Research Article

Existence of Weak Solution for a Free Boundary Lubrication Problem

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This paper is interested in a free boundary problem modelling a phenomenon of cavitation in hydrodynamic lubrication. We reformulate the problem (see Boukrouche, (1993)) in a large context by introducing two positive parameters, namely, N_0 and a . We build a weak formulation and establish the existence of the solution to the problem.

1. Introduction

The lubrication fields have many applications; one example is the study of the rotary mechanisms such as the bearing, joints. The study is concerned in looking for a moving free boundary problem related to the cavitation modelling in lubrication (see [1–4]). The experimental results make evidence of the occurrence of two distinct zones, one full of the fluid, namely, the saturated zone $\Omega(t)$; the other Ω^c ($\Omega^c = \Omega \setminus \Omega(t)$, where Ω is the global domain), is the cavitated zone (e.g., the mixture of fluid and air). Two approaches have been used to cope with phenomena. One of them [5] homogenizes the phenomena and considers it as a 2D phenomena, so introducing θ the saturation variable (lubricant concentration); the other one [6] takes full account of the three-dimensional character of this phenomena, with appearance of air bubbles and introduces in Ω^c the relative height as supplementary unknown (for more details, see [1, 3]). We use here the first approach but both approaches lead to the same mathematical problem. In this paper, we take the problem studied in [1–3] and rewrite it, here, in a large context, by introducing two positive parameters, namely, N_0 and a . This formulation of the problem gives as advantages a proportionality relation

Figure 1: A view of domains $\Omega(t)$ and Ω^c .

between N_0 and the pressure p , and the parameter a which allows the control of a squeezing effects. The mathematical modelling is made according to the model of Jakobsson-Floberg (see [1, 3]), where the lubricant is not defined only by a pressure p but also by a saturation variable θ . This variable θ characterizes the cavitation phenomena, where $\theta \equiv 1$ in $\Omega(t)$ and $0 \leq \theta < 1$ in Ω^c . The interface between $\Omega(t)$ and Ω^c constitutes the moving free boundary denoted by $\Gamma(t)$. The problem is a convection-diffusion problem type, and the Reynolds equation is elliptic in the saturated zone and hyperbolic in the other one. We note that the study of existence and uniqueness point of views to the problem has been established in [3] in the particular case of the $N_0 = 0$ and $a = 1$. For this, the author proved the existence of solution for this kind of problem by way of an approximation by an elliptic problem. In our work, we followed, exactly the same way, and the aim of this study is to construct a weak formulation and establish the existence of solution to the problem.

The plan of the paper is as follows. Section 2 proposes a state of the problem and a weak formulation. Section 3 introduces an elliptic nonlinear problem and gives the existence and uniqueness of the solution to this problem. Section 4 proposes an approximation of the elliptic nonlinear problem by a family of linear problems and proves a priori estimate. Last section gives a theorem of existence of the solution to the problem.

2. Notations and State of the Problem

2.1. Description of the Phenomenon

We consider a global domain Ω with border $\partial\Omega$. The fluid is injected at a given rate w over the fixed (internal) boundary Γ_I ($\Gamma_{\text{ex}} = \partial\Omega \setminus \Gamma_I$). For each $t \in]0; T[$, the experimental results make evidence of the occurrence of two distinct zones: one full of the fluid is the saturated zone $\Omega(t)$, where the pressure p ($p > 0$) and the saturation variable θ ($\theta \equiv 1$), and the other Ω^c ($\Omega^c = \Omega \setminus \Omega(t)$) is the cavitated zone, where the pressure is constant ($p = 0$) and $0 \leq \theta < 1$ (e.g., the mixture of fluid and air). The free boundary of the region $\Omega(t)$ containing fluid is $\Gamma(t)$ and the region $\Omega(0)$, with border Γ_I , occupied by the fluid at $t = 0$ being given (see Figure 1).

2.2. State of the Problem

The strong formulation of the problem described the phenomenon is written as follows.

Figure 2: Graph of saturation variable θ .

