Research Article

# Differential Equation and Recursive Formulas of Sheffer Polynomial Sequences 

Heekyung Youn and Yongzhi Yang

Department of Mathematics, University of St. Thomas, 2115 Summit Avenue, Saint Paul, MN 55105-1079, USA

Correspondence should be addressed to Heekyung Youn, hkyoun@stthomas.edu
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#### Abstract

We derive a differential equation and recursive formulas of Sheffer polynomial sequences utilizing matrix algebra. These formulas provide the defining characteristics of, and the means to compute, the Sheffer polynomial sequences. The tools we use are well-known Pascal functional and Wronskian matrices. The properties and the relationship between the two matrices simplify the complexity of the generating functions of Sheffer polynomial sequences. This work extends He and Ricci's work (2002) to a broader class of polynomial sequences, from Appell to Sheffer, using a different method. The work is self-contained.


## 1. Introduction

Sheffer polynomial sequences arise in numerous problems of applied mathematics, theoretical physics, approximation theory, and several other mathematical branches. In the past few decades, there has been a renewed interest in Sheffer polynomials. di Bucchianico recently summarized and documented more than five hundred old and new findings related to the study of Sheffer polynomial sequences in [1]. One aspect of such study is to find a differential equation and recursive formulas for Sheffer polynomial sequences. For instance, in [2], He and Ricci developed the differential equation and recursive formula for Appell polynomials, which is a subclass of Sheffer polynomial sequences. In this paper, we derive differential equation and recursive formulas for Sheffer polynomial sequences by using matrix algebra.

The remainder of the paper is organized as follows. In Section 2, we define Pascal functional and Wronskian matrices for analytic functions and derive their properties and the relationships between the functional matrices. In Section 3, we develop a differential
equation for Sheffer polynomial sequences and present differential equations for some wellknown Sheffer polynomials such as Laguerre, Lower factorial, Exponential, and Hermite polynomials. In Section 4, we discuss three recursive formulas for Sheffer polynomial sequences. An example will illustrate how these three forms of recursive formulas are useful in their own right.

## 2. Preliminaries

Let us start with the definitions of the generalized Pascal functional matrix of an analytic function [3] and the Wronskian matrix of several analytic functions. To avoid any unnecessary confusion, we use $f^{(k)}$ to stand for the $k$ th order derivative of $f$ and use $f^{k}$ to represent the $k$ th power of $f$ in the entire paper. In addition, $f^{(0)}=f$ and $f^{0}=1$.

Definition 2.1. Let $f(t)$ be an analytic function. The generalized Pascal functional matrix of $f(t)$, denoted by $D_{n}[f(t)]$, is an $(n+1)$ by $(n+1)$ matrix and is defined as

$$
\left(D_{n}[f(t)]\right)_{i, j}=\left\{\begin{array}{cl}
\binom{i}{j} f^{(i-j)}(t) & \text { if } i \geq j,  \tag{2.1}\\
0 & \text { otherwise, }
\end{array} \quad \text { for } i, j=0,1,2, \ldots, n\right.
$$

Definition 2.2. The $n$th order Wronskian matrix of $f_{1}(t), f_{2}(t), f_{3}(t), \ldots$, and $f_{m}(t)$ is an $(n+1)$ by $m$ matrix and is defined as

$$
\mathcal{W}_{n}\left[f_{1}(t), f_{2}(t), f_{3}(t), \ldots, f_{m}(t)\right]=\left[\begin{array}{ccccc}
f_{1}(t) & f_{2}(t) & f_{3}(t) & \cdots & f_{m}(t)  \tag{2.2}\\
f_{1}^{\prime}(t) & f_{2}^{\prime}(t) & f_{3}^{\prime}(t) & \cdots & f_{m}^{\prime}(t) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f_{1}^{(n)}(t) & f_{2}^{(n)}(t) & f_{3}^{(n)}(t) & \cdots & f_{m}^{(n)}(t)
\end{array}\right]
$$

We study the Pascal functional and Wronskian matrices in a neighborhood of $t=0$. Hence, when we mention analytic, we mean analytic near $t=0$.

Note 1. Often, readers will encounter expressions such as $P_{n}[f(x, t)]_{t=0}$ or $\mathcal{W}_{n}[f(x, t)]_{t=0}$. In this context, the variable $t$ is the working variable for the Pascal functional or the Wronkian matrix and the variable $x$ is merely a parameter.

In the following, we list some properties and relationships between the Pascal functional and Wronskian matrices that will be the main tool for our work.

Property 1. (a) $P_{n}[\cdot]$ and $\mathcal{W}_{n}[\cdot]$ are linear, that is, for any constants $a$ and $b$, and any analytic functions $f(t)$ and $g(t)$,

$$
\begin{align*}
& \mathcal{D}_{n}[a f(t)+b g(t)]=a \mathcal{D}_{n}[f(t)]+b \mathcal{D}_{n}[g(t)] \\
& \mathcal{W}_{n}[a f(t)+b g(t)]=a \mathcal{W}_{n}[f(t)]+b \mathcal{W}_{n}[g(t)] \tag{2.3}
\end{align*}
$$

(b) For any analytic functions $f(t)$ and $g(t)$,

$$
\begin{equation*}
D_{n}[f(t)] D_{n}[g(t)]=p_{n}[g(t)] D_{n}[f(t)]=D_{n}[f(t) g(t)] \tag{2.4}
\end{equation*}
$$

Furthermore, if $f(t) \neq 0$, then $\left(p_{n}[f(t)]\right)^{-1}=p_{n}\left[f^{-1}(t)\right]$, where $f^{-1}(t)$ denotes the multiplicative inverse of $f(t)$.
(c) For any analytic functions $f(t)$ and $g(t)$,

$$
\begin{equation*}
P_{n}[f(t)] \mathcal{W}_{n}[g(t)]=P_{n}[g(t)] \mathcal{W}_{n}[f(t)]=\mathcal{W}_{n}[(f g)(t)] \tag{2.5}
\end{equation*}
$$

Furthermore, for any analytic functions $g(t)$, and $f_{1}(t), f_{2}(t), \ldots$, and $f_{m}(t)$,

$$
\begin{equation*}
p_{n}[g(t)] \mathcal{W}_{n}\left[f_{1}(t), f_{2}(t), \ldots, f_{m}(t)\right]=\mathcal{W}_{n}\left[\left(g f_{1}\right)(t),\left(g f_{2}\right)(t), \ldots,\left(g f_{m}\right)(t)\right] \tag{2.6}
\end{equation*}
$$

