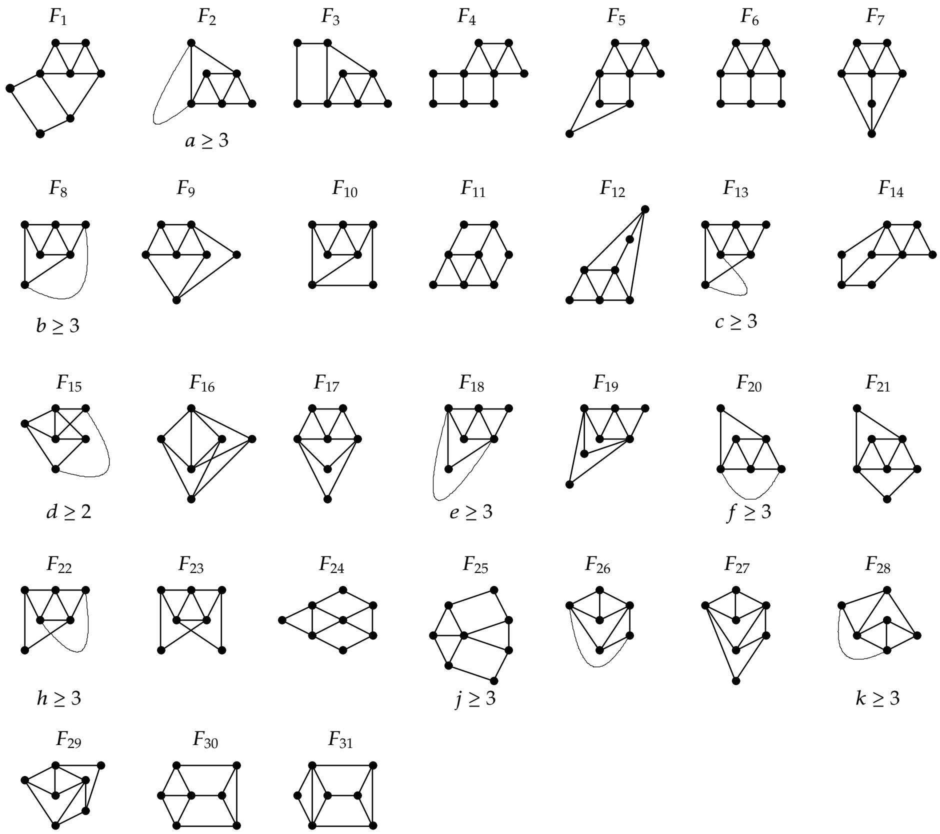


**Figure 1:** 31 types of 2-connected $(n, n+4)$-graphs with exactly three triangles and at least two induced 4-cycles. The light lines of the graphs refer to the paths of indicated length.

In what follows, we will use $F_i(\alpha)$, instead of $F_i$, to denote a graph of type $(i)$ that has a path of length $\alpha$. We now present our main results in the following theorem.

**Theorem 3.1.** (1) $H \in \langle F_1 \rangle$ *if and only if* $H$ *is of type* $F_1$.

(2) $H \in \langle F_2(a) \rangle$ *if and only if* $H$ *is of type* $F_2(a)$.

(3) $H \in \langle F_3 \rangle$ *if and only if* $H$ *is of type* $F_3$.

(4) $H \in \langle F_4 \rangle$ *if and only if* $H$ *is of type* $F_4$.

(5) $H \in \langle F_5 \rangle$ *if and only if* $H$ *is of type* $F_5$.

(6) $H \in \langle F_6 \rangle$ *if and only if* $H \cong F_6, F_{25}$ *or* $H$ *is of type* $F_{22}(3)$.

(7) $H \in \langle F_7 \rangle$ *if and only if* $H \cong F_7, F_{21}, F_{27}$ *or* $H$ *is of type* $F_{31}$.

(8) $\langle F_8(b) \rangle = \{F_8(b), F_{28}(b)\}$.

(9) $F_9$ *is* $\chi$*-unique.*

(10) $\langle F_{10} \rangle = \{F_{10}, F_{29}\}$.

(11) $H \in \langle F_{11} \rangle$ *if and only if* $H$ *is of type* $F_{11}$, $F_{13}(3)$, *or* $F_{24}$.

(12) *$H \in \langle F_{12} \rangle$ if and only if $H$ is of type $F_{12}$.*

(13) *$H \in \langle F_{13}(c) \rangle$ if and only if $H$ is of type $F_{13}(c)$ for $c \geq 4$, and $H \in \langle F_{13}(3) \rangle$ if and only if $H$ is of type $F_{11}, F_{13}(3)$, or $F_{24}$.*

(14) *$H \in \langle F_{14} \rangle$ if and only if $H$ is of type $F_{14}$ or $F_{18}(3)$.*

(15) *$F_{15}(d)$ is $\chi$-unique for $d \geq 3$, and $\langle F_{15}(2) \rangle = \{F_{15}(2), F_{31}\}$.*

(16) *$F_{16}$ is $\chi$-unique.*

(17) *$F_{17}$ is $\chi$-unique.*

(18) *$H \in \langle F_{18}(e) \rangle$ if and only if $H$ is of type $F_{18}(e)$ for $e \geq 4$, and $H \in \langle F_{18}(3) \rangle$ if and only if $H$ is of type $F_{14}$ or $F_{18}(3)$.*

(19) *$H \in \langle F_{19} \rangle$ if and only if $H$ is of type $F_{19}$.*

(20) *$\langle F_{20}(f) \rangle = \{F_{20}(f), F_{26}(f)\}$.*

(21) *$H \in \langle F_{21} \rangle$ if and only if $H \cong F_7$, $F_{21}$, $F_{27}$ or $H$ is of type $F_{31}$.*

