International Scholarly Research Network
ISRN Mathematical Analysis
Volume 2011, Article ID 485203, 9 pages
doi:10.5402/2011/485203

Research Article

Existence for Nonoscillatory Solutions of Higher-Order Nonlinear Differential Equations

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Received 8 August 2011; Accepted 22 September 2011

Academic Editor: Z. Dosla

The existence of nonoscillatory solutions of the higher-order nonlinear differential equation $[r(t)(x(t)+P(t)x(t-\tau))^{(n-1)}]' + \sum_{i=1}^{m} Q_i(t) f_i(x(t-\sigma_i)) = 0,\ t \geq t_0$, where $m \geq 1, n \geq 2$ are integers, $\tau > 0,\ \sigma_i \geq 0,\ r, P, Q_i \in C([t_0,\infty), R),\ f_i \in C(R,R)\ (i = 1,2,\ldots,m)$, is studied. Some new sufficient conditions for the existence of a nonoscillatory solution of above equation are obtained for general $Q_i(t)\ (i = 1,2,\ldots,m)$ which means that we allow oscillatory $Q_i(t)\ (i = 1,2,\ldots,m)$. In particular, our results improve essentially and extend some known results in the recent references.

1. Introduction

Consider the higher-order nonlinear neutral differential equation

$$\left[r(t)(x(t)+P(t)x(t-\tau))^{(n-1)}\right]' + \sum_{i=1}^{m} Q_i(t) f_i(x(t-\sigma_i)) = 0, \quad t \geq t_0. \tag{1.1}$$

With respect to (1.1), throughout, we shall assume the following:

(i) $m \geq 1,\ n \geq 2$ are integers, $\tau > 0,\ \sigma_i \geq 0$,

(ii) $r, P, Q_i \in C([t_0,\infty), R),\ r(t) > 0,\ f_i \in C(R,R),\ i = 1,2,\ldots,m$.

Let $\rho = \max_{1\leq i\leq m}\{\tau, \sigma_i\}$. By a solution of (1.1), we mean a function $x(t) \in C([t_1-\rho,\infty), R)$ for some $t_1 \geq t_0$ which has the property that $x(t)+P(t)x(t-\tau) \in C^{n-1}([t_1,\infty), R)$ and $r(t)(x(t)+P(t)x(t-\tau))^{(n-1)} \in C^1([t_1,\infty), R)$ and satisfies (1.1) on $[t_1,\infty)$.

A nontrivial solution of (1.1) is called oscillatory if it has arbitrarily large zeros, and, otherwise, it is nonoscillatory.

The existence of nonoscillatory solutions of higher-order nonlinear neutral differential equations received much less attention, which is due mainly to the technical difficulties arising in its analysis.

In 1998, Kulenovic and Hadziomerspahic [1] investigated the existence of nonoscillatory solutions of second-order nonlinear neutral differential equation

$$(x(t) + cx(t-\tau))'' + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0, \quad t \geq t_0, \tag{E_0}$$

where c is a constant.

In 2006, Zhang and Wang [2] investigated the second neutral delay differential equation with positive and negative coefficients:

$$\left[r(t)(x(t) + P(t)x(t-\tau))'\right]' + Q_1(t)f(x(t-\sigma_1)) - Q_2(t)g(x(t-\sigma_2)) = 0, \quad t \geq t_0, \tag{E}$$

where $\tau > 0$, $\sigma_i \geq 0, Q_1, Q_2 \in C([t_0,\infty), R^+)$, $f, g \in C(R,R)$, $xf(x) > 0$, $xg(x) > 0$, $(x \neq 0)$. By using Banach contraction mapping principle, they proved the following theorem which extends the results in [1].