For each $t \geq 0$, find a pair $(p, \theta) \in L^2(0, T; H^1(\Omega)) \times L^\infty(Q) \cap H^1(0, T; H^{-1}(\Omega))$ and $\Gamma(t)$ such that

$$\nabla(K(h, t, N_0)\nabla p) = \nabla(Vh\theta) + a\frac{\partial(h\theta)}{\partial t}, \quad (\theta \equiv 1) \text{ in } \Omega(t), \quad (2.1)$$

$$\nabla(Vh\theta) + a\frac{\partial(h\theta)}{\partial t} = 0 \quad \text{in } \Omega^c, \quad (2.2)$$

$$K(h, t, N_0)\frac{\partial p}{\partial n} = h(1 - \theta)(V - av)n \quad \text{on } \Gamma(t), \quad (2.3)$$

$$p = 0 \quad \text{on } \Gamma(t), \quad (2.4)$$

$$K(h, t, N_0)\frac{\partial p}{\partial \underline{n}} - h\theta V \underline{n} = w \quad \text{on } \Gamma_I, \quad (2.5)$$

$$p = 0 \quad \text{on } \Gamma_{\text{ex}}, \quad (2.6)$$

$$p(1 - \theta) = 0 \quad \text{in } \Omega, \quad (2.7)$$

$$\theta|_{t=0} = \theta_0, \quad (2.8)$$

where

$$K(h, t, N_0) = \frac{h^3}{1 - N_0^2} \Phi(h, t, N_0) \quad \text{with } 0 \leq N_0 < 1, \quad (2.9)$$

$$\Phi(h, t, N_0) = \frac{1}{12} + \frac{1}{4h^2(1 - N_0^2)} - \frac{1}{4h} \sqrt{\frac{N_0^2}{1 - N_0^2}} \coth\left(N_0 h \sqrt{1 - N_0^2}\right).$$

$h(t, x)$ is the thickness of the thin film supposed a regular and given function of the problem. V is the speed of the axis supposed being given. v is the moving free boundary (with $v = 0$ on Γ_I). n (resp., $\underline{n}$) is the normal vector along $\Gamma(t)$ (resp., Γ_I) exterior to $\Omega(t)$ (resp., $\Omega(0)$). The saturation variable θ can be represented by a graph (see, Figure 2).

In (2.1)–(2.3), there are the diffusion term $\nabla(K(h, t, N_0)\nabla p)$, the shearing term $\nabla(hV)$, and the squeezing term $\partial h / \partial t$.

2.3. Weak Formulation

Before starting the construction of a weak formulation of the problem (2.1)–(2.8), we denote by

$$\begin{aligned} Q &= \Omega \times]0, T[, \quad \Sigma_I = \Gamma_I \times]0, T[, \quad \Sigma_t = \Gamma(t) \times]0, T[, \quad \Sigma_{\text{ex}} = \Gamma_{\text{ex}} \times]0, T[, \\ E &= \left\{ \varphi \in H^1(Q) : \varphi = 0 \text{ on } \Sigma_{\text{ex}} \text{ and } t = 0 \right\}. \end{aligned} \quad (2.10)$$

Indeed, multiplying (2.1) by $\varphi \in E$ and integrating over $[0, T] \times \Omega(t)$, we obtain

$$\begin{aligned} & - \int_0^T \int_{\Omega(t)} K \nabla P \nabla \varphi \, dx \, dt + \int_0^T \int_{\Gamma(t)} K \nabla P \varphi n + \int_0^T \int_{\Gamma_I} K \nabla P \varphi \underline{n} \\ & = - \int_0^T \int_{\Omega(t)} V h \nabla \varphi \, dx \, dt + \int_0^T \int_{\Gamma(t)} V h \varphi n + \int_0^T \int_{\Gamma_I} V h \varphi \underline{n} + \int_0^T \int_{\Gamma_{\text{ex}}} V h \varphi \cdot n_{\text{ex}} \\ & \quad + a \left\{ \int_0^T \frac{d}{dt} \int_{\Omega(t)} h \varphi \, dx \, dt - \int_0^T \int_{\Gamma(t)} V h \Phi \cdot n \, dt - \int_0^T \int_{\Gamma_I} V h \varphi \underline{n} \, dt - \int_0^T \int_{\Omega(t)} h \frac{d\varphi}{dt} \, dx \, dt \right\}, \end{aligned} \quad (*)$$

where n_{ex} is the normal vector along $\partial\Omega$ exterior to Ω .

