International Scholarly Research Network
ISRN Geometry
Volume 2011, Article ID 479262, 16 pages
doi:10.5402/2011/479262

Research Article

Circle Numbers for Star Discs

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Received 20 March 2011; Accepted 8 April 2011

Academic Editors: A. Belhaj, A. Borowiec, and A. Fino

The notion of a generalized circle number which has recently been discussed for $l_{2,p}$-circles and ellipses will be extended here for star bodies and a class of unbounded star discs.

1. Introduction

Generalized circle numbers have been discussed for $l_{2,p}$-circles in Richter [1, 2] and for ellipses in Richter [3]. All these numbers correspond on the one hand to an area content property of the considered discs which is based upon the usual, that is, Euclidean, area content measure and a suitably adopted radius variable. On the other hand, they reflect a circumference property of the considered generalized circles with respect to a non-Euclidean length measure which is generated by a suitably chosen non-Euclidean disc. Several basic and specific properties of the circumference measure have been discussed in Richter [1–3]. We refer here to only two of them which are closely connected with each other by the main theorem of calculus. The first one is that the generalized circumference of the generalized circle coincides with the derivative of the area content function with respect to the adopted radius variable. The second one is that, vice versa, the area content of the circle disc equals the integral of the generalized circumferences of the circles within the disc with respect to the adopted radius variable. Integrating this way may be considered as a generalization of Cavalieri and Torricelli's method of indivisibles, where the indivisibles are now the generalized circles within the given disc and measuring them is based upon a non-Euclidean geometry. The far-reaching usefulness of this generalized method of indivisibles has been demonstrated in the work of Richter [1–4], where several applications are dealt with and where it was also shown that integrating usual, that is, Euclidean, lengths of the same indivisibles does not yield the area content, in general. In the present paper, we will prove that this method still applies when generalized circle numbers are derived for general star discs. In this sense, this paper deals with bounded and unbounded star discs. Notice that because we will not assume symmetry of the unit disc, distances will depend on directions in general.

To become more specific, let S be a star body in IR^2, and let its area content be defined as usual by its Lebesgue measure. Furthermore, let us call the boundary of ϱ times the star disc S the S-circle of S-radius ϱ, $\varrho > 0$ and denote it by $\mathfrak{C}_S(\varrho)$. If we define the perimeter of S by using different length measures, then we can observe different perimeter-to-(two-times-S-radius) ratios, and these ratios differ from the corresponding (area-content)-to-(squared-S-radius) ratio in general. If we choose, however, the length measure in a certain specific way, then the first ratio coincides with the second one for all $\varrho > 0$. In the most famous case when S is the Euclidean disc and measuring circumference is based upon Euclidean arc-length, the common constant value of the two ratios is the well-known circle number π.

If S is the symmetric and convex $l_{2,p}$-circle, centered at the origin and thus defining a norm then, according to Richter [1], the suitable arc-length measure is based upon the dual norm, that is, the norm which is generated by the l_{2,p^*}-circle $\{(x, y) : |x|^{p^*} + |y|^{p^*} = 1\}$ with $p^* \geq 1$ satisfying the equation $1/p + 1/p^* = 1$.

Similarly, if S is an $l_{2,p}$-circle, $p \in (0, 1)$, corresponding to an antinorm (see Moszyńska and Richter [5]) then, according to Richter [2], the suitable arc-length measure is based upon the star disc $S(p^{**}) = \{(x, y) \in \mathbb{R}^2 : |x|^{p^{**}} + |y|^{p^{**}} \geq 1\}$ with $p^{**} < 0$ satisfying $1/p + 1/p^{**} = 1$. The star disc $S(p^{**})$ corresponds to a specific semi-antinorm with respect to the canonical fan (see Moszyńska and Richter [5]).

The situation for ellipses has been discussed in Richter [3]. If $S = D_{a,b} = \{(x, y) \in \mathbb{R}^2 : (x/a)^2 + (y/b)^2 \leq 1\}$ is an elliptically contoured disc and $E_{(a,b)} = \partial S$ its boundary then the suitable arc-length measure on the Borel σ-field of subsets of the ellipse $E_{(a,b)}$ is based upon the disc $D_{(1/b,1/a)}$. Note that again $D_{(a,b)}$ and $D_{(1/b,1/a)}$ correspond to dual norms.

The arc-length measure used for defining the p-generalized circle number allows both for $p \geq 1$ and for $0 < p < 1$ the same additional interpretation in terms of the derivative of the area content function with respect to the p-radius variable. This way, the notion of the p-generalized circumference of the p-circle was introduced first in Richter [4] under more general circumstances and motivated there by several of its applications. Several geometric interpretations of this notion in cases of special norms and antinorms have been discussed so far. As just to refer to a few of them, let us recall that this notion is a basic one for the generalized method of indivisibles, that it allows to prove the so-called thin layers property of the Lebesgue measure and to think of a certain mixed area content in a new way and that it is closely connected with the solution to a certain isoperimetric problem. From a technical point of view, a basic difference between the two situations is that in the convex case, one uses triangle inequality for showing convergence of a sequence of suitably defined integral sums, and that one makes use of the reverse triangle inequality from Moszyńska and Richter [5] for proving such convergence if the p-generalized circle corresponds to an antinorm.

The arc-length measure used for defining ellipses numbers has also an interpretation in terms of the derivative of the area content function but with respect to a generalized radius variable corresponding to $E_{(a,b)}$. The common notion behind the different definitions of a generalized radius variable discussed so far in the literature is that of the Minkowski functional (or gauge function) of a star body, but looking onto the motivating applications, for example, from probability theory and mathematical statistics, let it become clear that further generalizations are desirable in future work.