(d) For any analytic functions $g(t)$ and $f(t)$ with $f(0)=0$ and $f^{\prime}(0) \neq 0$,

$$
\begin{equation*}
\mathcal{W}_{n}[g(f(t))]_{t=0}=\mathcal{W}_{n}\left[1, f(t), f^{2}(t), f^{3}(t), \ldots, f^{n}(t)\right]_{t=0} \Lambda_{n}^{-1} \mathcal{W}_{n}[g(t)]_{t=0} \tag{2.7}
\end{equation*}
$$

where $\Lambda_{n}=\operatorname{diag}[0!, 1!, 2!, \ldots, n!]$. The notational convention $\Lambda_{n}$ will be used throughout this paper.
Proof. The proofs of Property 1(a), 1(b), and 1(c) can be found in [4].
For Property 1(d), let us express the functions $f(t)$ and $g(t)$ as series around $t=0$; $f(t)=\sum_{j=0}^{\infty} f_{j} t^{j}$ and $g(t)=\sum_{j=0}^{\infty} g_{j} t^{j}$. Since $f(0)=0$ and $f^{\prime}(0) \neq 0$, the leading term of $f^{k}(t)$ is $t^{k}$. Therefore,

$$
\begin{equation*}
\left.\left(\frac{d}{d t}\right)^{(k)} g(f(t))\right|_{t=0}=\left.\left(\frac{d}{d t}\right)^{(k)} \sum_{j=0}^{\infty} g_{j} f^{j}(t)\right|_{t=0}=\left.\left(\frac{d}{d t}\right)^{(k)} \sum_{j=0}^{k} g_{j} f^{j}(t)\right|_{t=0} \tag{2.8}
\end{equation*}
$$

because $\left.(d / d t)^{(k)} f^{j}(t)\right|_{t=0}=0$ for all $j>k$. By (2.8) and noting $g_{k}=g^{(k)}(0) / k$ !, we have

$$
\begin{align*}
\mathcal{W}_{n}[g(f(t))]_{t=0} & =\left[\begin{array}{ccccc}
1 & 0 & 0 & \ldots & 0 \\
0 & f^{\prime}(0) & 0 & \cdots & 0 \\
0 & f^{\prime \prime}(0) & \left(f^{2}\right)^{\prime \prime}(0) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & f^{(n)}(0) & \left(f^{2}\right)^{(n)}(0) & \cdots & \left(f^{n}\right)^{(n)}(0)
\end{array}\right]\left[\begin{array}{c}
\frac{g^{(0)}(0)}{0!} \\
\frac{g^{(1)}(0)}{1!} \\
\frac{g^{(2)}(0)}{2!} \\
\vdots \\
\frac{g^{(n)}(0)}{n!}
\end{array}\right]  \tag{2.9}\\
& =\mathcal{W}_{n}\left[1, f(t), f^{2}(t), f^{3}(t), \ldots, f^{n}(t)\right]_{t=0} \Lambda_{n}^{-1} \mathcal{W}_{n}[g(t)]_{t=0} .
\end{align*}
$$

This completes the proof.

## 3. Sheffer Polynomial Sequence and Its Differential Equation

Let us first define a Sheffer polynomial sequence by a pair of generation functions $(g(t), f(t))$ as often done [5].

Definition 3.1. Let $g(t)$ be an invertible analytic function, that is, $g(0) \neq 0$, and $f(t)$ be analytic function with $f(0)=0$ and let $f^{\prime}(0) \neq 0$ that admits compositional inverse. Let $\bar{f}(t)$ denote the compositional inverse of $f(t)$. Then, $\left\{s_{n}(x)\right\}$ is the Sheffer polynomial sequence for $(g(t), f(t))$ if and only if

$$
\begin{equation*}
\frac{1}{g(\bar{f}(t))} e^{x \bar{f}(t)}=\sum_{k=0}^{\infty} \frac{s_{k}(x)}{k!} t^{k} \tag{3.1}
\end{equation*}
$$

Note 2. Since $(1 / g(\bar{f}(t))) e^{x \bar{f}(t)}$ is analytic, by Taylor's Theorem,

$$
\begin{equation*}
s_{k}(x)=\left.\left(\frac{d}{d t}\right)^{(k)} \frac{1}{g(\bar{f}(t))} e^{x \bar{f}(t)}\right|_{t=0} \tag{3.2}
\end{equation*}
$$

The family of Sheffer polynomial sequences contains two simpler subclasses of polynomial sequences, Appell and associated polynomial sequences [5]. An Appell polynomial sequence is a Sheffer polynomial sequence where $f(t)=t$. Hence, we say $\left\{a_{n}(x)\right\}$ is the Appell polynomial sequence for $g(t)$ if and only if $\left\{a_{n}(x)\right\}$ is the Sheffer polynomial sequence for $(g(t), t)$. Associated polynomial sequence is a Sheffer polynomial sequence where $g(t)=1$. Hence, we say $\left\{q_{n}(x)\right\}$ is the associated polynomial sequence for $f(t)$ if and only if $\left\{q_{n}(x)\right\}$ is the Sheffer polynomial sequence for $(1, f(t))$.

We define the Sheffer polynomial sequence in vector form to utilize Wronskian matrices.

Definition 3.2. The Sheffer vector for $(g(t), f(t))$, denoted by $\bar{S}_{n}(x)$, is defined as

$$
\bar{S}_{n}(x)=\left[\begin{array}{lllll}
s_{0}(x) & s_{1}(x) & s_{2}(x) & \cdots & s_{n}(x) \tag{3.3}
\end{array}\right]^{T},
$$

where $\left\{s_{n}(x)\right\}$ is the Sheffer polynomial sequence for $(g(t), f(t))$.
As noted in Note 2, the Sheffer vector can be expressed as

$$
\bar{S}_{n}(x)=\left[\begin{array}{lllll}
s_{0}(x) & s_{1}(x) & s_{2}(x) & \cdots & s_{n}(x) \tag{3.4}
\end{array}\right]^{T}=\mathcal{W}_{n}\left[\frac{1}{g(\bar{f}(t))} e^{x \bar{f}(t)}\right]_{t=0}
$$

In order to derive the differential equation for Sheffer polynomial sequence, we develop the following lemma.