In the same way, we apply to (2.2)

$$\begin{aligned} & - \int_0^T \int_{\Omega^c} V h \nabla \varphi \, dx \, dt + \int_0^T \int_{\Gamma(t)} V h \varphi \cdot n + \int_0^T \int_{\Gamma_I} V h \varphi \cdot \underline{n} + \int_0^T \int_{\Gamma_{\text{ex}}} V h \varphi \cdot n_{\text{ex}} \\ & \quad + a \left\{ \int_0^T \frac{d}{dt} \int_{\Omega^c} h \varphi \, dx \, dt - \int_0^T \int_{\Gamma(t)} V h \varphi \cdot n \, dt - \int_0^T \int_{\Gamma_I} V h \varphi \underline{n} \, dt - \int_0^T \int_{\Omega^c} h \frac{d\varphi}{dt} \, dx \, dt \right\} = 0. \end{aligned} \quad (**)$$

By adding (*) and (**) in all $\Omega = \Omega(t) \cup \Omega^c$, and using (2.3)–(2.6) then the weak formulation can be written as follows.

Find a pair $(p, \theta) \in L^2(0, T; H^1(\Omega)) \times L^\infty(Q) \cap H^1(0, T; H^{-1}(\Omega))$

$$0 \leq \theta < 1, \quad p = 0 \text{ on } \Sigma_{\text{ex}}, \quad p(1 - \theta) = 0, \quad \text{a.e. in } Q, \quad (2.11)$$

$$\begin{aligned} & \int_Q (K \nabla p - V h \theta) \nabla \varphi + a \left\{ \int_{\Omega(T)} h \theta \varphi - \int_{\Omega(0)} h \theta \varphi - \int_Q h \theta \frac{d\varphi}{dt} \right\} = \int_{\Sigma_I} w \varphi, \quad \forall \varphi \in E, \\ & \theta|_{t=0} = \theta_0 \in H^{-1}(\Omega). \end{aligned} \quad (2.12)$$

As $\varphi|_{t=0} = 0$ implies that $\int_{\Omega(0)} h \theta \varphi = 0$.

Figure 3: Graph of the nonlinear function β .

3. An Elliptic Nonlinear Problem

To solve the problem (2.11)-(2.12), we will approximate it by an elliptic problem in the same way as Boukrouche [3] and Gilardi [7].

Let β be a real function (see Figure 3) satisfying the following assumptions:

$$\begin{aligned} \beta &\in C^\infty(R) \quad \forall \xi \in R; \quad 0 \leq \beta(\xi) \leq 1, \quad \text{with } \beta'(\xi) \geq 0, \\ \beta(\xi) &= 0 \quad \text{for } \xi \leq 0. \end{aligned} \quad (3.1)$$

Put

$$\partial_0 Q = \Sigma_{\text{ex}} \cup \Omega_0. \quad (3.2)$$

consider now the problem, given $\varepsilon > 0$, find $\mathbf{p} \in H^1(Q)$ such that

$$\int_Q \varepsilon \frac{\partial \mathbf{p}}{\partial t} \frac{\partial \varphi}{\partial t} + \int_Q (K \nabla \mathbf{p} - V h \beta(\mathbf{p})) \nabla \varphi + a \left\{ \int_{\Omega(T)} h \beta(\mathbf{p}) \varphi - \int_Q h \beta(\mathbf{p}) \frac{d\varphi}{dt} \right\} = \int_{\Sigma_l} w \varphi, \quad (3.3)$$

$\mathbf{p} = 0$ on $\partial_0 Q$, for all $\varphi \in H^1(Q)$ vanishing on $\partial_0 Q$.

Introducing the operator τ is as follows:

$$\begin{aligned} \tau : H^1(Q) &\longrightarrow H^1(Q), \\ \mathbf{p} &\longmapsto q = \tau(\mathbf{p}). \end{aligned} \quad (3.4)$$

If $\mathbf{p} \in H^1(Q)$, $\tau(\mathbf{p})$ is a unique solution q to the linear problem

$$\begin{aligned} \forall q \in H^1(Q), \quad q &= 0, \quad \text{on } \partial_0 Q \\ \int_Q \varepsilon \frac{\partial q}{\partial t} \frac{\partial \varphi}{\partial t} + \int_Q K \nabla q \nabla \varphi - \int_Q h \beta(\mathbf{p}) \left(V \nabla \varphi + a \frac{d\varphi}{dt} \right) + a \int_{\Omega_T} h \beta(\mathbf{p}) \varphi &= \int_{\Sigma_l} w \varphi \end{aligned} \quad (3.5)$$

for every $\varphi \in H^1(Q)$ vanishing on $\partial_0 Q$.

Lemma 3.1. *The operator τ is continuous from $H^1(Q)$ with the weak topology into $H^1(Q)$ with the strong topology. Moreover $\tau(H^1(Q))$ is bounded in $H^1(Q)$.*

Proof. Let $\mathbf{p}_i \in H^1(Q)$ with $q_i = \tau(\mathbf{p}_i)$ for $i = 1, 2$.