Here, we start our consideration with the definition of the S-generalized circumference of an S-circle corresponding to a star body and will discuss in Section 2 both its general geometric meaning and its specific interpretation either when S is generated by an arbitrary norm or by an antinorm of special type.

It should be mentioned here that the Minkowski functional of a star body generates a distance which is not symmetric in general, that is, it does not assign a length in the usual sense to a generalized circle but a certain directed length.

In this sense, Section 2 deals mainly with generalized circle numbers for star bodies while Section 3 is devoted to a certain class of unbounded star discs.

Definition 1.1. Let λ_2 be the Lebesgue measure in $\mathbb{R}^2$, S a star body, and $A_S(\varrho) = \lambda_2(\varrho S)$ the corresponding area content function. Then,

$$\frac{d}{d\varrho} A_S(\varrho) =: \mathfrak{U}_S(\varrho), \quad \varrho > 0, \tag{1.1}$$

will be called the S-generalized circumference of ϱS or the S-generalized arc-length of the boundary $\mathfrak{C}_S(\varrho)$ of ϱS, $\varrho S = \{(\varrho x, \varrho y) \in \mathbb{R}^2 : (x, y) \in S\}$.

It follows from the properties of the Lebesgue measure that

$$\frac{A_S(\varrho)}{\varrho^2} = \frac{\mathfrak{U}_S(\varrho)}{2\varrho} = A_S(1), \quad \forall \varrho > 0. \tag{1.2}$$

The representation

$$A_S(\varrho) = \int_0^{\varrho} \mathfrak{U}_S(r) dr \tag{1.3}$$

may be understood as a generalized method of indivisibles for the Lebesgue measure, where the indivisibles are multiples of the boundary of S and measuring their circumferences is based upon $\mathfrak{U}_S$. Equations (1.2) may suggest on the one hand to call $A_S(1)$ the S-generalized circle number. On the other hand, one may consider at this stage of consideration a method of introducing generalized circle numbers which follows basically the idea of the main theorem of calculus being rather elementary if not even trivial. However, the papers of Richter [1–4] which are closely connected with this approach allow a new look onto a class of geometric measure representations or, similarly, onto a class of stochastic representations which are quite fruitful for many applications. Several of these applications, especially in probability theory and mathematical statistics, are discussed therein.

Clearly, there is always a necessity to give a geometric or otherwise mathematical interpretation of the circumference $\mathfrak{U}_S(\varrho)$. In other words, one naturally looks for a geometry such that the arc-length of $\mathfrak{C}_S(\varrho)$ with respect to this geometry coincides with the S-generalized circumference $\mathfrak{U}_S(\varrho)$. The non-Euclidean geometries being identified in this way may be considered as geometries "being close to the Euclidean one" as those were discussed in Hilbert [6] in connection with his fourth problem. If we can uniquely identify a geometry such that the arc-length measure of S, $\mathrm{AL}_{S,S^*}(\varrho)$, which is based upon the geometry's unit ball S^*, satisfies

$$\mathrm{AL}_{S,S^*}(\varrho) = \mathfrak{U}_S(\varrho), \tag{1.4}$$

then we can observe already the nontrivial situation that

$$\frac{A_S(\varrho)}{\varrho^2} = \frac{\mathrm{AL}_{S,S^*}}{2\varrho} = A_S(1), \qquad \forall \varrho > 0. \tag{1.5}$$

At such stage of investigation, it will then be already much more motivated that the area content of the unit star, $A_S(1)$, is called the S-generalized circle number, $\pi(S)$.

In this sense, the considerations in Richter [1–3] deal with restrictions of the function $S \to \pi(S)$ to $l_{2,p}$-balls, $p > 0$, and to axes aligned ellipses.

2. Star Bodies

A subset S from $\mathbb{R}^2$ is called a star body if it is star-shaped with respect to the origin and compact and has the origin in its interior. A set of this type has the property that for every $z \in \mathbb{R}^2$ there exists a uniquely determined $\varrho > 0$ such that $z/\varrho \in \partial S$, where ∂S denotes the boundary of the set S. This ϱ equals the value of the Minkowski functional with respect to the reference set S

$$h_S(x, y) = \inf\{\lambda > 0 : (x, y) \in \lambda S\}, \tag{2.1}$$

at any point $(x, y) \in \partial S$. The function h_S is often called the gauge function of S (see, e.g., in Webster [7]) and coincides, for $(x, y) \neq (0, 0)$, with the reciprocal of the radial function (see, e.g., in Thompson [9] and Moszyńska [8]),

$$\varrho_S((x, y)) = \sup\{\lambda \geq 0 : \lambda(x, y) \in S\}. \tag{2.2}$$

The special cases that h_S is a norm or an antinorm are of particular interest and will be separately dealt with in Examples 2.13 and 2.14.

With a star body S, the pair $(\mathbb{R}^2, h_S)$ may be considered as a generalized Minkowski plane. The star disc and the star circle of S-radius ϱ will be defined then by $K_S(\varrho) = \{r \cdot s,\ s \in S,\ 0 \leq r \leq \varrho\}$ and $\mathfrak{C}_S(\varrho) = \partial K_S(\varrho)$, respectively. The set S will be called the unit star in this plane.

Let T be another star disc in $\mathbb{R}^2$ which will be specified later. Whenever possible, we may define the T-arc-length of the curve $\mathfrak{C}_S(\varrho)$ as follows.