Lemma 3.3. Let $\left\{s_{n}(x)\right\}$ be the Sheffer polynomial sequence for $(g(t), f(t))$. Then,

$$
\begin{align*}
& \mathcal{W}_{n}\left[s_{0}(x), s_{1}(x), \ldots, s_{n}(x)\right]^{T} \Lambda_{n}^{-1} \\
& \quad=\mathcal{W}_{n}\left[1, \bar{f}(t), \bar{f}^{2}(t), \ldots, \bar{f}^{n}(t)\right]_{t=0} \Lambda_{n}^{-1} p_{n}\left[\frac{1}{g(t)}\right]_{t=0} p_{n}\left[e^{x t}\right]_{t=0} . \tag{3.5}
\end{align*}
$$

Proof. Using (3.4) and Property 1(d), we have

$$
\begin{equation*}
\bar{S}_{n}(x)=\mathcal{W}_{n}\left[\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right]_{t=0}=\mathcal{W}_{n}\left[1, \bar{f}(t), \bar{f}^{2}(t), \ldots, \bar{f}^{n}(t)\right]_{t=0} \Lambda_{n}^{-1} \mathcal{W}_{n}\left[\frac{e^{x t}}{g(t)}\right]_{t=0} \tag{3.6}
\end{equation*}
$$

Using Property 1 (c) and noting $\mathcal{W}_{n}\left[e^{x t}\right]_{t=0}=\left[\begin{array}{lllll}1 & x & x^{2} & \cdots & x^{n}\end{array}\right]^{T}$, we obtain

$$
\bar{S}_{n}(x)=\mathcal{W}_{n}\left[1, \bar{f}(t), \bar{f}^{2}(t), \ldots, \bar{f}^{n}(t)\right]_{t=0} \Lambda_{n}^{-1} p_{n}\left[\frac{1}{g(t)}\right]_{t=0}\left[\begin{array}{lll}
1 & x & x^{2} \cdots \tag{3.7}
\end{array} x^{n}\right]^{T}
$$

Taking the $k$ th order derivative with respect to $x$ on both sides of (3.7) and dividing by $k$ ! yields

$$
\begin{align*}
\frac{1}{k!}\left[s_{0}^{(k)}(x) s_{1}^{(k)}(x) s_{2}^{(k)}(x) \cdots s_{n}^{(k)}(x)\right]^{T}= & \mathcal{W}_{n}\left[1, \bar{f}(t), \bar{f}^{2}(t), \ldots, \bar{f}^{n}(t)\right]_{t=0} \Lambda_{n}^{-1} p_{n}\left[\frac{1}{g(t)}\right]_{t=0} \\
& \times\left[\begin{array}{llll}
0 \cdots & \cdots & \left.1\binom{k+1}{k} x\binom{k+2}{k} x^{2} \cdots\binom{n}{k} x^{n-k}\right]^{T}
\end{array} .\right. \tag{3.8}
\end{align*}
$$

The left-hand of (3.8) is the $k$ th column of $\mathcal{W}_{n}\left[s_{0}(x), s_{1}(x), \ldots, s_{n}(x)\right]^{T} \Lambda_{n}^{-1}$, and the right-hand of (3.8) is the $k$ th column of $\mathcal{W}_{n}\left[1, \bar{f}(t), \bar{f}^{2}(t), \ldots, \bar{f}^{n}(t)\right]_{t=0} \Lambda_{n}^{-1} p_{n}[1 / g(t)]_{t=0}$ $D_{n}\left[e^{x t}\right]_{t=0}$.

After the introduction of definitions and the lemma, we are ready to develop differential equations for Sheffer polynomials.

Theorem 3.4. Let $\left\{s_{n}(x)\right\}$ be the Sheffer polynomial sequence for $(g(t), f(t))$. Then, it satisfies the following differential equation:

$$
\begin{equation*}
\sum_{k=1}^{n}\left(\beta_{k} x+\alpha_{k}\right) \frac{s_{n}^{(k)}(x)}{k!}-n s_{n}(x)=0 \tag{3.9}
\end{equation*}
$$

where $\beta_{k}=\left.\left(f(t) / f^{\prime}(t)\right)^{(k)}\right|_{t=0}$ and $\alpha_{k}=\left.\left(-g^{\prime}(t) f(t) / g(t) f^{\prime}(t)\right)^{(k)}\right|_{t=0}$.
Proof. Let us consider $\mathcal{W}_{n}\left[t(d / d t)\left(e^{x \bar{f}(t)} / g(\bar{f}(t))\right)\right]_{t=0}$. On one hand, by Property $1(\mathrm{c})$, we have

$$
\begin{align*}
\mathcal{W}_{n}\left[t \frac{d}{d t}\left(\frac{e^{\bar{x}(t)}}{g(\bar{f}(t))}\right)\right]_{t=0} & =p_{n}[t]_{t=0} w_{n}\left[\frac{d}{d t}\left(\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right)\right]_{t=0} \\
& =\left[\begin{array}{cccccccc}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
& & & \ddots & \ddots & \\
0 & 0 & 0 & 0 & \cdots & n-1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & n & 0
\end{array}\right]\left[\begin{array}{c}
s_{1}(x) \\
s_{2}(x) \\
s_{3}(x) \\
\vdots \\
s_{n}(x) \\
s_{n+1}(x)
\end{array}\right] . \tag{3.10}
\end{align*}
$$

On the other hand, by Properties 1(c) and 1(d) and Lemma 3.3, we have

$$
\begin{align*}
\mathcal{W}_{n}\left[t \frac{d}{d t}\left(\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right)\right]_{t=0}= & \mathcal{W}_{n}\left[\left(x-\frac{g^{\prime}(\bar{f}(t))}{g(\bar{f}(t))}\right) \frac{t}{f^{\prime}(\bar{f}(t))} \frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right]_{t=0} \\
= & \mathcal{W}_{n}\left[1, \bar{f}(t), \ldots, \bar{f}^{n}(t)\right]_{t=0} \Lambda_{n}^{-1} \mathcal{W}_{n}\left[\left(x-\frac{g^{\prime}(t)}{g(t)}\right) \frac{f(t)}{f^{\prime}(t)} \frac{e^{x t}}{g(t)}\right]_{t=0} \\
= & \mathcal{W}_{n}\left[1, \bar{f}(t), \ldots, \bar{f}^{n}(t)\right]_{t=0} \Lambda_{n}^{-1} p_{n}\left[\frac{1}{g(t)}\right]_{t=0} p_{n}\left[e^{x t}\right]_{t=0} \\
& \times \mathcal{W}_{n}\left[x \frac{f(t)}{f^{\prime}(t)}-\frac{g^{\prime}(t) f(t)}{g(t) f^{\prime}(t)}\right]_{t=0} \\
= & \mathcal{W}_{n}\left[s_{0}(x), s_{1}(x), \ldots, s_{n}(x)\right]^{T} \Lambda_{n}^{-1} \mathcal{W}_{n}\left[x \frac{f(t)}{f^{\prime}(t)}-\frac{g^{\prime}(t) f(t)}{g(t) f^{\prime}(t)}\right]_{t=0} . \tag{3.11}
\end{align*}
$$