(22) *$H \in \langle F_{22}(h) \rangle$ if and only if $H$ is of type $F_{22}(h)$ for $h \geq 4$, and $H \in \langle F_{22}(3) \rangle$ if and only if $H \cong F_6, F_{25}$ or $H$ is of type $F_{22}(3)$.*

(23) *$H \in \langle F_{23} \rangle$ if and only if $H$ is of type $F_{23}$.*

(24) *$H \in \langle F_{24} \rangle$ if and only if $H$ is of type $F_{11}$, $F_{13}(3)$, or $F_{24}$.*

(25) *$H \in \langle F_{25} \rangle$ if and only if $H \cong F_6, F_{25}$ or $H$ is of type $F_{22}(3)$.*

(26) *$\langle F_{26}(j) \rangle = \{F_{20}(j), F_{26}(j)\}$.*

(27) *$H \in \langle F_{27} \rangle$ if and only if $H \cong F_7$, $F_{21}$, $F_{27}$ or $H$ is of type $F_{31}$.*

(28) *$\langle F_{28}(k) \rangle = \{F_8(k), F_{28}(k)\}$.*

(29) *$\langle F_{29} \rangle = \{F_{10}, F_{29}\}$.*

(30) *$\langle F_{30} \rangle = \{F_{15}(2), F_{30}\}$.*

(31) *$H \in \langle F_{31} \rangle$ if and only if $H \cong F_7$, $F_{21}$, $F_{27}$ or $H$ is of type $F_{31}$.*

## 4. Chromatic Polynomials of the Graphs

Before proving our main result, we present here some useful information about the chromatic polynomial of $F_i$ $(1 \leq i \leq 31)$. Let $W(n,k)$ denote the graph of order $n$ obtained from a wheel $W_n$ by deleting all but $k$ consecutive spokes. Also let $W_m(5,3)$ denote the graph obtained from $W(5,3)$ by identifying the end-vertices of a path $P_m$ to two non-adjacent degree 3 vertices of $W(5,3)$. Using Software Maple or Lemmas 2.1, 2.2 and 2.5, it is easy to obtain the chromatic polynomial of each graph in $\mathcal{F}$ as shown in the following lemma.

**Lemma 4.1.** (1)

$$P(F_1) = \lambda(\lambda - 1)N_1(\lambda), \tag{4.1}$$

*where* $N_1(\lambda) = (\lambda - 2)(\lambda^2 - 3\lambda + 3)(\lambda^3 - 6\lambda^2 + 13\lambda - 11)$ *and* $N_1(1) = 3$.

(2)

$$P(F_2(a)) = \frac{(\lambda-2)P(C_{a+1})P(W(5,3))}{\lambda(\lambda-1)} = \lambda(\lambda-1)(\lambda-2)^2\left(\lambda^2-4\lambda+5\right)Q_{a+1}(\lambda) = \lambda(\lambda-1)N_2(\lambda), \quad (4.2)$$

*where* $N_2(\lambda) = (\lambda-2)^2(\lambda^2-4\lambda+5)Q_{a+1}(\lambda)$ *and* $N_2(1) = (-1)^2(1-4+5)(-1)^{a+1} = 2(-1)^{a+1}$.

(3)

$$P(F_3) = \lambda(\lambda-1)N_3(\lambda), \quad (4.3)$$

*where* $N_3(\lambda) = (\lambda-2)^2(\lambda^2-4\lambda+5)(\lambda^2-3\lambda+3)$ *and* $N_3(1) = 2$.

(4)

$$P(F_4) = \lambda(\lambda-1)N_4(\lambda), \quad (4.4)$$

*where* $N_4(\lambda) = (\lambda-2)^3(\lambda^2-3\lambda+3)^2$ *and* $N_4(1) = -1$.

(5)

$$P(F_5) = \lambda(\lambda-1)N_5(\lambda), \quad (4.5)$$

*where* $N_5(\lambda) = (\lambda-2)^3(\lambda^3-5\lambda^2+10\lambda-7)$ *and* $N_5(1) = 1$.

(6)

$$P(F_6) = \lambda(\lambda-1)N_6(\lambda), \quad (4.6)$$

*where* $N_6(\lambda) = (\lambda-2)(\lambda^2-4\lambda+5)(\lambda^3-5\lambda^2+9\lambda-7)$ *and* $N_6(1) = 4$.

(7)

$$P(F_7) = \lambda(\lambda-1)N_7(\lambda), \quad (4.7)$$

*where* $N_7(\lambda) = (\lambda-2)^2(\lambda^3-6\lambda^2+14\lambda-13)$ *and* $N_7(1) = (-1)^2(1-6+14-13) = -4$.

(8)

$$\begin{aligned} P(F_8(b)) &= (\lambda-2)^3 P(C_{b+2}) - (\lambda-3) P(W(b+3,3)) \\ &= (\lambda-2)^3 P(C_{b+2}) - (\lambda-2)(\lambda-3)[P(C_{b+2}) - P(C_{b+1})] \\ &= \lambda(\lambda-1)(\lambda-2)\left[\left(\lambda^2-5\lambda+7\right)Q_{b+2}(\lambda) + (\lambda-3)Q_{b+1}(\lambda)\right] \\ &= \lambda(\lambda-1)N_8(\lambda), \end{aligned} \tag{4.8}$$

*where* $N_8(\lambda) = (\lambda-2)[(\lambda^2-5\lambda+7)Q_{b+2}(\lambda) + (\lambda-3)Q_{b+1}(\lambda)]$ *and* $N_8(1) = (-1)[3(-1)^{b+2} + (-2)(-1)^{b+1}] = 5(-1)^{b+1}$.