Definition 2.1. If $\mathfrak{Z}_n = \{z_0, z_1, \ldots, z_n = z_0\}$ denotes a successive and positive (anticlockwise) oriented partition of $\mathfrak{C}_S(\varrho)$, then the positive directed T-arc-length of $\mathfrak{C}_S(\varrho)$ is defined by

$$\mathrm{AL}_{ST}(\varrho) := \lim_{n \to \infty} \sum_{j=1}^{n} h_T(z_j - z_{j-1}), \tag{2.3}$$

if the limit exists for and is independent of all described partitions of $\mathfrak{C}_S(\varrho)$ with $F(\mathfrak{Z}_n) = \max_{1 \leq j \leq n} h_T(z_j - z_{j-1})$ tending to zero as $n \to \infty$.

Using triangle inequality or its reverse, one can show that if h_S is a norm or antinorm then taking the limit may be changed with taking the supremum or the infimum, respectively.

Notice that because h_T is in general not a symmetric function, the orientation in the partition may have essential influence onto the value of $\mathrm{AL}_{S,T}(\varrho)$ and is, therefore, assumed here always to be positive.

For studying $\mathrm{AL}_{S,T}(\varrho)$, let a parameter representation of the unit-S-circle $\mathfrak{C}_S(1)$ be given by $\mathfrak{C}_S(1) = \{R_S(\varphi)(\cos\varphi, \sin\varphi)^T,\ 0 \le \varphi < 2\pi\}$. Later on, we will assume that R_S is a.e. differentiable. From the relation

$$h_S\left((x,y)^T\right) = 1, \quad \forall (x,y)^T \in \mathfrak{C}_S(1), \tag{2.4}$$

it follows that

$$h_S\left((\cos\varphi, \sin\varphi)^T\right) = \frac{1}{R_S(\varphi)}, \quad 0 \le \varphi < 2\pi. \tag{2.5}$$

In other words, with the notation $M_S(\varphi) = h_S((\cos\varphi, \sin\varphi)^T)$, we have

$$\mathfrak{C}_S(1) = \left\{ \left(\frac{\cos\varphi}{M_S(\varphi)}, \frac{\sin\varphi}{M_S(\varphi)} \right)^T,\ 0 \le \varphi < 2\pi \right\}. \tag{2.6}$$

This motivates the following definition which generalizes more particular notions from earlier considerations.

Definition 2.2. For an arbitrary star body S, the S-generalized sine and cosine functions are

$$\sin_S(\varphi) = \frac{\sin\varphi}{M_S(\varphi)}, \qquad \cos_S(\varphi) = \frac{\cos\varphi}{M_S(\varphi)}, \quad \varphi \in [0, 2\pi), \tag{2.7}$$

respectively.

Notice that there is an elementary geometric interpretation of these generalized trigonometric functions when one considers a right-angled triangl $\mathrm{Tr} = ((0,0)^T, (x,0)^T, (x,y)^T)$ with $x > 0$ and $y > 0$ as follows. The S-generalized sine and cosine of the angle $\varphi \in [0, 2\pi)$ between the directions of the positive x-axes and the line through the points $(0,0)^T$ and $(x,y)^T$ are

$$\sin_S(\varphi) = \frac{y}{h_S\left((x,y)^T\right)}, \qquad \cos_S(\varphi) = \frac{x}{h_S\left((x,y)^T\right)}, \tag{2.8}$$

respectively. These functions satisfy

$$h_S(\cos_S(\varphi), \sin_S(\varphi)) = 1, \tag{2.9}$$

generalizing the well-known formula $\cos^2\varphi + \sin^2\varphi = 1$.

Definition 2.3. The S-generalized polar coordinate transformation

$$\mathrm{Pol}_S : [0,\infty) \times [0,2\pi) \to \mathbb{R}^2 \tag{2.10}$$

is defined by

$$x = r\cos_S(\varphi), \quad y = r\sin_S(\varphi), \quad 0 \le \varphi < 2\pi,\ 0 \le r < \infty. \tag{2.11}$$

Let us denote the quadrants in $\mathbb{R}^2$ in the usual anticlockwise ordering by Q_1 up to Q_4.

Theorem 2.4. *The map* Pol_S *is almost one-to-one, for* $x \neq 0$*, its inverse* Pol_S^{-1} *is given by*

$$r = h_S(x,y), \quad \arctan\left(\left|\frac{y}{x}\right|\right) = \varphi \text{ in } Q_1,\ \pi - \varphi \text{ in } Q_2,\ \varphi - \pi \text{ in } Q_3,\ 2\pi - \varphi \text{ in } Q_4, \tag{2.12}$$

and its Jacobian satisfies

$$J(r,\varphi) = \frac{D(x,y)}{D(r,\varphi)} = \frac{r}{M_S^2(\varphi)}. \tag{2.13}$$

Proof. The proof follows that of Theorem 8 in Richter [3] and makes essentially use of the fact that the derivatives of the S-generalized trigonometric functions $\sin_S$ and $\cos_S$ allow the representations

$$\begin{aligned} \sin_S'(\varphi) &= \frac{1}{M_S^2(\varphi)}\left[\cos\varphi M_S(\varphi) - \sin\varphi M_S'(\varphi)\right], \\ \cos_S'(\varphi) &= \frac{1}{M_S^2(\varphi)}\left[-\sin\varphi M_S(\varphi) - \cos\varphi M_S'(\varphi)\right]. \end{aligned} \tag{2.14}$$