Thus,

$$
\mathcal{W}_{n}\left[t \frac{d}{d t}\left(\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right)\right]_{t=0}=\left[\begin{array}{ccccc}
s_{0}(x) & 0 & 0 & \cdots & 0  \tag{3.12}\\
s_{1}(x) & \frac{s_{1}^{\prime}(x)}{1!} & 0 & \cdots & 0 \\
s_{2}(x) & \frac{s_{2}^{\prime}(x)}{1!} & \frac{s_{2}^{\prime \prime}(x)}{2!} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s_{n}(x) & \frac{s_{n}^{\prime}(x)}{1!} & \frac{s_{n}^{\prime \prime}(x)}{2!} & \cdots & \frac{s_{n}^{(n)}(x)}{n!}
\end{array}\right]\left[\begin{array}{c}
\beta_{0} x+\alpha_{0} \\
\beta_{1} x+\alpha_{1} \\
\beta_{2} x+\alpha_{2} \\
\vdots \\
\beta_{n} x+\alpha_{n}
\end{array}\right] .
$$

Equating the last rows of (3.10) and (3.12), we get

$$
\begin{equation*}
n s_{n}(x)=\sum_{k=0}^{n} \frac{s_{n}^{(k)}(x)}{k!}\left(\beta_{k} x+\alpha_{k}\right) \tag{3.13}
\end{equation*}
$$

Since $f(0)=0, \alpha_{0}=\beta_{0}=0$, a rearrangement of (3.13) produces the desired result.
The following corollaries are immediate consequences of Theorem 3.4. When $g(t)=1$, $\alpha_{k}=0$ for all $k>0$, and we have a differential equation for associated polynomials.

Corollary 3.5. Let $\left\{q_{n}(x)\right\}$ be the associated polynomial sequence for $f(t)$. Then, it satisfies the following differential equation:

$$
\begin{equation*}
\sum_{k=1}^{n} \beta_{k} x \frac{q_{n}^{(k)}(x)}{k!}-n q_{n}(x)=0 \tag{3.14}
\end{equation*}
$$

where $\beta_{k}=\left.\left(f(t) / f^{\prime}(t)\right)^{(k)}\right|_{t=0}$.
Setting $f(t)=t$ in Theorem 3.4, we get a differential equation for Appell polynomials.
Corollary 3.6. Let $\left\{a_{n}(x)\right\}$ be the Appell polynomial sequence for $g(t)$. Then, it satisfies the following differential equation:

$$
\begin{equation*}
\sum_{k=2}^{n} \frac{\bar{\alpha}_{k-1}}{(k-1)!} a_{n}^{(k)}(x)+\left(x+\bar{\alpha}_{0}\right) a_{n}^{\prime}(x)-n a_{n}(x)=0 \tag{3.15}
\end{equation*}
$$

where $\bar{\alpha}_{k}=\left.\left(-g^{\prime}(t) / g(t)\right)^{(k)}\right|_{t=0}$.
Proof. Since $f(t)=t, \beta_{1}=1$ and $\beta_{k}=0$ for all $k \neq 1$. Furthermore,

$$
\begin{equation*}
\alpha_{k}=\left.\left(\frac{-\operatorname{tg}^{\prime}(t)}{g(t)}\right)^{(k)}\right|_{t=0}=\left.k\left(\frac{-g^{\prime}(t)}{g(t)}\right)^{(k-1)}\right|_{t=0}=k \bar{\alpha}_{k-1} \tag{3.16}
\end{equation*}
$$

Hence, the above differential equation follows from Theorem 3.4.

Remark 3.7. The above differential equation for Appell is equivalent to the one in Theorem 2.1 in [2].

Let us apply Theorem 3.4 to derive differential equations of some well-known Sheffer polynomials. For the sake of brevity of context, we leave the detailed calculations of examples in the paper to interested readers.

Example 3.8. Let $L_{n}^{\{a\}}(x)$ denote the Laguerre polynomial of order $a$, which is the Sheffer polynomial for $(g(t), f(t))=\left((1-t)^{-a-1}, t /(t-1)\right) . L_{n}^{\{a\}}(x)=\sum_{k=0}^{n}\binom{n+a}{n-k}(n!/ k!)(-x)^{k}$ satisfies the following differential equation:

$$
\begin{equation*}
x\left(L_{n}^{\{a\}}(x)\right)^{\prime \prime}-(x-a-1)\left(L_{n}^{\{a\}}(x)\right)^{\prime}+n L_{n}^{\{a\}}(x)=0 \tag{3.17}
\end{equation*}
$$

In particular for $a=0, L_{n}^{\{0\}}(x)=L_{n}(x)$, generally known as the Laguerre polynomial, satisfies

$$
\begin{equation*}
x L_{n}^{\prime \prime}(x)-(x-1) L_{n}^{\prime}(x)+n L_{n}(x)=0 \tag{3.18}
\end{equation*}
$$

Example 3.9. Let $p_{n}(x)$ denote the Poisson-Charlier polynomial of order $a$, which is the Sheffer polynomial for $(g(t), f(t))=\left(e^{a\left(e^{t}-1\right)}, a\left(e^{t}-1\right)\right) \cdot p_{n}(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} a^{-k}(x)_{k}$, where $(x)_{k}=$ $x(x-1)(x-2) \cdots(x-k+1)$ is lower factorial polynomial, satisfies the following differential equation:

$$
\begin{equation*}
\sum_{k=1}^{n}\left((-1)^{k+1} x-a\right) \frac{p_{n}^{(k)}(x)}{k!}-n p_{n}(x)=0 \tag{3.19}
\end{equation*}
$$

Example 3.10. The Actuarial polynomial, denoted by $A_{n}(x)$, is the Sheffer polynomial for $(g(t), f(t))=\left((1-t)^{-b}, \ln (1-t)\right) . A_{n}(x)=\sum_{k=0}^{n}\binom{b}{k}\left(\sum_{j=k}^{n} S(n, j)(j)_{k}(-x)^{j-k}\right)$, where $S(n, k)$ is the Stirling number of the second kind and $(j)_{k}$ is the lower factorial polynomial. Then,

$$
\begin{equation*}
\sum_{k=2}^{n}\left(\frac{x}{k(k-1)}+\frac{b}{k}\right) A_{n}^{(k)}(x)-(x-b) A_{n}^{\prime}(x)+n A_{n}(x)=0, \quad \text { for } n \geq 2 \tag{3.20}
\end{equation*}
$$

Example 3.11. The Lower factorial polynomial is the associated polynomial sequence for $f(t)=e^{a t}-1$. Lower factorial polynomial $w_{n}(x)=(x / a)_{n}=(x / a)(x / a-1) \cdots(x / a-n+1)$ satisfies the following differential equation:

$$
\begin{equation*}
x \sum_{k=1}^{n} \frac{(-a)^{k-1}}{k!} w_{n}^{(k)}(x)-n w_{n}(x)=0 \tag{3.21}
\end{equation*}
$$