Taking $\varphi = (q_1 - q_2)$ in (3.5), we have

$$\begin{aligned} \int_Q \varepsilon \left(\frac{\partial}{\partial t} (q_1 - q_2) \right)^2 + \int_Q K |\nabla (q_1 - q_2)|^2 &= \int_Q h(\beta(\mathbf{p}_1) - \beta(\mathbf{p}_2)) \left(V \nabla (q_1 - q_2) + a \frac{d(q_1 - q_2)}{dt} \right) \\ &\quad - a \left\{ \int_{\Omega(T)} h(\beta(\mathbf{p}_1) - \beta(\mathbf{p}_2)) (q_1 - q_2) \right\}. \end{aligned} \quad (3.6)$$

Using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} \alpha \|q_1 - q_2\|_{H^1(Q)}^2 &\leq C_1 \left\{ \|q_1 - q_2\|_{H^1(Q)} \|\beta(\mathbf{p}_1) - \beta(\mathbf{p}_2)\|_{L^2(Q)} \right\} \\ &\quad + C_2 \left\{ \|q_1 - q_2\|_{L^2(\Omega(T))} \|\beta(\mathbf{p}_1) - \beta(\mathbf{p}_2)\|_{L^2(\Omega(T))} \right\}, \end{aligned} \quad (3.7)$$

where α is constant depending on h , N_0 , and ε . C_1 and C_2 are two constants depending on h , V , and a .

As

$$\begin{aligned} H^{1/2}(\Omega(T)) &\longrightarrow L^2(\Omega(T)), & H^1(Q) &\longrightarrow H^{1/2}(\Omega(T)), \\ H^{1/2}(\Sigma_I) &\longrightarrow L^2(\Sigma_I), & H^1(Q) &\longrightarrow H^{1/2}(\Sigma_I) \end{aligned} \quad (3.8)$$

and β is Lipschitz continuous function, there exists a constant C depending on h , V , N_0 , a , and ε such that

$$\|q_1 - q_2\|_{H^1(Q)} \leq C \left\{ \|\mathbf{p}_1 - \mathbf{p}_2\|_{L^2(Q)} + \|\mathbf{p}_1 - \mathbf{p}_2\|_{L^2(\Omega(T))} \right\}. \quad (3.9)$$

If $\mathbf{p}_\varepsilon \rightarrow \mathbf{p}$ converge weakly in $H^1(Q)$, then $\mathbf{p}_\varepsilon \rightarrow \mathbf{p}$, $L^2(Q)$ and $\mathbf{p}_\varepsilon(T) \rightarrow \mathbf{p}(T)$, $L^2(\Omega(T))$. Thus $\tau(\mathbf{p}_\varepsilon) \rightarrow \tau(\mathbf{p})$, then the continuity of τ is shown.

Taking $\varphi = q$ in (3.5) and using Cauchy-Schwarz's inequality, we obtain $\|q\|_{H^1(Q)} \leq k_1$, where k_1 is a constant depending on h , V , N_0 , β , T , a , and ε . $\square$

Theorem 3.2. *If the function β satisfies hypothesis (3.1), then, for every $\varepsilon > 0$, there exists a solution to the problem (3.3).*

Proof. Use Lemma 3.1 and Schauder fixed-point theorem. $\square$

Theorem 3.3 (cf. [2, 3]). *If the function β satisfies hypothesis (3.1), then for every $\varepsilon > 0$, the solution of the problem (3.3) is unique.*

Figure 4: Graph of Heaviside function H_ϵ .

4. Approximating Problems

In order to solve the problem (2.11)-(2.12), we consider a new family of problems of type (3.3) in which the function β is an approximation of the Heaviside function (see Figure 4). Therefore we consider a family of functions

$$H_\epsilon : [0, +\infty[\longrightarrow \mathbb{R} \setminus H_\epsilon \in C^\infty(\mathbb{R}), \quad (4.1)$$

$$0 \leq H_\epsilon \leq 1, \quad H'_\epsilon \geq 0, \quad (4.2)$$

$$H_\epsilon(0) = 0, \quad \lim_{\epsilon \rightarrow 0} \inf \{ \zeta > 0 : H_\epsilon(\zeta) = 1 \} = 0, \quad (4.3)$$

$$\lim_{\epsilon \rightarrow 0} L_\epsilon \sqrt{\epsilon} = 0, \quad \text{where } L_\epsilon = \sup \{ H'_\epsilon(\zeta) : \zeta > 0 \}. \quad (4.4)$$

Consider now the following approximating problem.