□

Using S-generalized polar coordinates, we can write

$$\mathfrak{C}_S(\varrho) = \left\{ \left(\varrho\cos_S(\varphi), \varrho\sin_S(\varphi)\right)^T,\ 0 \le \varphi < 2\pi,\ \varrho > 0 \right\}. \tag{2.15}$$

We assume from now on that h_T is positively homogeneous, put

$$h_T(z_j - z_{j-1}) = h_T\left(\left(\Delta_j x, \Delta_j y\right)^T\right), \tag{2.16}$$

and consider

$$h_T(z_j - z_{j-1}) = h_T\left(\left(\frac{\Delta_j x(\varphi)}{\Delta_j\varphi}, \frac{\Delta_j y(\varphi)}{\Delta_j\varphi}\right)^T\right)\Delta_j\varphi, \tag{2.17}$$

for sufficiently thin partition $\mathfrak{Z}_n$ and $\Delta_j\varphi > 0$. We get in the limit, which was assumed in Definition 2.1 to be uniquely determined

$$\begin{aligned} \mathrm{AL}_{S,T}(\varrho) &= \int_0^{2\pi} h_T(x'(\varphi), y'(\varphi))\,d\varphi = \varrho\int_0^{2\pi} h_T\left(\frac{x'(\varphi)}{\varrho}, \frac{y'(\varphi)}{\varrho}\right)d\varphi \\ &= \varrho\int_0^{2\pi} h_T\left(\left(\cos_S'(\varphi), \sin_S'(\varphi)\right)^T\right)d\varphi = \varrho\mathrm{AL}_{S,T}(1). \end{aligned} \tag{2.18}$$

It follows from the proof of Theorem 2.4 that

$$\mathrm{AL}_{S,T}(\varrho) = \varrho\int_0^{2\pi} R_S^2(\varphi)h_T\left(M_S(\varphi)\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix} + M_S'(\varphi)\begin{pmatrix}-\cos\varphi\\ -\sin\varphi\end{pmatrix}\right)d\varphi, \tag{2.19}$$

that is,

$$\mathrm{AL}_{S,T}(\varrho) = \varrho\int_0^{2\pi} R_S^2(\varphi)h_T(O(\varphi)\mathfrak{x}_S(\varphi))\,d\varphi, \tag{2.20}$$

with

$$O(\varphi) = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}, \qquad \mathfrak{x}_S(\varphi) = \begin{pmatrix}\dfrac{R_S'(\varphi)}{R_S^2(\varphi)}\\ \dfrac{1}{R_S(\varphi)}\end{pmatrix}. \tag{2.21}$$

The following lemmas and corollaries represent certain steps towards a reformulation of formula (2.20).

Lemma 2.5. *In the case of their existence, the partial derivatives $h_{S,x}$ and $h_{S,y}$ of the function $(x,y) \to h_S(x,y)$ satisfy the representation*

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}h_{S,x}(r\cos_s(\varphi), r\sin_s(\varphi))\\ h_{S,y}(r\cos_s(\varphi), r\sin_s(\varphi))\end{pmatrix} = \mathfrak{O}(\varphi)\mathfrak{x}_S(\varphi). \tag{2.22}$$

Proof. It follows from the relation

$$h_S(r\cos_S(\varphi), r\sin_S(\varphi)) = r \tag{2.23}$$

that the partial derivatives $h_{S,x}$ and $h_{S,y}$ satisfy the equation system

$$\frac{\partial}{\partial\varphi}h_S(\cos_S(\varphi), \sin_S(\varphi)) = 0, \qquad \frac{\partial}{\partial r}h_S(r\cos_S(\varphi), r\sin_S(\varphi)) = 1. \tag{2.24}$$

Solving this differential equation system, we get

$$\begin{aligned} h_{S,x}(r\cos_s(\varphi), r\sin_s(\varphi)) &= \frac{1}{R_S^2(\varphi)}\left(R_S(\varphi)\cos(\varphi) + R_S'(\varphi)\right)\sin(\varphi), \\ h_{S,y}(r\cos_s(\varphi), r\sin_s(\varphi)) &= \frac{1}{R_S^2(\varphi)}\left(R_S(\varphi)\sin(\varphi) - R_S'(\varphi)\right)\cos(\varphi). \end{aligned} \tag{2.25}$$

Hence,

$$\begin{pmatrix} h_{S,x}(r\cos_s(\varphi), r\sin_s(\varphi)) \\ h_{S,y}(r\cos_s(\varphi), r\sin_s(\varphi)) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathfrak{O}(\varphi)\mathfrak{x}_S(\varphi). \tag{2.26}$$

□

Let B be a 2×2-matrix and $BT = \{B(x,y)^T : (x,y)^T \in T\}$. Clearly, multiplying the set T by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ causes an anticlockwise rotation of T through the angle $\pi/2$. Hence, if T is a star disc, then BT is a star disc too.