Example 3.12. The exponential polynomial is the associated polynomial sequence for $f(t)=$ $\ln (1+t)$. Let $\phi_{n}(x)$ denote the exponential polynomial and $\phi_{n}(x)=\sum_{k=0}^{n} S(n, k) x^{k}$, where $S(n, k)$ is the Stirling number of the second kind. Then,

$$
\begin{equation*}
x \sum_{k=2}^{n} \frac{(-1)^{k} \phi_{n}^{(k)}(x)}{k(k-1)}+x \phi_{n}^{\prime}(x)-n \phi_{n}(x)=0, \quad \text { for } n \geq 2 \tag{3.22}
\end{equation*}
$$

Example 3.13. The Hermite polynomial of order $v$ is the Appell polynomial sequence for $g(t)=e^{v t^{2} / 2}$. Let us denote the Hermite polynomial of order $v$ as $H_{n}^{\{v\}}(x)$ and $H_{n}^{\{v\}}(x)=$ $\sum_{k=0}^{\lfloor n / 2\rfloor}(-v / 2)^{k}\left((n)_{2 k} / k!\right) x^{n-2 k}$. Then

$$
\begin{equation*}
v\left(H_{n}^{\{v\}}(x)\right)^{\prime \prime}-x\left(H_{n}^{\{v\}}(x)\right)^{\prime}+n H_{n}^{\{v\}}(x)=0 \tag{3.23}
\end{equation*}
$$

## 4. Recurrence Relations for the Sheffer Polynomials

Finding recursive formulas is one of main interests on study of the Sheffer polynomial sequences. For instance, Lehmer in [6] developed six recursive relations for the Bernoulli polynomial sequence (one of Appell polynomial sequences). In this section, we derive three recursive formulas. The first formula expresses $s_{n+1}(x)$ in terms of $s_{n}(x)$ and its derivatives, and the second and third formulas express $s_{n+1}(x)$ in terms of $s_{k}(x)$ for $k=0,1, \ldots, n$.

### 4.1. The First Recursive Formula for the Sheffer Polynomials

Theorem 4.1 (Recursive Formula I). Let $\left\{s_{n}(x)\right\}$ denote the Sheffer polynomial sequence for $(g(t), f(t))$. Then, $s_{0}(x)=1 / g(0)$ and

$$
\begin{equation*}
s_{n+1}(x)=\sum_{k=0}^{n}\left(\gamma_{k} x+\delta_{k}\right) \frac{s_{n}^{(k)}(x)}{k!}, \quad \text { for } n \geq 0 \tag{4.1}
\end{equation*}
$$

where $\gamma_{k}=\left.\left(1 / f^{\prime}(t)\right)^{(k)}\right|_{t=0}$ and $\delta_{k}=\left.\left(-g^{\prime}(t) / g(t) f^{\prime}(t)\right)^{(k)}\right|_{t=0}$.
Proof. The proof is similar to the proof of Theorem 3.4. Let us consider $\mathcal{W}_{n}\left[(d / d t)\left(e^{x \bar{f}(t)} / g(\bar{f}(t))\right)\right]_{t=0}$.
On one hand,

$$
\mathcal{W}_{n}\left[\frac{d}{d t}\left(\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right)\right]_{t=0}=\left[\begin{array}{lllll}
s_{1}(x) & s_{2}(x) & s_{3}(x) & \cdots & s_{n+1}(x) \tag{4.2}
\end{array}\right]^{T}
$$

On the other hand, by Properties $1(\mathrm{c})$ and $1(\mathrm{~d})$ and Lemma 3.3, we have

$$
\begin{align*}
\mathcal{W}_{n}\left[\frac{d}{d t}\left(\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right)\right]_{t=0}= & \mathcal{W}_{n}\left[\left(x-\frac{g^{\prime}(\bar{f}(t))}{g(\bar{f}(t))}\right) \frac{1}{f^{\prime}(\bar{f}(t))} \frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right]_{t=0} \\
= & \mathcal{W}_{n}\left[1, \bar{f}(t), \ldots, \bar{f}^{n}(t)\right]_{t=0} \Lambda_{n}^{-1} p_{n}\left[\frac{1}{g(t)}\right]_{t=0} p_{n}\left[e^{x t}\right]_{t=0}  \tag{4.3}\\
& \times \mathcal{W}_{n}\left[x \frac{1}{f^{\prime}(t)}-\frac{g^{\prime}(t)}{g(t) f^{\prime}(t)}\right]_{t=0} \\
= & \mathcal{W}_{n}\left[s_{0}(x), s_{1}(x), \ldots, s_{n}(x)\right]^{T} \Lambda_{n}^{-1} \mathcal{W}_{n}\left[x \frac{1}{f^{\prime}(t)}-\frac{g^{\prime}(t)}{g(t) f^{\prime}(t)}\right]_{t=0}
\end{align*}
$$

Thus,

$$
\mathcal{W}_{n}\left[\frac{d}{d t}\left(\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right)\right]_{t=0}=\left[\begin{array}{ccccc}
s_{0}(x) & 0 & 0 & \cdots & 0  \tag{4.4}\\
s_{1}(x) & \frac{s_{1}^{\prime}(x)}{1!} & 0 & \cdots & 0 \\
s_{2}(x) & \frac{s_{2}^{\prime}(x)}{1!} & \frac{s_{2}^{\prime \prime}(x)}{2!} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s_{n}(x) & \frac{s_{n}^{\prime}(x)}{1!} & \frac{s_{n}^{\prime \prime}(x)}{2!} & \cdots & \frac{s_{n}^{(n)}(x)}{n!}
\end{array}\right]\left[\begin{array}{c}
r_{0} x+\delta_{0} \\
r_{1} x+\delta_{1} \\
r_{2} x+\delta_{2} \\
\vdots \\
r_{n} x+\delta_{n}
\end{array}\right] .
$$

Equating the last rows of (4.2) and (4.4), we get the desired result.
Example 4.2. For $A_{n}(x)=\sum_{k=0}^{n}\binom{b}{k}\left(\sum_{j=k}^{n} S(n, j)(j)_{k}(-x)^{j-k}\right)$, the Actuarial polynomial,

$$
\begin{equation*}
A_{0}(x)=1, \quad A_{n+1}(x)=(-x+b) A_{n}(x)+x A_{n}^{\prime}(x), \quad \text { for } n \geq 0 \tag{4.5}
\end{equation*}
$$