(9)

$$P(F_9) = \lambda(\lambda-1)N_9(\lambda), \tag{4.9}$$

*where* $N_9(\lambda) = (\lambda-2)^2(\lambda^3-6\lambda^2+14\lambda-14)$ *and* $N_9(1) = -5$.

(10)

$$P(F_{10}) = \lambda(\lambda-1)N_{10}(\lambda), \tag{4.10}$$

*where* $N_{10}(\lambda) = (\lambda-2)(\lambda^4-8\lambda^3+26\lambda^2-41\lambda+27)$ *and* $N_{10}(1) = -5$.

(11)

$$P(F_{11}) = \lambda(\lambda-1)N_{11}(\lambda), \tag{4.11}$$

*where* $N_{11}(\lambda) = (\lambda-2)^2(\lambda^4-7\lambda^3+20\lambda^2-28\lambda+17)$ *and* $N_{11}(1) = 3$.

(12)

$$P(F_{12}) = \lambda(\lambda-1)N_{12}(\lambda), \tag{4.12}$$

*where* $N_{12}(\lambda) = (\lambda-2)^3(\lambda^2-4\lambda+6)$ *and* $N_{12}(1) = -3$.

(13)

$$\begin{aligned} P(F_{13}(c)) &= (\lambda-2)\left[(\lambda-2)^2 P(C_{c+2}) - \frac{P(K_4)P(C_{c+1})}{\lambda(\lambda-1)}\right] \\ &= (\lambda-2)^3 P(C_{c+2}) - (\lambda-2)^2(\lambda-3)P(C_{c+1}) \\ &= \lambda(\lambda-1)(\lambda-2)^2[(\lambda-2)Q_{c+2}(\lambda) - (\lambda-3)Q_{c+1}(\lambda)] \\ &= \lambda(\lambda-1)N_{13}(\lambda), \end{aligned} \tag{4.13}$$

*where* $N_{13}(\lambda) = (\lambda-2)^2[(\lambda-2)Q_{c+2}(\lambda) - (\lambda-3)Q_{c+1}(\lambda)]$ *and* $N_{13}(1) = (-1)^2[(-1)(-1)^{c+2} - (-2)(-1)^{c+1}] = 3(-1)^{c+1}$.

(14)

$$P(F_{14}) = \lambda(\lambda-1)N_{14}(\lambda), \tag{4.14}$$

*where* $N_{14}(\lambda) = (\lambda-2)^4(\lambda^2 - 3\lambda + 4)$ *and* $N_{14}(1) = 2$.

(15)

$$\begin{aligned} P(F_{15}(d)) &= (\lambda-2)P(W(d+4,3)) - (\lambda-3)P(W(d+3,3)) \\ &= (\lambda-2)^2[P(C_{d+3}) - P(C_{d+2})] - (\lambda-2)(\lambda-3)[P(C_{d+2}) - P(C_{d+1})] \\ &= \lambda(\lambda-1)(\lambda-2)[(\lambda-2)Q_{d+3}(\lambda) - (2\lambda-5)Q_{d+2}(\lambda) + (\lambda-3)Q_{d+1}(\lambda)] \\ &= \lambda(\lambda-1)N_{15}(\lambda), \end{aligned} \tag{4.15}$$

*where* $N_{15}(\lambda) = (\lambda-2)[(\lambda-2)Q_{d+3}(\lambda) - (2\lambda-5)Q_{d+2}(\lambda) + (\lambda-3)Q_{d+1}(\lambda)]$ *and* $N_{15}(1) = (-1)[(-1)(-1)^{d+3} - (-3)(-1)^{d+2} + (-2)(-1)^{d+1}] = 6(-1)^{d+1}$.

(16)

$$P(F_{16}) = \lambda(\lambda-1)N_{16}(\lambda), \tag{4.16}$$

*where* $N_{16}(\lambda) = (\lambda-2)(\lambda^3 - 7\lambda^2 + 19\lambda - 19)$ *and* $N_{16}(1) = 6$.

(17)

$$P(F_{17}) = \lambda(\lambda-1)N_{17}(\lambda), \tag{4.17}$$

*where* $N_{17}(\lambda) = (\lambda-2)(\lambda^4 - 8\lambda^3 + 26\lambda^2 - 41\lambda + 25)$ *and* $N_{17}(1) = -3$.

(18)

$$\begin{aligned} P(F_{18}(e)) &= (\lambda-2)[(\lambda-1)P(W(e+3,3)) - (\lambda-2)(\lambda-3)P(C_{e+1})] \\ &= (\lambda-1)(\lambda-2)^2[P(C_{e+2}) - P(C_{e+1})] - (\lambda-2)^2(\lambda-3)P(C_{e+1}) \\ &= (\lambda-1)(\lambda-2)^2P(C_{e+2}) - 2(\lambda-2)^3P(C_{e+1}) \\ &= \lambda(\lambda-1)(\lambda-2)^2[(\lambda-1)Q_{e+2}(\lambda) - 2(\lambda-2)Q_{e+1}(\lambda)] \\ &= \lambda(\lambda-1)N_{18}(\lambda), \end{aligned} \tag{4.18}$$

*where* $N_{18}(\lambda) = (\lambda-2)^2[(\lambda-1)Q_{e+2}(\lambda) - 2(\lambda-2)Q_{e+1}(\lambda)]$ *and* $N_{18}(1) = (-1)^2[0 - 2(-1)(-1)^{e+1}] = 2(-1)^{e+1}$.

(19)

$$P(F_{19}) = \lambda(\lambda-1)N_{19}(\lambda), \tag{4.19}$$

*where* $N_{19}(\lambda) = (\lambda-2)^2(\lambda^3 - 6\lambda^2 + 14\lambda - 11)$ *and* $N_{19}(1) = -2$.