For fixed ϵ and a (where $\epsilon, a \in]0; 1[$), find p_ϵ such that

$$p_\epsilon \in H^1(Q), \quad p_\epsilon = 0 \quad \text{on } \partial_0 Q, \quad (4.5)$$

$$\int_Q \epsilon \frac{\partial p_\epsilon}{\partial t} \frac{\partial \varphi}{\partial t} + \int_Q K \nabla p_\epsilon \nabla \varphi - \int_Q h H_\epsilon(p_\epsilon) \left(V \nabla \varphi + a \frac{d\varphi}{dt} \right) + a \int_{\Omega(T)} h H_\epsilon(p_\epsilon) \varphi = \int_{\Sigma_I} w \varphi$$

for every $\varphi \in H^1(Q)$ vanishing on $\partial_0 Q$.

From Theorems 3.2 and 3.3, we deduce the following theorem.

Theorem 4.1 (cf., [3]). *For every ϵ and $a \in]0, 1[$, there exists at least one solution to the problem (4.5). Moreover if h and V are sufficiently regular, every solution belongs to $H^1(Q) \cap L^\infty(Q)$.*

Lemma 4.2. *If the function H_ϵ verifies (4.1)-(4.2) and $dh/dt \geq 0$, then one has*

$$\int_{\Omega(T)} h H_\epsilon(p_\epsilon) p_\epsilon - \int_Q h H_\epsilon(p_\epsilon) \frac{dp_\epsilon}{dt} \geq 0. \quad (4.6)$$

Proof. Putting $\widetilde{H}_\epsilon(\zeta) = \int_0^\zeta H_\epsilon(\tau) d\tau$, we have then $\widetilde{H}_\epsilon(\zeta) \leq \zeta H_\epsilon(\zeta)$ and $\int_{\Omega(T)} h H_\epsilon(p_\epsilon) p_\epsilon - \int_Q h H_\epsilon(p_\epsilon) (dp_\epsilon/dt) = \int_{\Omega(T)} h H_\epsilon(p_\epsilon) p_\epsilon - \int_Q h (d/dt)(\widetilde{H}_\epsilon(p_\epsilon))$.

Using the integration by parts, we obtain

$$\int_{\Omega(T)} hH_\varepsilon(p_\varepsilon)p_\varepsilon - \int_Q h \frac{d}{dt} (\widetilde{H}_\varepsilon(p_\varepsilon)) = \int_{\Omega(T)} h [H_\varepsilon(p_\varepsilon)p_\varepsilon - \widetilde{H}_\varepsilon(p_\varepsilon)] + \int_Q \frac{dh}{dt} \widetilde{H}_\varepsilon(\zeta) \quad (4.7)$$

as $H_\varepsilon(p_\varepsilon)p_\varepsilon - \widetilde{H}_\varepsilon(p_\varepsilon) \geq 0$, then Lemma 4.2 is shown. $\square$

The following proposition gives some a priori estimates for pressure p_ε .

Proposition 4.3. *There exists a constant C independent of ε , a , N_0 , and p_ε*

$$\int_Q \varepsilon \left| \frac{\partial p_\varepsilon}{\partial t} \right|^2 + \int_Q K |\nabla p_\varepsilon|^2 + a \left\{ \int_{\Omega(T)} hH_\varepsilon p_\varepsilon - \int_Q hH_\varepsilon \frac{dp_\varepsilon}{dt} \right\} \leq C. \quad (4.8)$$

Proof. Taking $\varphi = p_\varepsilon$ in (4.5), we obtain

$$\int_Q \varepsilon \left| \frac{\partial p_\varepsilon}{\partial t} \right|^2 + \int_Q K |\nabla p_\varepsilon|^2 + a \left\{ \int_{\Omega(T)_T} hH_\varepsilon p_\varepsilon - \int_Q hH_\varepsilon \frac{dp_\varepsilon}{dt} \right\} = \int_Q hVH_\varepsilon(p_\varepsilon) \nabla p_\varepsilon + \int_{\Sigma_I} wp_\varepsilon. \quad (4.9)$$

By using Lemma 4.2, we have

$$\int_Q \varepsilon \left| \frac{\partial p_\varepsilon}{\partial t} \right|^2 + \int_Q K |\nabla p_\varepsilon|^2 \leq \text{mes}(Q) \beta \|\nabla p_\varepsilon\|_{L^2(Q)} \|V\|_{(L^\infty)^2} + \|p_\varepsilon\|_{L^2(\Sigma_I)} \|w\|_{L^2(\Sigma_I)}. \quad (4.10)$$

Applying Poincare's inequality, we have

$$\|p_\varepsilon\|_{L^2(Q)}^2 = \int_0^T \int_\Omega |p_\varepsilon|^2 \leq \alpha_1 \|\nabla p_\varepsilon\|_{L^2(Q)}^2, \quad (4.11)$$

where α_1 is constant depending on Ω .