Example 4.3. Let $L_{n}(x)$ denote the Laguerre polynomial. Then,

$$
\begin{equation*}
L_{0}(x)=1, \quad L_{n+1}(x)=(-x+1) L_{n}(x)+(2 x-1) L_{n}^{\prime}(x)-x L_{n}^{\prime \prime}(x), \quad \text { for } n \geq 0 \tag{4.6}
\end{equation*}
$$

Example 4.4. Hermite polynomial is defined as the Sheffer polynomial sequence for $(g(t), f(t))=\left(e^{t^{2} / 4}, t / 2\right)$ Let us denote the polynomial as $H_{n}(x)$. Then,

$$
\begin{equation*}
H_{0}(x)=1, \quad H_{n+1}(x)=2 x H_{n}(x)-H_{n}^{\prime}(x), \quad \text { for } n \geq 0 \tag{4.7}
\end{equation*}
$$

Corollary 4.5 (Recursive Formula I for associated polynomial sequences). Let $\left\{q_{n}(x)\right\}$ denote the associated polynomial sequence for $f(t)$. Then, $q_{0}(x)=1$ and

$$
\begin{equation*}
q_{n+1}(x)=x \sum_{k=0}^{n} \frac{\gamma_{k}}{k!} q_{n}^{(k)}(x), \quad \text { for } n \geq 0 \tag{4.8}
\end{equation*}
$$

where $\gamma_{k}=\left.\left(1 / f^{\prime}(t)\right)^{(k)}\right|_{t=0}$.
Proof. It follows from Theorem 4.1 since $g(t)=1$ and hence $\delta_{k}=0$ for all $k$.
Example 4.6. For the exponential polynomial $\phi_{n}(x)$,

$$
\begin{equation*}
\phi_{0}(x)=1, \quad \phi_{n+1}(x)=x\left[\phi_{n}(x)+\phi_{n}^{\prime}(x)\right], \quad \text { for } n \geq 0 \tag{4.9}
\end{equation*}
$$

Corollary 4.7 (Recursive Formula I for Appell polynomial sequences). Let $\left\{a_{n}(x)\right\}$ denote the Appell polynomial sequence for $g(t)$. Then, $a_{0}(x)=1 / g(0)$ and

$$
\begin{equation*}
a_{n+1}(x)=x a_{n}(x)+\sum_{k=0}^{n} \frac{\delta_{k}}{k!} a_{n}^{(k)}(x), \quad \text { for } n \geq 0 \tag{4.10}
\end{equation*}
$$

where $\delta_{k}=\left.\left(-g^{\prime}(t) / g(t)\right)^{(k)}\right|_{t=0}$.
Proof. Since $f(t)=t, \delta_{k}=\left.\left(-g^{\prime}(t) / g(t) f^{\prime}(t)\right)^{(k)}\right|_{t=0}=\left.\left(-g^{\prime}(t) / g(t)\right)^{(k)}\right|_{t=0}, \quad \gamma_{0}=1$, and $\gamma_{k}=0$ for all other $k$ in Theorem 4.1.

Example 4.8. For the Hermite polynomial of order $v$, denoted by $H_{n}^{\{v\}}(x)$,

$$
\begin{equation*}
H_{0}^{\{v\}}(x)=1, \quad H_{n+1}^{\{v\}}(x)=x H_{n}^{\{v\}}(x)-v\left(H_{n}^{\{v\}}\right)^{\prime}(x), \quad \text { for } n \geq 0 \tag{4.11}
\end{equation*}
$$

Example 4.9. Let us consider the Stirling polynomial sequence $S_{n}(x)$, which is the Sheffer polynomial sequence for $(g(t), \bar{f}(t))=\left(e^{-t}, \ln \left(t / 1-e^{-t}\right)\right)$. To obtain the recursive formula for $S_{n}(x)$ by Theorem 4.1, we have to find $f(t)$, that is, solve the insolvable transcendental equation

$$
\begin{equation*}
e^{t}=\frac{y}{1-e^{-y}} \tag{4.12}
\end{equation*}
$$

for $y$.
This example shows a type of Sheffer sequences for which Theorem 4.1 fails to produce a recursive formula. This motivate us to develop other recursive formulas, which represent $s_{n+1}(x)$ in term of its previous terms $s_{k}(x)$ and the derivatives of $\bar{f}(t)$ and $g(t)$.

### 4.2. The Second Recursive Formula for the Sheffer Polynomials

Theorem 4.10 (Recursive Formula II). Let $\left\{s_{n}(x)\right\}$ be the Sheffer polynomial sequence for $(g(t), f(t))$. Then, $s_{0}(x)=1 / g(0)$ and

$$
\begin{equation*}
\epsilon_{0} s_{n+1}(x)=x s_{n}(x)+\sum_{k=0}^{n}\binom{n}{k} \theta_{k} s_{n-k}(x)-\sum_{k=1}^{n}\binom{n}{k} \epsilon_{k} s_{n+1-k}(x), \quad \text { for } n \geq 0, \tag{4.13}
\end{equation*}
$$

where $\epsilon_{k}=\left.\left(f^{\prime}(\bar{f}(t))\right)^{(k)}\right|_{t=0}=\left.\left(1 / \bar{f}^{\prime}(t)\right)^{(k)}\right|_{t=0}$ and $\theta_{k}=\left.\left(-g^{\prime}(\bar{f}(t)) / g(\bar{f}(t))\right)^{(k)}\right|_{t=0}$.
Proof. Let us consider $\mathcal{W}_{n}\left[f^{\prime}(\bar{f}(t))(d / d t)\left(e^{x \bar{f}(t)} / g(\bar{f}(t))\right)\right]_{t=0}$. On one hand, applying Property 1(c),

$$
\begin{align*}
\mathcal{W}_{n}\left[f^{\prime}(\bar{f}(t)) \frac{d}{d t}\left(\frac{e^{\bar{x}(t)}}{g(\bar{f}(t))}\right)\right]_{t=0}= & p_{n}\left[\frac{d}{d t}\left(\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right)\right]_{t=0} \mathcal{W}_{n}\left[f^{\prime}(\bar{f}(t))\right]_{t=0} \\
& =\left[\begin{array}{ccccc}
s_{1}(x) & 0 & 0 & \cdots & 0 \\
s_{2}(x) & s_{1}(x) & 0 & \cdots & 0 \\
s_{3}(x) & \binom{2}{1} s_{2}(x) & s_{1}(x) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s_{n+1}(x) & \binom{n}{1} s_{n}(x) & \binom{n}{2} s_{n-1}(x) & \cdots & s_{1}(x)
\end{array}\right]\left[\begin{array}{c}
\epsilon_{0} \\
\epsilon_{1} \\
\epsilon_{2} \\
\vdots \\
\epsilon_{n}
\end{array}\right] . \tag{4.14}
\end{align*}
$$