(20)

$$\begin{aligned}
P(F_{20}(f)) &= P(W_{f+1}(5,3)) - \frac{P(W(5,3))P(C_{f+1})}{\lambda(\lambda-1)} \\
&= (\lambda-2)P(\theta_{f+1,2,2}) - (\lambda-2)^2 P(C_{f+2}) - (\lambda-2)\left(\lambda^2 - 4\lambda + 5\right)P(C_{f+1}) \\
&= \lambda(\lambda-1)\left[(\lambda-2)M_{f+1,2,2}(\lambda) - (\lambda-2)^2 Q_{f+2}(\lambda) - (\lambda-2)\left(\lambda^2 - 4\lambda + 5\right)Q_{f+1}(\lambda)\right] \\
&= \lambda(\lambda-1)N_{20}(\lambda),
\end{aligned} \tag{4.20}$$

*where* $N_{20}(\lambda) = (\lambda-2)M_{f+1,2,2}(\lambda) - (\lambda-2)^2 Q_{f+2}(\lambda) - (\lambda-2)(\lambda^2 - 4\lambda + 5)Q_{f+1}(\lambda)$ *and* $N_{20}(1) = (-1)(-1)^f - (-1)^f - (-1)(2)(-1)^{f+1} = 4(-1)^{f+1}$.

(21)

$$P(F_{21}) = \lambda(\lambda-1)N_{21}(\lambda), \tag{4.21}$$

*where* $N_{21}(\lambda) = (\lambda-2)^2(\lambda^3 - 6\lambda^2 + 14\lambda - 13)$ *and* $N_{21}(1) = -4$.

(22)

$$\begin{aligned}
P(F_{22}(h)) &= \frac{P(W(h+3,3))P(W(5,3))}{P(K_3)} \\
&= (\lambda-2)\left(\lambda^2 - 4\lambda + 5\right)[P(C_{h+2}) - P(C_{h+1})] \\
&= \lambda(\lambda-1)(\lambda-2)\left(\lambda^2 - 4\lambda + 5\right)[Q_{h+2}(\lambda) - Q_{h+1}(\lambda)] \\
&= \lambda(\lambda-1)N_{22}(\lambda),
\end{aligned} \tag{4.22}$$

*where* $N_{22}(\lambda) = (\lambda-2)(\lambda^2 - 4\lambda + 5)[P(Q_{h+2}(\lambda) - P(Q_{h+1}(\lambda)]$ *and* $N_{22}(1) = (-1)(2)[(-1)^{h+2} - (-1)^{h+1}] = 4(-1)^{h+1}$.

(23)

$$P(F_{23}) = \lambda(\lambda-1)N_{23}(\lambda), \tag{4.23}$$

*where* $N_{23}(\lambda) = (\lambda-2)(\lambda^2 - 4\lambda + 5)^2$ *and* $N_{23}(1) = -4$.

(24)

$$P(F_{24}) = \lambda(\lambda-1)N_{24}(\lambda), \tag{4.24}$$

*where* $N_{24}(\lambda) = (\lambda-2)^2(\lambda^4 - 7\lambda^3 + 20\lambda^2 - 28\lambda + 17)$ *and* $N_{24}(1) = 3$.

(25)

$$P(F_{25}) = \lambda(\lambda-1)N_{25}(\lambda), \tag{4.25}$$

*where* $N_{25}(\lambda) = (\lambda-2)(\lambda^2 - 4\lambda + 5)(\lambda^3 - 5\lambda^2 + 9\lambda - 7)$ *and* $N_{25}(1) = 4$.

(26)

$$\begin{aligned} P(F_{26}(j)) &= P(W_{j+1}(5,3)) - \frac{P(W(5,3))P(C_{j+1})}{\lambda(\lambda-1)} \\ &= (\lambda-2)P(\theta_{j+1,2,2}) - (\lambda-2)^2 P(C_{j+2}) - (\lambda-2)\left(\lambda^2 - 4\lambda + 5\right)P(C_{j+1}) \\ &= \lambda(\lambda-1)\left[(\lambda-2)M_{j+1,2,2}(\lambda) - (\lambda-2)^2 Q_{j+2}(\lambda) - (\lambda-2)\left(\lambda^2 - 4\lambda + 5\right)Q_{j+1}(\lambda)\right] \\ &= \lambda(\lambda-1)N_{26}(\lambda), \end{aligned} \tag{4.26}$$

*where* $N_{26}(\lambda) = (\lambda-2)M_{j+1,2,2}(\lambda) - (\lambda-2)^2 Q_{j+2}(\lambda) - (\lambda-2)(\lambda^2 - 4\lambda + 5)Q_{j+1}(\lambda)$ *and* $N_{26}(1) = (-1)(-1)^j - (-1)^j - (-1)(2)(-1)^{j+1} = 4(-1)^{j+1}$.

(27)

$$P(F_{27}) = \lambda(\lambda-1)N_{27}(\lambda) \tag{4.27}$$

*where* $N_{27}(\lambda) = (\lambda-2)^2(\lambda^3 - 6\lambda^2 + 14\lambda - 13)$ *and* $N_{27}(1) = -4$.