Corollary 2.6. *For positively homogeneous h_T, differentiable h_S, formula* (2.20) *may be rewritten as*

$$\mathrm{AL}_{S,T}(\varrho) = \varrho \int_0^{2\pi} R_S^2(\varphi) h_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)T}\left(\nabla h_S(x,y)|_{(x,y)=\mathrm{Pol}_S(r,\varphi)}\right) d\varphi. \tag{2.27}$$

Proof. Based upon Lemma 2.5, formula (2.20) may be reformulated as

$$\mathrm{AL}_{S,T}(\varrho) = \varrho \int_0^{2\pi} R_S^2(\varphi) h_T\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h_{S,x}(r\cos_s(\varphi), r\sin_s(\varphi)) \\ h_{S,y}(r\cos_s(\varphi), r\sin_s(\varphi)) \end{pmatrix}\right) d\varphi. \tag{2.28}$$

Because of

$$\begin{aligned} h_T\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) &= \inf\left\{\lambda > 0 : \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \lambda T\right\} \\ &= \inf\left\{\lambda > 0 : \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T\right\} \\ &= h_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)T}\left(\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right), \end{aligned} \tag{2.29}$$

it follows the assertion. □

Remark 2.7. The plug-in version $\nabla h_S(x,y)|_{(x,y)=\mathrm{Pol}_S(r,\varphi)}$ of the gradient $\nabla h_S(x,y)$ coincides with the image of the gradient $\nabla h_S(x,y)$ after changing Cartesian with S-generalized polar coordinates, $\mathrm{Pol}_S(\nabla h_S(x,y))(r,\varphi)$.

Proof. Changing Cartesian coordinates (x, y) with S-generalized polar coordinates (r, φ), we have $r = h_S(x, y)$ and $\varphi = \arctan(y/x)$. Starting from

$$\frac{\partial}{\partial x} h_S(x, y) = \frac{\partial}{\partial r} h_S(x, y) \frac{\partial}{\partial x} r + \frac{\partial}{\partial \varphi} h_S(x, y) \frac{\partial}{\partial x} \varphi, \tag{2.30}$$

and the analogous one for $(\partial/\partial y) h_S(x, y)$, and taking into account that

$$\begin{aligned} \frac{\partial}{\partial r} h_S(x, y)|_{(x,y)=\mathrm{Pol}_S(r,\varphi)} &= \frac{\partial}{\partial r} h_S(r \cos_S(\varphi), r \sin_S(\varphi)) = 1, \\ \frac{\partial}{\partial \varphi} h_S(x, y)|_{(x,y)=\mathrm{Pol}_S(r,\varphi)} &= \frac{\partial}{\partial \varphi} h_S(r \cos_S(\varphi), r \sin_S(\varphi)) = 0, \end{aligned} \tag{2.31}$$

it follows

$$\begin{aligned} \frac{\partial}{\partial x} h_S(x, y) &= \frac{\partial r}{\partial x}|_{(x,y)=\mathrm{Pol}_S(r,\varphi)} = h_{S,x}(r \cos_S(\varphi), r \sin_S(\varphi)), \\ \frac{\partial}{\partial y} h_S(x, y) &= \frac{\partial r}{\partial y}|_{(x,y)=\mathrm{Pol}_S(r,\varphi)} = h_{S,y}(r \cos_S(\varphi), r \sin_S(\varphi)). \end{aligned} \tag{2.32}$$

□

Remark 2.8. For positively homogeneous h_T and differentiable Minkowski functional h_S of the star disc S, formula (2.20) may be rewritten as

$$\mathrm{AL}_{S,T}(\varrho) = \varrho \int_0^{2\pi} R_S^2(\varphi) h_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) T}(\mathrm{Pol}_S(\nabla h_S(x, y))(r, \varphi)) d\varphi. \tag{2.33}$$

Definition 2.9. A star body S and a star disc T satisfy the rotated gradient condition if

$$h_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) T}(\nabla h_S(x, y)|_{(x,y)=\mathrm{Pol}_S(r,\varphi)}) = 1, \quad \text{a.e.} \tag{2.34}$$

Let us notice that at the point (x, y) from $\mathfrak{C}_S(\varrho)$, the gradient $\nabla h_S(x, y)$ is normal to the level set $\mathfrak{C}_S(\varrho)$, $\varrho > 0$ of h_S. The following lemma is a consequence of the above consideration.

Lemma 2.10. *For a star body S and a star disc S^* satisfying the rotated gradient condition* (2.34)*, the positive directed S^*-arc-length of $\mathfrak{C}_S(\varrho)$ allows the representation*

$$\mathrm{AL}_{S,S^*}(\varrho) = \varrho \int_0^{2\pi} R_s^2(\varphi) d\varphi. \tag{2.35}$$

We consider now the area function

$$A_S(\varrho) = \varrho^2 A_S(1) = \frac{\varrho^2}{2}\int_0^{2\pi} R_S^2(\varphi)\,d\varphi, \tag{2.36}$$

where $A_S(1)$ denotes the area content of the unit disc $K_S(1)$. The derivative of the area function satisfies obviously

$$\frac{d}{d\varrho}A_S(\varrho) = \varrho\int_0^{2\pi} R_S^2(\varphi)\,d\varphi. \tag{2.37}$$

The following theorem has thus been proved.

Theorem 2.11. *If the star body S and the star disc S^* satisfy the rotated gradient condition* (2.34), *then*

$$\mathfrak{U}_S(\varrho) = \mathrm{AL}_{S,S^*}(\varrho), \tag{2.38}$$

that is, the S-generalized circumference of S coincides with the positive directed S^*-circumference of S.