As $\|p_\varepsilon\|_{L^2(\Sigma_I)} \leq \alpha_2 \|\nabla p_\varepsilon\|_{L^2(\Sigma_I)}^2$, we finally deduce the result. $\square$

Proposition 4.4. *For every φ nonnegative and $\varphi \in D(\Omega)$, there exists a constant $C(\varphi)$ such that*

$$\int_\Omega \varphi \left| \varepsilon \frac{\partial p_\varepsilon(0, x)}{\partial t} \right|^2 \leq C(\varphi) (1 + L_\varepsilon) \sqrt{\varepsilon}. \quad (4.12)$$

Proof. From (4.5), we have

$$\begin{aligned}
\varepsilon \frac{\partial^2 P_\varepsilon}{\partial t^2} + \nabla \cdot (K \nabla p_\varepsilon) &= \nabla \cdot (V h H_\varepsilon(p_\varepsilon)) + a \frac{\partial}{\partial t} (h H_\varepsilon(p_\varepsilon)) \quad \text{in } \Omega, \\
K(h, t, N_0) \nabla p_\varepsilon n - h H_\varepsilon V n &= w \quad \text{on } \Sigma_I, \\
p_\varepsilon &= 0 \quad \text{on } \partial_0 Q, \\
\frac{\partial p_\varepsilon}{\partial t}(T) &= 0 \quad \text{on } \Omega.
\end{aligned} \tag{4.13}$$

Multiplying (4.13) by $\varphi(\partial p_\varepsilon / \partial t)$ and integrating over $]0, t[\times \Omega$, we have

$$\begin{aligned}
\frac{\varepsilon}{2} \int_\Omega \varphi \left| \frac{\partial p_\varepsilon}{\partial t}(0, x) \right|^2 + \frac{1}{2} \int_\Omega K \varphi |\nabla p_\varepsilon|^2 + \int_0^t \int_\Omega \varphi h H'_\varepsilon(p_\varepsilon) \left| \frac{\partial p_\varepsilon}{\partial t}(t) \right|^2 \\
= - \int_0^t \int_\Omega \left\{ K \nabla p_\varepsilon \nabla \varphi + \varphi \nabla (h H_\varepsilon(p_\varepsilon) V) + a \varphi H_\varepsilon(p_\varepsilon) \frac{\partial h}{\partial t} \right\} \frac{\partial p_\varepsilon}{\partial t}(t) \\
+ \frac{\varepsilon}{2} \int_\Omega \varphi \left| \frac{\partial p_\varepsilon}{\partial t}(t) \right|^2 + \frac{1}{2} \int_0^t \int_\Omega \varphi \frac{\partial}{\partial t} (K) |\nabla p_\varepsilon|^2.
\end{aligned} \tag{4.14}$$

Taking

$$\begin{aligned}
A &= \frac{\varepsilon}{2} \int_\Omega \varphi \left| \frac{\partial p_\varepsilon}{\partial t}(t) \right|^2 + \frac{1}{2} \int_0^t \int_\Omega \varphi \frac{\partial}{\partial t} (K) |\nabla p_\varepsilon|^2, \\
B &= - \int_0^t \int_\Omega \left\{ K \nabla p_\varepsilon \nabla \varphi + \varphi \nabla (h H_\varepsilon(p_\varepsilon) V) + a \varphi H_\varepsilon(p_\varepsilon) \frac{\partial h}{\partial t} \right\} \frac{\partial p_\varepsilon}{\partial t}(t),
\end{aligned} \tag{4.15}$$