(28)

$$\begin{aligned} P(F_{28}(k)) &= (\lambda-2)^3 P(C_{k+2}) - (\lambda-3)P(W(k+3,3)) \\ &= (\lambda-2)^3 P(C_{k+2}) - (\lambda-2)(\lambda-3)[P(C_{k+2}) - P(C_{k+1})] \\ &= \lambda(\lambda-1)(\lambda-2)\left[\left(\lambda^2 - 5\lambda + 7\right)Q_{k+2}(\lambda) + (\lambda-3)Q_{k+1}(\lambda)\right] \\ &= \lambda(\lambda-1)N_{28}(\lambda), \end{aligned} \tag{4.28}$$

*where* $N_{28}(\lambda) = (\lambda-2)[(\lambda^2 - 5\lambda + 7)Q_{k+2}(\lambda) + (\lambda-3)Q_{k+1}(\lambda)]$ *and* $N_{28}(1) = (-1)[3(-1)^{k+2} + (-2)(-1)^{k+1}] = 5(-1)^{k+1}$.

(29)

$$P(F_{29}) = \lambda(\lambda - 1)N_{29}(\lambda), \tag{4.29}$$

*where* $N_{29}(\lambda) = (\lambda - 2)(\lambda^4 - 8\lambda^3 + 26\lambda^2 - 41\lambda + 27)$ *and* $N_{29}(1) = -5$.

(30)

$$P(F_{30}) = \lambda(\lambda - 1)N_{30}(\lambda), \tag{4.30}$$

*where* $N_{30}(\lambda) = (\lambda - 2)(\lambda^4 - 8\lambda^3 + 26\lambda^2 - 42\lambda + 29)$ *and* $N_{30}(1) = -6$.

(31)

$$P(F_{31}) = \lambda(\lambda - 1)N_{31}(\lambda), \tag{4.31}$$

*where* $N_{31}(\lambda) = (\lambda - 2)^2(\lambda^3 - 6\lambda^2 + 14\lambda - 13)$ *and* $N_{31}(1) = -4$.

**Lemma 4.2.** *Let* $\mathcal{F}_1 = \{F_4, F_5\}$*,* $\mathcal{F}_2 = \{F_2, F_3, F_{14}, F_{18}, F_{19}\}$*,* $\mathcal{F}_3 = \{F_1, F_{11}, F_{12}, F_{13}, F_{17}, F_{24}\}$*,* $\mathcal{F}_4 = \{F_6, F_7, F_{20}, F_{21}, F_{22}, F_{23}, F_{25}, F_{26}, F_{27}, F_{31}\}$*,* $\mathcal{F}_5 = \{F_8, F_9, F_{10}, F_{28}, F_{29}\}$*, and* $\mathcal{F}_6 = \{F_{15}, F_{16}, F_{30}\}$*. Then, for each* $F \in \mathcal{F}_i$*,* $i = 1, 2, 3, 4, 5, 6$*,* $H \sim F$ *implies that* $H$ *must be of type* $F$ *or* $F'$ *for an* $F'$ *in* $\mathcal{F}_i$*.*

*Proof.* It follows directly from Lemma 4.1 that if $i \neq j$, $F_p \in \mathcal{F}_i$ and $F_q \in \mathcal{F}_j$, then $|N_p(1)| = i \neq j = |N_q(1)|$. □

From Lemmas 2.3 and 4.1, we also get the following lemma directly.

**Lemma 4.3.** (1) $F_6 \sim F_{25}$.

(2) $F_7 \sim F_{21} \sim F_{27} \sim F_{31}$.

(3) $F_8(b) \sim F_{28}(k)$ *if and only if* $b = k$.

(4) $F_{10} \sim F_{29}$.

(5) $F_{11} \sim F_{24}$.

(6) $F_{20}(f) \sim F_{26}(j)$ *if and only if* $f = j$.

## 5. Proof of the Main Theorem

We are now ready to prove our main theorem.

(1) Let $H \sim F_1$. By Lemma 4.2, $H$ is of type (1), (11), (12), (13), (17), or (24). If $H = F_1$, then $H$ is of type $F_1$. Lemma 4.1 further implies that $P(F_1, \lambda) \neq P(F_i, \lambda)$, $i = 11, 12, 17, 24$. Hence, $H$ cannot be of type (11), (12), (17), or (24). If $H = F_{13}(c)$, by Lemma 2.3, $c = 3$. Using Software Maple, we have

$$\begin{aligned} P(F_{13}(3)) &= \lambda(\lambda - 1)(\lambda - 2)^2\left(\lambda^4 - 7\lambda^3 + 20\lambda^2 - 28\lambda + 17\right) \\ &\neq (\lambda - 2)\left(\lambda^2 - 3\lambda + 3\right)\left(\lambda^3 - 6\lambda^2 + 13\lambda - 11\right) \\ &= P(F_1). \end{aligned} \tag{5.1}$$

Thus, $H$ must be of type $F_1$.

(2) Let $H \sim F_2$. By Lemma 4.2, $H$ is of type (2), (3), (14), (18), or (19). If $H = F_2(a')$, then by Lemma 2.3, $a' = a$. Thus, $H$ must be of type $F_2$. Since $F_2(a)$ has two induced $C_4$s while each of $F_3$ and $F_{19}$ has at least three induced $C_4$s, by Lemma 2.3, $H$ cannot be of type (3) or (19). Since $P(F_{14})$ is divisible by $(\lambda-2)^4$ but not $P(F_2(a))$, $H$ cannot be of type (14). If $H = F_{18}(e)$, then by Lemma 2.3, $e = a$. Note that

$$\begin{aligned} P(F_2(a)) &= (\lambda-1)(\lambda-2)^3 P(C_{a+1}) - (\lambda-2)^2(\lambda-3)P(C_{a+1}), \\ P(F_{18}(a)) &= (\lambda-1)(\lambda-2)P(W(a+3,3)) - (\lambda-2)^2(\lambda-3)P(C_{a+1}). \end{aligned} \tag{5.2}$$

This implies that $(\lambda-2)^2 P(C_{a+1}) = P(W(a+3,3))$, a contradiction since $P(W(a+3,3))$ is not divisible by $(\lambda-2)^2$. Thus, $H \in \langle F_2(a)\rangle$ if and only if $H$ is of type $F_2(a)$.