On the other hand, by Properties 1(a) and 1(c),

$$
\begin{align*}
& \mathcal{W}_{n}\left[f^{\prime}(\bar{f}(t)) \frac{d}{d t}\left(\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right)\right]_{t=0} \\
& =\mathcal{W}_{n}\left[\left(x-\frac{g^{\prime}(\bar{f}(t))}{g(\bar{f}(t))}\right) \frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right]_{t=0} \\
& =x \mathcal{W}_{n}\left[\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right]_{t=0}-p_{n}\left[\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right]_{t=0} \mathcal{W}_{n}\left[\frac{g^{\prime}(\bar{f}(t))}{g(\bar{f}(t))}\right]_{t=0} \\
& =x\left[\begin{array}{c}
s_{0}(x) \\
s_{1}(x) \\
s_{2}(x) \\
\vdots \\
s_{n}(x)
\end{array}\right]+\left[\begin{array}{ccccc}
s_{0}(x) & 0 & 0 & \cdots & 0 \\
s_{1}(x) & s_{0}(x) & 0 & \cdots & 0 \\
s_{2}(x) & \binom{2}{1} s_{1}(x) & s_{0}(x) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s_{n}(x) & \binom{n}{1} s_{n-1}(x) & \binom{n}{2} s_{n-2}(x) & \cdots & s_{0}(x)
\end{array}\right]\left[\begin{array}{c}
\theta_{0} \\
\theta_{1} \\
\theta_{2} \\
\vdots \\
\theta_{n}
\end{array}\right] . \tag{4.15}
\end{align*}
$$

Equating the last rows of (4.14) and (4.15) leads to

$$
\begin{equation*}
\sum_{k=0}^{n}\binom{n}{k} s_{n+1-k}(x) \epsilon_{k}=x s_{n}(x)+\sum_{k=0}^{n}\binom{n}{k} s_{n-k}(x) \theta_{k} \tag{4.16}
\end{equation*}
$$

A rearrangement of the above yields the desired result.

Example 4.11. Let $L_{n}^{\{a\}}(x)$ be the Laguerre polynomial of order $a, p_{n}(x)$ the Poisson-Charlier polynomial of order $a$, and $M_{n}(x)$ the Meixner polynomial of the first kind of order $(b, c)$, which is the Sheffer polynomial sequence for $(g(t), f(t))=\left(\left((1-c) /\left(1-c e^{t}\right)\right)^{b},\left(1-e^{t}\right) /\left(c^{-1}-\right.\right.$ $\left.e^{t}\right)$ ). Then,

$$
\begin{align*}
L_{0}^{\{a\}}(x)=1, \quad L_{n+1}^{\{a\}}(x)= & (2 n+a+1-x) L_{n}^{\{a\}}(x)-\left(n^{2}+n a\right) L_{n-1}^{\{a\}}(x), \quad \text { for } n \geq 0, \\
p_{0}(x)=1, \quad a p_{n+1}(x)= & (x-n-a) p_{n}(x)-n p_{n-1}(x), \quad \text { for } n \geq 0,  \tag{4.17}\\
M_{0}(x)=1, \quad c M_{n+1}(x)= & ((c-1) x+n(c+1)+b c) M_{n}(x) \\
& -(n(n-1)+n b) M_{n-1}(x), \quad \text { for } n \geq 0 .
\end{align*}
$$

Corollary 4.12 (Recursive Formula II for associated polynomial sequences). Let $\left\{q_{n}(x)\right\}$ denote the associated polynomial sequence for $f(t)$. Then, $q_{0}(x)=1$ and

$$
\begin{equation*}
\epsilon_{0} q_{n+1}(x)=x q_{n}(x)-\sum_{k=1}^{n}\binom{n}{k} \epsilon_{k} q_{n+1-k}(x), \quad \text { for } n \geq 0 \tag{4.18}
\end{equation*}
$$

where $\epsilon_{k}=\left.\left(f^{\prime}(\bar{f}(t))\right)^{(k)}\right|_{t=0}$.
Example 4.13. Let $\phi_{n}(x)$ denote the exponential polynomial. Then,

$$
\begin{equation*}
\phi_{0}(x)=1, \quad \phi_{n+1}(x)=x \phi_{n}(x)+\sum_{k=0}^{n-1}\binom{n}{k+1}(-1)^{k} \phi_{n-k}(x), \quad \text { for } n \geq 0 \tag{4.19}
\end{equation*}
$$

Corollary 4.14 (Recursive Formula II for Appell polynomial sequences). Let $\left\{a_{n}(x)\right\}$ denote the Appell polynomial sequence for $g(t)$. Then, $a_{0}(x)=1 / g(0)$ and

$$
\begin{equation*}
a_{n+1}(x)=x a_{n}(x)+\sum_{k=0}^{n}\binom{n}{k} \theta_{k} a_{n-k}(x), \quad \text { for } n \geq 0 \tag{4.20}
\end{equation*}
$$

where $\theta_{k}=\left.\left(-g^{\prime}(t) / g(t)\right)^{(k)}\right|_{t=0}$.
Remark 4.15. Theorem 2.2 and Theorem 2.4 in [2] are special cases of Corollary 4.14.
Example 4.16. Let $H_{n}^{\{v\}}(x)$ denote the Hermite polynomial of order $v$. Then,

$$
\begin{equation*}
H_{0}^{\{v\}}(x)=1, \quad H_{n+1}^{\{v\}}(x)=x H_{n}^{\{v\}}(x)-n v H_{n-1}^{\{v\}}(x), \quad \text { for } n \geq 0 \tag{4.21}
\end{equation*}
$$

Here, we would like to revisit Example 4.9. In order to obtain the recursive formula for the Stirling polynomial sequence $\left\{S_{n}(x)\right\}$ by Theorem 4.10, we need to compute

$$
\begin{equation*}
\epsilon_{k}=\left.\left(f^{\prime}(\bar{f}(t))\right)^{(k)}\right|_{t=0}=\left.\left(\frac{1}{\bar{f}^{\prime}(t)}\right)^{(k)}\right|_{t=0}=\left.\left(\frac{t\left(e^{t}-1\right)}{e^{t}-1-t}\right)^{(k)}\right|_{t=0} \tag{4.22}
\end{equation*}
$$

Equation (4.22) seems difficult to evaluate and does not lead to any nice result. This is a motivation for us to develop yet another formula in Theorem 4.17.