then we have

$$\begin{aligned}
\frac{\varepsilon}{2} \int_\Omega \varphi \left| \frac{\partial p_\varepsilon}{\partial t}(0, x) \right|^2 &\leq A + B, \\
\frac{\varepsilon^2}{2} T \int_\Omega \varphi \left| \frac{\partial p_\varepsilon}{\partial t}(0, x) \right|^2 &\leq \int_0^T (\varepsilon A + \varepsilon B) dt, \\
\varepsilon \int_0^T A dt &= \frac{\varepsilon^2}{2} \int_0^T \int_\Omega \varphi \left| \frac{\partial p_\varepsilon}{\partial t}(t) \right|^2 + \frac{\varepsilon}{2} \int_0^T \int_\Omega \varphi \frac{\partial}{\partial t} (K) |\nabla p_\varepsilon|^2 \\
&\leq \|\varphi\|_{L^\infty} \frac{\varepsilon}{2} \int_0^T \int_\Omega \left| \frac{\partial p_\varepsilon}{\partial t}(t) \right|^2 + \varepsilon \left\| \frac{\partial}{\partial t} (K) \right\|_{L^\infty} \|\varphi\|_{L^\infty} \int_0^T \int_\Omega |\nabla p_\varepsilon|^2.
\end{aligned} \tag{4.16}$$

Using Proposition 4.3., we obtain

$$\varepsilon \int_0^T A dt \leq c_1(\varphi) \sqrt{\varepsilon}, \tag{4.17}$$

where $c_1(\varphi) = C \max\{(1/2)\|\varphi\|_{L^\infty}, \|(\partial/\partial t)(K)\|_{L^\infty}\|\varphi\|_{L^\infty}\}$ and C is constant of Proposition 4.3:

$$\begin{aligned} \varepsilon \int_0^T B dt = -\sqrt{\varepsilon} \left\{ \int_0^T \int_\Omega K \nabla \varphi \nabla p_\varepsilon \sqrt{\varepsilon} \frac{\partial p_\varepsilon}{\partial t} + \varphi h V H'_\varepsilon(p_\varepsilon) \left| \frac{\partial p_\varepsilon}{\partial t}(t) \right|^2 \sqrt{\varepsilon} \right. \\ \left. + \int_0^T \int_\Omega \varphi H_\varepsilon(p_\varepsilon) \nabla(hV) \sqrt{\varepsilon} \frac{\partial p_\varepsilon}{\partial t} + a \varphi H_\varepsilon(p_\varepsilon) \frac{\partial h}{\partial t} \frac{\partial p_\varepsilon}{\partial t} \sqrt{\varepsilon} \right\}. \end{aligned} \quad (4.18)$$

We obtain finally

$$\frac{T}{2} \int_\Omega \varphi \left| \varepsilon \frac{\partial p_\varepsilon}{\partial t}(0, x) \right|^2 \leq \int_0^T (\varepsilon A + \varepsilon B) dt \leq C(\varphi)(1 + L_\varepsilon) \sqrt{\varepsilon}. \quad (4.19)$$

□

5. An Existence Theorem of the Problem (2.11)-(2.12)

Theorem 5.1. *There exists at least one solution to the problem (2.11)-(2.12).*

Proof. If $a \in]0, 1]$ and from Proposition 4.3, we can extract a subsequence of p_ε , still denoted by (p_ε) , such that

$$p_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} p \quad \text{in } L^2(0, T; H^1(\Omega)) \text{ weakly,} \quad (5.1)$$

$$\sqrt{\varepsilon} \frac{\partial p_\varepsilon}{\partial t} \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^2(Q) \text{ weakly.} \quad (5.2)$$

Moreover $H_\varepsilon(p_\varepsilon) \in [0, 1]$, for all $\varepsilon > 0$, then there exists $\theta \in L^\infty(Q)$ such that

$$H_\varepsilon(p_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \theta \quad \text{in } L^2(Q) \text{ weakly.} \quad (5.3)$$

We can now proof that the (p, θ) given in (5.1) and (5.3) is solution of the problem (2.11)-(2.12).

We denote by

$$\begin{aligned} \mathbb{k} &= \left\{ \varphi \in L^2(0, T; H^1(\Omega)) : \varphi = 0 \text{ on } \Sigma_{\text{ex}} \text{ and } \varphi \geq 0 \text{ in } Q \right\}, \\ \mathbb{k}' &= \left\{ \varphi \in L^2(Q) : 0 \leq \varphi \leq 1 \right\}, \end{aligned} \quad (5.4)$$

$\mathbb{k} \times \mathbb{k}'$ is convex space of $L^2(0, T; H^1(\Omega)) \times L^2(Q)$, then $\mathbb{k} \times \mathbb{k}'$ is weakly closed.

Therefore $p \in L^2(0, T; H^1(\Omega))$, $p \geq 0$ a.e. in Q and $\theta \in L^2(Q) \cap L^\infty(Q)$ with $\theta \in [0, 1]$ a.e. in Q .