If relation (2.38) holds, then

$$\mathrm{AL}_{S,S^*}(1) = 2A_S(1). \tag{2.39}$$

Consequently, the ratios $A_S(\varrho)/\varrho^2$ and $\mathrm{AL}_{S,S^*}(\varrho)/2\varrho$ satisfy the relations

$$\frac{A_S(\varrho)}{\varrho^2} \overset{(a)}{=} A_S(1) \overset{(c)}{=} \frac{\mathrm{AL}_{S,S^*}(\varrho)}{2\varrho}, \qquad \forall\varrho > 0. \tag{2.40}$$

In this sense, the geometry and the arc-length measure generated by S^* fulfill our expectations. The following definition is thus well motivated if a star body S and a star disc S^* are chosen in such a way that the limit in Definition 2.1 is uniquely determined, h_T is positively homogeneous, h_S is a.e. differentiable, and the rotated gradient condition (2.34) is satisfied.

Definition 2.12. (a) The properties of the star bodies $K_S(\varrho)$, $\varrho > 0$, which are expressed by the equations (a) and (c) in (2.40) are called the area content and the S-generalized circumference properties of the discs, respectively.

(b) The quantity $A_S(1) =: \pi(S)$ is called the S-generalized circle number of the star bodies $K_S(\varrho)$, $\varrho > 0$.

We may write now (2.40) as

$$\mathrm{AL}_{S,S^*}(\varrho) = 2\pi(S)\varrho, \qquad A_S(\varrho) = \pi(S)\varrho^2. \tag{2.41}$$

Notice that the circle number function $S \to \pi(S)$ assigns a generalized circle number to any star body $K_S(\varrho)$ satisfying assumption (2.34). The restrictions of this function to $l_{2,p}$-balls or axes aligned ellipses were considered in Richter [1–3].

Example 2.13. Here, we consider a first, rather general case, where the rotated gradient condition (2.34) is satisfied. Let $\|\cdot\|_{(p)}$ and $\|\cdot\|_{(d)}$ denote a (primary) C^1-norm in $\mathbb{R}^n$ and the corresponding dual one, respectively. It is proved in Yang [10] that

$$\left\| \nabla \|\mathfrak{x}\|_{(p)} \right\|_{(d)} = 1, \quad \forall \mathfrak{x} \in \mathbb{R}^n. \tag{2.42}$$

Hence, if S is a convex body, that is, $h_S(\mathfrak{x}) = \| \mathfrak{x} \|_{(p)}$ is a (primary) norm, and if

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T = \left\{ \mathfrak{x} \in \mathbb{R}^n : \| \mathfrak{x} \|_{(d)} \leq 1 \right\} = S^* \tag{2.43}$$

is the unit ball with respect to the corresponding dual norm, then the condition (2.34) is satisfied.

For determining the actual value of a generalized circle number $\pi(S) = A_S(1)$ we may refer, for example, to Pisier [11], where volumes of convex bodies are dealt with. Alternatively, one may use S-generalized polar coordinates for making the respective calculations in given cases. The particular results for $l_{2,p}$-circles with $p \geq 1$ and of axes aligned ellipses as well as the corresponding generalized circle numbers have been dealt with in Richter [1, 3].

Example 2.14. We consider now the nonconvex $l_{2,p}$-circles $C_p = \{(x,y) \in \mathbb{R}^2 : |x|^p + |y|^p = 1\}$ with $0 < p < 1$. Such generalized circles correspond to antinorms. A suitable arc-length measure for measuring C_p is based upon the star disc $S(p^{**}) = \{(x,y) \in \mathbb{R}^2 : |x|^{p^{**}} + |y|^{p^{**}} \geq 1\}$ with $p^{**} < 0$ satisfying $1/p + 1/p^{**} = 1$. The star discs $S(p^{**})$ are closely related to specific semiantinorms with respect to the canonical fan. The corresponding generalized circle numbers have been determined in Richter [2]. As because this was done without referring explicitely to (2.34), we may state here the following problem.

Problem 1. Give a general description of sets T satisfying condition (2.34) for sets S being generated by antinorms.

As was indicated in Richter [2], p-generalized circle numbers for $0 < p < 1$ may occur, for example, within certain combinatorial formulae. Notice further that the reciprocal values of the coefficients of the binomial series expansion

$$\frac{1}{4\sqrt{1-4u}} = \sum_{n=0}^{\infty} \frac{1}{\pi(1/n)} \cdot u^n, \quad u \in \left(0, \frac{1}{4}\right), \tag{2.44}$$

are just the generalized circle numbers corresponding to the nonconvex $l_{2,1/n}$-circles. One could also ask for a (possibly elementary geometric?) explanation of this fact.

3. Unbounded Star Discs

In this section, we consider a class of (truncated) unbounded Orlicz discs. More generally than in the preceding section, a star-shaped subset of IR^2 is called a star disc if all its intersections with balls centered at the origin are star bodies. The boundary of a star disc is called a star circle. Notice that a star circle is not necessarily bounded. Special sets of this type will be studied in this section. To be more specific, let us consider, for arbitrary $p < 0$, the function

$$(x, y) \longrightarrow \left|(x, y)\right|_p = \left(|x|^p + \left|y\right|^p\right)^{1/p}, \quad x \neq 0,\ y \neq 0, \tag{3.1}$$

which denotes a semi-antinorm, and the p-generalized circle

$$C_p = \left\{(x, y) \in \mathbb{R}^2 : \left|(x, y)\right|_p = 1\right\}. \tag{3.2}$$

The pairs of straight lines $|y| = 1$ and $|x| = 1$ represent asymptotes for the circle C_p as $|x| \to \infty$ or $|y| \to \infty$, respectively. The intersection point of the p-circle C_p with the line $y = x$ is for positive coordinates $(x_0, y_0) = 2^{-1/p} \cdot (1, 1)$.