### 4.3. The Third Recursive Formula for the Sheffer Polynomials

Theorem 4.17 (Recursive Formula III). Let $\left\{s_{n}(x)\right\}$ denote the Sheffer polynomial sequence for $(g(t), f(t))$. Then, $s_{0}(x)=1 / g(0)$ and

$$
\begin{equation*}
s_{n+1}(x)=\sum_{k=0}^{n}\binom{n}{k}\left(x \alpha_{k}+\beta_{k}\right) s_{n-k}(x), \quad \text { for } n \geq 0 \tag{4.23}
\end{equation*}
$$

where $\alpha_{k}=\left.\left(1 / f^{\prime}(\bar{f}(t))\right)^{(k)}\right|_{t=0}=\left.(\bar{f}(t))^{(k+1)}\right|_{t=0}$ and $\beta_{k}=\left.\left(-g^{\prime}(\bar{f}(t)) / g(\bar{f}(t)) f^{\prime}(\bar{f}(t))\right)^{(k)}\right|_{t=0}$.
Proof. We have

$$
\left[\begin{array}{lllll}
s_{1}(x) & s_{2}(x) & s_{3}(x) & \cdots & s_{n+1}(x) \tag{4.24}
\end{array}\right]^{T}=\mathcal{W}_{n}\left[\frac{d}{d t}\left(\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right)\right]_{t=0}
$$

Also,

$$
\begin{aligned}
& \mathcal{W}_{n}\left[\frac{d}{d t}\left(\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right)\right]_{t=0} \\
& \quad=\mathcal{W}_{n}\left[\left(x-\frac{g^{\prime}(\bar{f}(t))}{g(\bar{f}(t))}\right) \frac{1}{f^{\prime}(\bar{f}(t))} \frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right]_{t=0} \\
& \quad=P_{n}\left[\left(x-\frac{g^{\prime}(\bar{f}(t))}{g(\bar{f}(t))}\right) \frac{1}{f^{\prime}(\bar{f}(t))}\right]_{t=0} \mathcal{W}_{n}\left[\frac{e^{x \bar{f}(t)}}{g(\bar{f}(t))}\right]_{t=0}
\end{aligned}
$$

$$
=\left[\begin{array}{ccccc}
x \alpha_{0}+\beta_{0} & 0 & 0 & \cdots & 0  \tag{4.25}\\
x \alpha_{1}+\beta_{1} & x \alpha_{0}+\beta_{0} & 0 & \cdots & 0 \\
x \alpha_{2}+\beta_{2} & 2\left(x \alpha_{1}+\beta_{1}\right) & x \alpha_{0}+\beta_{0} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x \alpha_{n}+\beta_{n}\binom{n}{1}\left(x \alpha_{n-1}+\beta_{n-1}\right) & \binom{n}{2}\left(x \alpha_{n-2}+\beta_{n-2}\right) & \cdots & x \alpha_{0}+\beta_{0}
\end{array}\right]\left[\begin{array}{c}
s_{0}(x) \\
s_{1}(x) \\
s_{2}(x) \\
\vdots \\
s_{n}(x)
\end{array}\right] .
$$

Equating the last rows of (4.24) and (4.25), we get the desired result.
Example 4.18. Let $A_{n}(x)$ be the Actuarial polynomial. Then,

$$
\begin{equation*}
A_{0}(x)=1, \quad A_{n+1}(x)=-x \sum_{k=0}^{n}\binom{n}{k} A_{k}(x)+b A_{n}(x), \quad \text { for } n \geq 0 \tag{4.26}
\end{equation*}
$$

Corollary 4.19 (Recursive Formula III for associated polynomial sequences). Let $\left\{q_{n}(x)\right\}$ denote the associated polynomial sequence for $f(t)$. Then, $q_{0}(x)=1$ and

$$
\begin{equation*}
q_{n+1}(x)=x \sum_{k=0}^{n}\binom{n}{k} \alpha_{k} q_{n-k}(x), \quad \text { for } n \geq 0 \tag{4.27}
\end{equation*}
$$

where $\alpha_{k}=\left.\left(1 / f^{\prime}(\bar{f}(t))\right)^{(k)}\right|_{t=0}=\left.(\bar{f}(t))^{(k+1)}\right|_{t=0}$.
Example 4.20. Let $\phi_{n}(x)$ denote the exponential polynomial. By Corollary 4.19, we have the following well-known result

$$
\begin{equation*}
\phi_{0}(x)=1, \quad \phi_{n+1}(x)=x \sum_{k=0}^{n}\binom{n}{k} \phi_{k}(x), \quad \text { for } n \geq 0 \tag{4.28}
\end{equation*}
$$

Finally, let us finish up Example 4.9 and conclude this paper.
To obtain the recursive formula for the Stirling polynomial sequence $\left\{S_{n}(x)\right\}$ by Theorem 4.17, we need to compute

$$
\begin{gather*}
\alpha_{k}=\left.\left(\frac{1}{f^{\prime}(\bar{f}(t))}\right)^{(k)}\right|_{t=0}=\left.(\bar{f}(t))^{(k+1)}\right|_{t=0}=\left.\left(\frac{1}{t}-\frac{1}{e^{t}-1}\right)^{(k)}\right|_{t=0}  \tag{4.29}\\
\beta_{k}=\left.\left(\frac{-g^{\prime}(\bar{f}(t))}{g(\bar{f}(t)) f^{\prime}(\bar{f}(t))}\right)^{(k)}\right|_{t=0}=\left.\left(\frac{1}{t}-\frac{1}{e^{t}-1}\right)^{(k)}\right|_{t=0} .
\end{gather*}
$$

Noting $t /\left(e^{t}-1\right)$ is the exponential generating function for Bernoulli number sequence $\left\{B_{\mathrm{n}}\right\}$ in [7], we can easily compute (4.29) to get

$$
\begin{align*}
\alpha_{k}=\beta_{k} & =\left.\left.\left(\frac{1}{t}-\frac{1}{e^{t}-1}\right)^{(k)}\right|_{t=0}\left(\frac{1}{t}\left[1-\frac{t}{e^{t}-1}\right]\right)^{(k)}\right|_{t=0} \\
& =\left.\left(-\sum_{j=0}^{\infty} \frac{B_{j+1}}{(j+1)!} t^{j}\right)^{(k)}\right|_{t=0}=\frac{-B_{k+1}}{k+1} \tag{4.30}
\end{align*}
$$

Therefore, the recursive formula for the Stirling polynomial $S_{n}(x)$ is $S_{0}(x)=1$ and

$$
\begin{equation*}
S_{n+1}(x)=-\sum_{k=0}^{n}\binom{n}{k}(x+1) \frac{B_{k+1}}{k+1} S_{n-k}(x)=-\frac{(x+1)}{n+1} \sum_{k=0}^{n}\binom{n+1}{k+1} B_{k+1} S_{n-k}(x) . \tag{4.31}
\end{equation*}
$$

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