Thus if $\varepsilon \rightarrow 0$ in (4.5), we find the problem (2.11)-(2.12). In order to prove $0 \leq \theta < 1$ and $p(1 - \theta) = 0$, a.e. in Q , we give a brief demonstration (for more details, see [7, pages 1113-1114]).

First, we need to prove $\int_Q p(1 - \theta) = 0$. We notice that $\lim \int_Q p_\varepsilon(1 - H_\varepsilon(p_\varepsilon)) = 0$

$$\text{since } 0 \leq \lim \int_Q p_\varepsilon(1 - H_\varepsilon(p_\varepsilon)) \leq \text{mes } Q \cdot \sup\{\zeta \geq 0 : H_\varepsilon(p_\varepsilon) < 1\}. \quad (5.5)$$

Now we have to prove that

$$\int_Q p(1 - \theta) = \lim \int_Q p_\varepsilon(1 - H_\varepsilon(p_\varepsilon)). \quad (5.6)$$

We define $w_\varepsilon = \varepsilon(\partial p_\varepsilon / \partial t)p_\varepsilon - H_\varepsilon(p_\varepsilon)$ and use (4.13).

Then the couple (p, θ) is solution of the problem (2.11)-(2.12). $\square$

Theorem 5.2. *If $w \geq 0$ on $\Gamma_I \times [0, T]$ and $\theta_0 \geq 0$, then $p \geq 0$ a.e. in $\Omega \times [0, T]$.*

Proof. Following [7], we construct a sequence solution of the problem

$$\begin{aligned} \varepsilon \frac{\partial p_\varepsilon}{\partial t} + p_\varepsilon &= p^- \quad \text{on } [0, T], \\ p_\varepsilon(T) &= 0, \end{aligned} \quad (5.7)$$

where $p^- = \sup(-p, 0)$, $p^+ = \sup(p, 0)$ and $p = p^+ - p^-$.

From the classical Cauchy-Lipschitz-Picard theorem [2], there exists a unique solution $p_\varepsilon \in C^1[0, T]$. Multiplying (5.7) by p_ε (resp., by $\varepsilon(\partial p_\varepsilon / \partial t)$) and integrating over Q , we obtain

$$\|p_\varepsilon\|_{L^2(Q)} \leq \|p^-\|_{L^2(Q)}, \quad (5.8)$$

respectively,

$$\left\| \varepsilon \frac{\partial p_\varepsilon}{\partial t} \right\|_{L^2(Q)} \leq \|p^-\|_{L^2(Q)}. \quad (5.9)$$

Deriving (5.7) with respect to x , multiplying by ∇p_ε , and integrating over Q , we obtain

$$\|\nabla p_\varepsilon\|_{L^2(Q)} \leq \|\nabla p^-\|_{L^2(Q)}. \quad (5.10)$$

We deduce that there exists $\tilde{p} \in L^2(0, T; H^1(\Omega))$ and $\zeta \in L^2(Q)$ such that

$$p_\varepsilon \rightharpoonup p^- \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \quad (5.11)$$

$$\varepsilon \frac{\partial p_\varepsilon}{\partial t} \rightharpoonup \zeta \quad \text{weakly in } L^2(0, T; H^1(\Omega)). \quad (5.12)$$

From (5.11) we have

$$\varepsilon p_\varepsilon \longrightarrow 0 \quad \text{in } L^2(0, T; H^1(\Omega)), \quad \text{therefore } \zeta = 0. \quad (5.13)$$

Passing to the limit in (5.7), we deduce that $\tilde{p} = p^-$ a.e. in Q .

As $E = H_0^1(0, T; V')$ is dense in $L^2(0, T; H^1(\Omega))$, (2.12) can be rewritten in the following form:

$$a \int_0^T \langle v, \frac{\partial(\theta h)}{\partial t} \rangle, \quad \varphi > V + \int_Q (K \nabla \mathbf{p} - V h \theta) \nabla \varphi = \int_{\Sigma_t} w \varphi, \quad \forall \varphi \in V, \text{ a.e. in } [0, T]. \quad (5.14)$$

Taking now p_ε as test function in (5.14) and passing to the limit over ε , we deduce

$$\int_Q K |\nabla p^-|^2 \geq \int_\Sigma w p^- \quad (5.15)$$

as $w \geq 0$, therefore $p^- = 0$ a.e. in Q , that is, $p \geq 0$ a.e. in Q . $\square$

Next work will consist in finding some existence of relationship between the pressure p and parameter N_0 and in completing numerical analysis study to the problem (2.1)–(2.8).

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