Let further $C_p(r) = r \cdot C_p$, $r > 0$ denote the p-generalized circle of p-generalized radius $r > 0$. It is the boundary of the unbounded p-generalized disc of p-generalized radius r,

$$K_p(r) = \left\{(x, y) \in R^2 : \left|(x, y)\right|_p \leq r\right\} = rK_p, \qquad K_p = K_p(1). \tag{3.3}$$

As because

$$\begin{aligned} \left|(x, y)\right|_p \leq r &\Longleftrightarrow |x|^p + \left|y\right|^p \geq r^p \\ &\Longleftrightarrow f(|x|) + f\left(\left|y\right|\right) \geq f(r) \quad \text{for } f(\lambda) = \lambda^p,\ \lambda \geq 0, \end{aligned} \tag{3.4}$$

one may call $K_p(r)$ a two-dimensional Orlicz antiball corresponding to the Young function f. The disc K_p is a star-shaped but noncompact set and, therefore, not a star body. For any $(x, y) \in C_p(r)$ one may think of r as the value of the Minkowski functional with respect to the reference set K_p. The area content and the Euclidean circumference of the unit p-circle are obviously unbounded. That is why we consider from now on truncated p-circles. To this end, let us introduce truncation cones

$$\begin{aligned} C(x_1) &:= \left\{(x, y) \in \mathbb{R}^2 : \frac{\|(x, y) - \Pi_1(x, y)\|}{\|\Pi_1(x, y)\|} < \frac{\|(x_1, y_1) - \Pi_1(x_1, y_1)\|}{\|\Pi_1(x_1, y_1)\|}\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 : \frac{\left|x - y\right|}{\left|x + y\right|} < \frac{\left|x_1 - y_1\right|}{\left|x_1 + y_1\right|}\right\}, \end{aligned} \tag{3.5}$$

where $\|\cdot\|$ denotes Euclidean norm, $1 = (1, 1)$, x_1 is chosen according to $x_1 > x_0 = 2^{-1/p}$ and $|y_1| = (1 - |x_1|^p)^{1/p} < 1$.

The question of interest is now whether we may define in a reasonable way circle numbers for the truncated p-discs $K_p^{x_1}(r) = rK_p^{x_1}$, the boundaries of which are the p-circles $C_p^{x_1}(r) = rC_p^{x_1}$ of p-radius r, where

$$K_p^{x_1} := K_p \cap C(x_1), \qquad C_p^{x_1} := C_p \cap C(x_1). \tag{3.6}$$

To this end, let $\mathfrak{Z}_n = (z_0, z_1, \ldots, z_n)$ be an arbitrary successive anticlockwise-oriented partition of the truncated circle $C_p^{x_1}$ satisfying $z_0 = (x_0, (1-x_0^p)^{1/p})$ and $z_n = (x_1, (1-x_1^p)^{1/p})$. We consider the sum

$$S(\mathfrak{Z}_n) = \sum_{j=1}^{n} \left|z_j - z_{j-1}\right|_q = \sum_{j=1}^{n} \left(\left|x_j - x_{j-1}\right|^q + \left|y_j - y_{j-1}\right|^q\right)^{1/q}, \quad q \in (0,1), \tag{3.7}$$

and observe that due to the reverse triangle inequality it decreases monotonously as

$$F(\mathfrak{Z}_n) := \sup_{1 \le j \le n} \left|z_j - z_{j-1}\right|_q \longrightarrow 0. \tag{3.8}$$

According to the symmetry of $C_p^{x_1}$, the following remark is justified.

Remark 3.1. For $q \in (0,1)$, the $l_{2,q}$-arc-length of the truncated circle $C_p^{x_1}$ is defined as

$$\mathrm{AL}_{p,q}^{x_1} = 8 \lim_{F(\mathfrak{Z}_n) \to 0} S(\mathfrak{Z}_n). \tag{3.9}$$

If $x \to y(x)$ denotes an arbitrary parameter representation of the truncated circle $C_p^{x_1}$, then

$$\frac{1}{8}\mathrm{AL}_{p,q}^{x_1} = \lim_{F(\mathfrak{Z}_n) \to 0} \sum_{j=1}^{n} \left(1 + \left|\frac{\Delta y_j}{\Delta x_j}\right|^q\right)^{1/q} \Delta x_j = \int_{x_0}^{x_1} \left(1 + \left|y'(x)\right|^q\right)^{1/q} dx. \tag{3.10}$$

Let us denote the usual Euclidean area content of the truncated circle disc $K_p^{x_1}$ by $A_p^{x_1}$.

Lemma 3.2. *Let for arbitrary $p < 0$ the number $p^* \in (0,1)$ be uniquely defined by the equation $1/p + 1/p^* = 1$. Then,*

$$A_p^{x_1} = \frac{1}{2}\mathrm{AL}_{p,p^*}^{x_1}. \tag{3.11}$$

Proof. With

$$y(x) = \left(1 - |x|^p\right)^{1/p} = (1 - x^p)^{1/p}, \qquad y'(x) = -(1 - x^p)^{1/p-1} x^{p-1}, \tag{3.12}$$

it follows from the above formulae that

$$\mathrm{AL}_{p,q}^{x_1} = 8 \int_{1/2^{1/p}}^{x_1} \left(1 + (1 - x^p)^{(1/p-1)q} x^{(p-1)q}\right)^{1/q} dx. \tag{3.13}$$