(3) Let $H \sim F_3$. By Lemma 4.2 and the above result, $H$ is of type (3), (14), (18), or (19). If $H = F_3$, then $H$ is of type $F_3$. By Lemma 4.1, $F_3 \not\sim F_{14}$ and $F_{19}$. If $H = F_{18}(e)$, by Lemma 2.3, $e = 3$. Using Software Maple, we have

$$\begin{aligned} P(F_{18}(3),\lambda) &= \lambda(\lambda-1)(\lambda-2)^4\left(\lambda^2-3\lambda+4\right) \\ &\neq (\lambda-2)^2\left(\lambda^2-4\lambda+5\right)\left(\lambda^2-3\lambda+3\right) = P(F_3,\lambda). \end{aligned} \tag{5.3}$$

Thus, $H$ must be of type $F_3$.

(4) Let $H \sim F_4$. By Lemma 4.2, $H$ is of type (4) or (5). If follows directly from Lemma 4.1 that $F_4 \not\sim F_5$. Thus, $H$ must be of type $F_4$.

(5) Let $H \sim F_5$. By Lemma 4.2 and the above result, $H$ must be of type (5). Thus, $H$ must be of type $F_5$.

(6) By Lemma 4.2, $H$ is of type (6), (7), (20), (21), (22), (23), (25), (26), (27), or (31). If $H = F_6$, then $H \cong F_6$. Note that Lemma 4.1 implies that $F_6 \not\sim F_i$, $i = 7, 21, 23, 27, 31$. If $H = F_{20}(f)$, $F_{22}(h)$, or $F_{26}(j)$, by Lemma 2.3, $f = h = j = 3$. Using Software Maple, we have

$$\begin{aligned} P(F_{20}(3),\lambda) = P(F_{26}(3),\lambda) &= \lambda(\lambda-1)(\lambda-2)^2\left(\lambda^4-7\lambda^3+20\lambda^2-28\lambda+18\right) \\ &\neq \lambda(\lambda-1)(\lambda-2)\left(\lambda^2-4\lambda+5\right)\left(\lambda^3-5\lambda^2+9\lambda-7\right) \\ &= P(F_{22}(3),\lambda) = P(F_6,\lambda). \end{aligned} \tag{5.4}$$

Thus, by Lemma 4.3, $H \in \langle F_6\rangle$ if and only if $H \cong F_6, F_{25}$ or of type $F_{22}(3)$.

(7) Let $H \sim F_7$. By Lemma 4.2 and the above results, $H$ is of type (7), (20), (21), (22) where $h \geq 4$, (23), (26), (27), or (31). If $H = F_i$, $i = 7, 21, 27, 31$, Lemma 4.3 implies that $H \cong F_7$, $F_{21}$, $F_{27}$, or $H$ is of type $F_{31}$. Lemma 4.1 further implies that $H$ cannot be of type (20), (22), (23), or (26). Thus, $H \in \langle F_7\rangle$ if and only if $H \cong F_7$, $F_{21}$, $F_{27}$, or $H$ is of type $F_{31}$.

(8) Let $H \sim F_8(b)$. By Lemma 4.2, $H$ is of type (8), (9), (10), (28), or (29). If $H = F_8(b')$, by Lemma 2.3, $b' = b$. Thus, $H \cong F_8(b)$. Since $F_8(b)$ is of order at least 8 but $F_i$, $i = 9, 10, 29$ is of order 7, by Lemma 2.3, $P(F_8(b)) \neq P(F_i)$, $i = 9, 10, 29$. By Lemma 4.3, $P(F_8(b)) = P(F_{28}(b))$. Hence, $\langle F_8(b)\rangle = \{F_8(b), F_{28}(b)\}$.

(9) Let $H \sim F_9$. By Lemma 4.2 and the above results, $H$ is of type (9), (10), or (29). By Lemma 4.1, $F_9 \not\sim F_{10}, F_{29}$. Thus, $H \cong F_9$ and $F_9$ is $\chi$-unique.

(10) Let $H \sim F_{10}$. By Lemma 4.2 and the above result, $H$ is of type (10) or (29). By Lemma 4.3, $\langle F_{10}\rangle = \{F_{10}, F_{29}\}$.

(11) Let $H \sim F_{11}$. By Lemma 4.2 and the above result, $H$ is of type (11), (12), (13), (17), or (24). If $H = F_{11}$ or $F_{24}$, by Lemma 4.3, $H$ must be of type $F_{11}$ or $F_{24}$. Lemma 4.1 further implies that $P(F_{11}, \lambda) \neq P(F_{12}, \lambda)$ and $P(F_{17}, \lambda)$. Hence, $H$ cannot be of type (12) or (17). If $H = F_{13}(c)$, Lemma 2.3 implies that $c = 3$. Using Software Maple, we have

$$\begin{aligned} P(F_{13}(3), \lambda) &= \lambda(\lambda - 1)(\lambda - 2)^2\left(\lambda^4 - 7\lambda^3 + 20\lambda^2 - 28\lambda + 17\right) \\ &= P(F_{11}, \lambda). \end{aligned} \tag{5.5}$$

Hence, $H \in \langle F_{11}\rangle$ if and only if $H$ is of type $F_{11}$, $F_{13}(3)$, or $F_{24}$.

(12) Let $H \sim F_{12}$. By Lemma 4.2 and the above result, $H$ is of type (12), (13) with $c \geq 4$ or (17). Since $F_{12}$ and $F_{13}(c)$ have different order, Lemma 2.3 implies that $F_{12} \nsim F_{13}$. Lemma 4.1 also implies that $F_{12} \nsim F_{17}$. Thus, $H$ must be of type $F_{12}$.