Changing variables $u = x^p,\ dx = du/(pu^{(p-1)/p})$ causes a change of the limits of integration:

$$\begin{aligned}\mathrm{AL}_{p,q}^{x_1} &= \frac{8}{-p}\int_{x_1^p}^{1/2}\left(1+(1-u)^{((1-p)/p)q}u^{((p-1)/p)q}\right)^{1/q}\frac{du}{u^{(p-1)/p}}\\ &= \frac{8}{|p|}\int_{x_1^p}^{1/2}\left(u^{((1-p)/p)q}+(1-u)^{((1-p)/p)q}\right)^{1/q}du.\end{aligned} \tag{3.14}$$

Assuming now $1/p + 1/q = 1$, or equivalently $q = p/(p-1) =: p^*$, it follows that

$$\mathrm{AL}_{p,p^*}^{x_1} = \frac{8}{|p|}\int_{x_1^p}^{1/2}\left(u^{-1}+(1-u)^{-1}\right)^{(p-1)/p}du = \frac{8}{|p|}\int_{x_1^p}^{1/2}\left(\frac{1-u+u}{u(1-u)}\right)^{1-1/p}du. \tag{3.15}$$

Hence,

$$\mathrm{AL}_{p,p^*}^{x_1} = \frac{8}{|p|}\int_{x_1^p}^{1/2}u^{1/p-1}(1-u)^{1/p-1}du. \tag{3.16}$$

Now, what about the area content of the truncated circle $C_p^{x_1}$? The $l_{2,p}$-generalized standard triangle coordinate transformation Tr from Richter [4] is defined by

$$\mathrm{Tr}_p(r,\mu) = (x,y) \quad \text{with } x = r\mu,\ y = +(-)r\left(1-|\mu|^p\right)^{1/p}. \tag{3.17}$$

Because of

$$\begin{aligned}&\left\{(x,y) : |x|^p + |y|^p = 1, x_0 \le x \le x_1\right\} = \mathrm{Tr}_p(\{1\}\times[x_0,x_1]),\\ &r\left\{(x,y) : |x|^p + |y|^p = 1,\ x_0 \le x \le x_1\right\}\\ &\quad = \left\{(rx,ry) : |x|^p + |y|^p = 1,\ x_0 \le x \le x_1\right\}\\ &\quad = \left\{(\xi,\eta) : \left|\frac{\xi}{r}\right|^p + \left|\frac{\eta}{r}\right|^p = 1,\ x_0 \le \frac{\xi}{r} \le x_1\right\}\\ &\quad = \left\{(x,y) : |x|^p + |y|^p = r^p,\ x_0 \le \frac{x}{r}(=:\mu) \le x_1\right\}\\ &\quad = \mathrm{Tr}_p(\{r\}\times[x_0,\ x_1]),\end{aligned} \tag{3.18}$$

it follows

$$\bigcup_{0\le r\le 1} r\left\{(x,y) : |x|^p + |y|^p = 1,\ x_0 \le x \le x_1\right\} = \mathrm{Tr}_p([0,1]\times[x_0,x_1]) = K_p^{x_1}. \tag{3.19}$$

That is,

$$K_p^{x_1} = \mathrm{Tr}_p([0,1]\times[x_0,\ x_1]). \tag{3.20}$$

Changing Cartesian with standard triangle coordinates in the integral

$$A_p^{x_1} = \int_{K_p^{x_1}} d(x,y), \tag{3.21}$$

we get

$$A_p^{x_1} = 8\int_{r=0}^{1}\left(\int_{\mu=x_0}^{x_1} r\left(1-\mu^p\right)^{(1-p)/p} d\mu\right)dr = \frac{8}{2}\int_{2^{-1/p}}^{x_1}\left(1-\mu^p\right)^{1/p-1} d\mu. \tag{3.22}$$

Substituting $y=\mu^p, dy/d\mu = py^{(p-1)/p}$, it follows that

$$A_p^{x_1} = \frac{4}{p}\int_{1/2}^{x_1^p}(1-y)^{1/p-1}y^{1/p-1}dy = \frac{4}{|p|}\int_{x_1^p}^{1/2} y^{1/p-1}(1-y)^{1/p-1}dy. \tag{3.23}$$

Hence, the lemma is proved. □

Remark 3.3. Because of $p<0$,

$$A_p^{x_1} \longrightarrow \infty \quad \text{as } x_1 \longrightarrow \infty. \tag{3.24}$$

The following corollary and definition are now quite obvious and well motivated.

Corollary 3.4. *For arbitrary $x_1 > x_0 = 2^{-1/p}$, the truncated star discs $K_p^{x_1}$ have the area content and p-generalized circumference properties* (a*) *and* (c*), *respectively,*

$$\frac{A_p^{x_1}(r)}{r^2} \overset{(a^*)}{=} A_p^{x_1} \overset{(c^*)}{=} \frac{\mathrm{AL}_{p,p*}^{x_1}(r)}{2r}, \tag{3.25}$$

from which it follows immediately

$$\frac{d}{dr}A_p^{x_1}(r) = \mathrm{AL}_{p,p^*}^{x_1}(r). \tag{3.26}$$

Remark 3.5. One may think of (3.26) as reflecting a generalized method of indivisibles for each $x_1 > x_0$, where the truncated circles $C_p^{x_1}$ are the indivisibles and measuring them is based upon the geometry generated by the disc K_{p^*}.

Definition 3.6. For arbitrary $x_1 > 2^{-1/p}$, the quantity $A_p^{x_1} =: \pi^{x_1}(p)$ will be called the circle number of the truncated circle $C_p^{x_1}$.

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