(13) Let $H \sim F_{13}(c), c \geq 4$. By Lemma 4.2 and the above result, $H$ is of type (13) with $c \geq 4$ or (17). If $H = F_{13}(c')$, then $c' = c$. Since $F_{13}(c)$ and $F_{17}$ have different order, Lemma 2.3 implies that $F_{13}(c) \nsim F_{17}$. Thus, $H \in \langle F_{13}(c)\rangle$ if and only if $H$ is of type $F_{13}(c)$ for $c \geq 4$ and $H \in \langle F_{13}(3)\rangle$ if and only if $H$ is of type $F_{11}$, $F_{13}(3)$, or $F_{24}$.

(14) Let $H \sim F_{14}$. By Lemma 4.2 and the above result, $H$ is of type (14), (18) or (19). If $H = F_{14}$, then $H$ is of type $F_{14}$. If $H = F_{18}(e)$, by Lemma 2.3, $e = 3$. Using Software Maple, we have

$$P(F_{18}(3), \lambda) = \lambda(\lambda - 1)(\lambda - 2)^4\left(\lambda^2 - 3\lambda + 4\right) = P(F_{14}, \lambda). \tag{5.6}$$

By Lemma 4.1, we also have $F_{14} \nsim F_{19}$. Hence, $H \in \langle F_{14}\rangle$ if and only if $H$ is of type $F_{14}$ or $F_{18}(3)$.

(15) Let $H \sim F_{15}(d)$. By Lemma 4.2, $H$ must be of type (15), (16), or (30). If $H = F_{15}(d')$, by Lemma 2.3, $d' = d$. Thus, $H \cong F_{15}$. Since $F_{16}$ has exactly six induced $C_4$s while $F_{15}(d)$ has only two induced $C_4$s, by Lemma 2.3, $H$ cannot be of type (16). If $H = F_{31}$, by Lemma 2.3, $d = 2$. Using Software Maple, we have

$$\begin{aligned} P(F_{15}(2)) &= \lambda(\lambda - 1)(\lambda - 2)\left(\lambda^4 - 8\lambda^3 + 26\lambda^2 - 42\lambda + 29\right) \\ &= P(F_{30}). \end{aligned} \tag{5.7}$$

Thus, $\langle F_{15}(2)\rangle = \{F_{15}(2), F_{30}\}$ and $F_{15}(d)$ is $\chi$-unique for $d \geq 3$.

(16) Let $H \sim F_{16}$. By Lemma 4.2 and the above results, $H \cong F_{16}$. Thus, $F_{16}$ is $\chi$-unique.

(17) Let $H \sim F_{17}$. By Lemma 4.2 and the above results, $H \cong F_{17}$. Thus, $F_{17}$ is $\chi$-unique.

(18) Let $H \sim F_{18}(e)$, $e \geq 4$. By Lemma 4.2 and the above results, $H$ must be of of type (18) with $e \geq 4$, or (19). If $H = F_{18}(e')$, Lemma 2.3 implies that $e' = e$. Since $F_{18}(e)$ and $F_{19}$ are of different order, it follows that $H$ cannot be of type (19). Thus, $H \in \langle F_{18}(e)\rangle$ if and only if $H$ is of type $F_{18}(e)$ for $e \geq 4$, and $H \in \langle F_{18}(3)\rangle$ if and only if $H$ is of type $F_{14}$ or $F_{18}(3)$.

(19) Let $H \sim F_{19}$. By Lemma 4.2 and the above results, $H$ must be of type $F_{19}$.

(20) Let $H \sim F_{20}(f)$. By Lemma 4.2 and the above results, $H$ must be of type (20), (22) where $h \geq 4$, (23) or (26). If $H = F_{20}(f')$, Lemma 2.3 implies that $f' = f$. If $H = F_{22}(h)$, Lemma 2.3 implies that $h = f$. Note that

$$P(F_{20}(f)) = (\lambda - 1)P(W(f+4,4)) - (\lambda - 3)P(W(f+3,3)),$$
$$P(F_{22}(f)) = (\lambda - 1)(\lambda - 2)P(W(f+3,3)) - (\lambda - 3)P(W(f+3,3)). \tag{5.8}$$

This implies that $P(W(f+4,4)) = (\lambda-2)P(W(f+3,3))$, a contradiction since $P(W(f+4,4))$ is not divisible by $(\lambda-2)^2$ but $(\lambda-2)P(W(f+3,3))$ is divisible by $(\lambda-2)^2$. Since $F_{20}$ and $F_{23}$ are of different order, Lemma 2.3 further implies that $H$ cannot be of type (23). Lemma 4.3 then implies that $\langle F_{20}(f)\rangle = \{F_{20}(f), F_{26}(f)\}$.

(21) The result follows directly from (7) above.

(22) Let $H \sim F_{22}(h)$, $h \geq 4$. By Lemma 4.2 and the above result, $H$ is of type (22) with $h \geq 4$, or (23). If $H = F_{22}(h')$, Lemma 2.3 implies that $h' = h$. Since $F_{22}(h)$ and $F_{23}$ are of different order, Lemma 2.3 further implies that $H$ cannot be of type (23). Thus, $H \in \langle F_{22}(h)\rangle$ if and only if $H$ is of type $F_{22}(h)$ for $h \geq 4$, and $H \in \langle F_{22}(3)\rangle$ if and only if $H \cong F_6$, $F_{25}$ or $H$ is of type $F_{22}(3)$.

(23) Let $H \sim F_{23}$. By Lemma 4.2 and the above results, $H$ must be of type $F_{23}$. Thus, $H \in \langle F_{23}\rangle$ if and only if $H$ is of type $F_{23}$.

(24) The result follows directly from (11) above.

(25) The result follows directly from (6) above.

(26) The result follows directly from (20) above.

(27) The result follows directly from (7) above.

(28) The result follows directly from (8) above.

(29) The result follows directly from (10) above.

(30) The result follows directly from (15) above.

(31) The result follows directly from (7) above.

This completes the proof of our main theorem.

## 6. Further Research

The above results and the main results in [6, 7] completely determined the chromaticity of all 2-connected $(n, n+4)$-graphs with (i) exactly 3 triangles (and at least one induced 4-cycle) and (ii) at least 4 triangles. However, the study of the chromaticity of 2-connected $(n, n+4)$-graphs with exactly 3 triangles is far from completion although all 23 $\chi_r$-closed families of such graphs have been obtained in [8] as shown in Figure 2. Base on the above results, it is expected that many different families of 2-connected $(n, n+4)$-graphs with exactly 3 triangles are $\chi$-equivalent. Perhaps, the approach used in the study of the chromaticity of $K_4$-homeomorphs (see [13]) or a more efficient approach of comparing the chromatic polynomials of graphs can be applied in solving the following problem.

*Problem 1.* Determine the chromatic uniqueness of all 2-connected $(n, n+4)$-graphs with exactly 3 triangles.

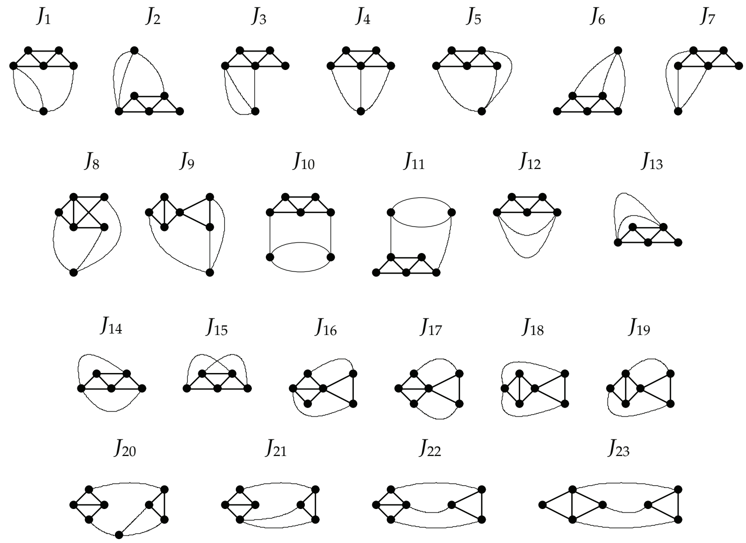


**Figure 2:** Relative-closed family of 2-connected $(n, n+4)$-graphs with exactly 3 triangles. The light lines of the graphs refer to the paths with edges not belong to any triangles.

## Acknowledgment

The authors would like to thank the referee for the valuable comments.

## References

[1] C. Y. Chao and E. G. Whitehead Jr., "On chromatic equivalence of graphs," in *Theory and Applications of Graph*, Y. Alavi and D. R. Lick, Eds., vol. 642 of *Lecture Notes in Mathematics*, pp. 121–131, Springer, New York, NY, USA, 1978.
[2] B. Loerinc, "Chromatic uniqueness of the generalized $\theta$-graph," *Discrete Mathematics*, vol. 23, no. 3, pp. 313–316, 1978.
[3] F. M. Dong, K. L. Teo, and K. M. Koh, "A note on the chromaticity of some 2-connected $(n, n+3)$-graphs," *Discrete Mathematics*, vol. 243, no. 1–3, pp. 217–221, 2002.
[4] K. M. Koh and K. L. Teo, "Chromatic classes of 2-connected $(n, n+3)$-graphs with at least two triangles," *Discrete Mathematics*, vol. 127, no. 1–3, pp. 243–258, 1994.
[5] K. L. Teo and K. M. Koh, "Chromatic classes of certain 2-connected $(n, n+2)$-graphs," *Ars Combinatoria*, vol. 32, pp. 65–76, 1991.
[6] Y. H. Peng and G. C. Lau, "Chromatic classes of 2-connected $(n, n+4)$-graphs with at least four triangles," *Discrete Mathematics*, vol. 278, no. 1–3, pp. 209–218, 2004.
[7] Y. H. Peng and G. C. Lau, "Chromatic classes of 2-connected $(n, n+4)$-graphs with three triangles and one induced 4-cycle," *Discrete Mathematics*, vol. 309, no. 10, pp. 3092–3101, 2009.
[8] G. C. Lau and Y. H. Peng, "Relative-closed family of graphs with exactly three triangles," Technical Report 2, Institute for Mathematical Research, Putra, Malaysia, 2010.
[9] M. Behzad, G. Chartrand, and L. Lesniak-Foster, *Graphs and Digraphs*, Wadsworth, Belmont, Calif, USA, 1979.
[10] H. Whitney, "The coloring of graphs," *Annals of Mathematics*, vol. 33, pp. 688–718, 1932.
[11] A. A. Zykov, "On some properties of linear complexes," *Matematicheskii Sbornik*, vol. 24, pp. 163–188, 1949, Translation in American Mathematical Society Translations, no. 79.
[12] E. G. Whitehead Jr. and L. C. Zhao, "Cutpoints and the chromatic polynomial," *Journal of Graph Theory*, vol. 8, pp. 371–377, 1984.
[13] X. F. Li, "A family of chromatically unique $K_4$-homeomorphs," *Journal of Anhui University of Technology and Science*, vol. 32, no. 4, pp. 18–21, 2008.