Theorem A ([2, Theorem 2.3]). *Assume that*

(H_1) *f and g satisfy local Lipschitz condition and $xf(x) > 0$, $xg(x) > 0$, for $x \neq 0$;*

(H_2) *$Q_i(t) \geq 0$, $i = 1, 2$, $aQ_1(t) - Q_2(t)$ is eventually nonnegative for every $a > 0$;*

(H_3) *$\int_{t_0}^{\infty}\int_{t_0}^{t}(Q_i(t)/r(s))ds\,dt < \infty$, $i = 1, 2$ hold*

if one of the following two conditions is satisfied:

(H_4) *$P(t) > 1$ eventually, and $0 < P_2 \leq P_1 < P_2^2 < +\infty$,*

(H_5) *$P(t) < -1$ eventually, and $-\infty < P_2 \leq P_1 < -1$,*

where $P_1 = \limsup_{t\to\infty} P(t)$, $P_2 = \liminf_{t\to\infty} P(t)$, then (1.1) *has a nonoscillatory solution.*

In 2007, Zhou [3] studies the existence of nonoscillatory solution of the following second-order nonlinear differential equation.

$$\left[r(t)(x(t) + P(t)x(t-\tau))'\right]' + \sum_{i=1}^{m} Q_i(t)f_i(x(t-\sigma_i)) = 0, \quad t \geq t_0, \tag{E'}$$

where $f_i \in C(R,R)$ $(i = 1, 2, \ldots, m)$. By using Krasnoselskii's fixed point theorem, they proved the following theorem.

Theorem B ([3, Theorem 1]). *Assume that there exist nonnegative constants c_1 and c_2 such that $c_1 + c_2 < 1$, $-c_2 \leq P(t) \leq c_1$. Further, assume that*

$$\int_{t_0}^{\infty}\int_{t_0}^{t}\frac{|Q_i(t)|}{r(s)}ds\,dt < \infty, \quad i = 1, 2, \ldots, m. \tag{1.2}$$

Then (1.1) *has a bounded nonoscillatory solution.*

In this paper, by using Krasnoselskii's fixed point theorem and some new techniques, we obtain some sufficient conditions for the existence of a nonoscillatory solution of (1.1) for general $Q_i(t)$ $(i = 1, 2, \ldots, m)$ which means that we allow oscillatory $Q_i(t)$ $(i = 1, 2, \ldots, m)$. Meanwhile, we extend the main results of [2, 3].

2. Main Result

The following fixed point theorem will be used to prove the main results in this section.

Lemma 2.1 (see [3, Krasnoselskii's fixed point theorem]). *Let X be a Banach space, let Ω be a bounded closed convex subset of X, and let S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contraction and S_2 is completely continuous, then the equation*

$$S_1x + S_2x = x \tag{2.1}$$

has a solution in Ω.

Theorem 2.2. *Assume that there exist nonnegative constants c_1 and c_2 such that $c_1 + c_2 < 1$, $-1 < -c_2 \le P(t) \le c_1 < 1$. Further, assume that*

$$\int_{t_0}^{\infty} \int_{t_0}^{t} \frac{s^{n-2}|Q_i(t)|}{r(s)} ds\, dt < \infty, \quad i = 1, 2, \ldots, m. \tag{2.2}$$

Then (1.1) *has a bounded nonoscillatory solution.*

Proof. By interchanging the order of integral, we note that (2.2) is equivalent to

$$\int_{t_0}^{\infty} s^{n-2} \int_{s}^{\infty} \frac{|Q_i(t)|}{r(s)} ds\, dt < \infty, \quad i = 1, 2, \ldots, m. \tag{2.3}$$

By (2.3), we choose $T > t_0$ sufficiently large such that

$$\frac{1}{(n-2)!} \int_{T}^{\infty} s^{n-2} \int_{s}^{\infty} \frac{M}{r(s)} \sum_{i=1}^{m} |Q_i(u)| du\, ds < \frac{1 - c_1 - c_2}{4}, \tag{2.4}$$

where $M = \max_{((1-c_1-c_2)/2) \le x \le 1} \{|f_i(x)| : 1 \le i \le m\}$.

Let $C([t_0, \infty), R)$ be the set of all continuous functions with the norm $\|x\| = \sup_{t \ge t_0} |x(t)| < \infty$. Then $C([t_0, \infty), R)$ is a Banach space. We define a bounded, closed, and convex subset Ω of $C([t_0, \infty), R)$ as follows:

$$\Omega = \left\{ x = x(t) \in C([t_0, \infty), R) : \frac{1 - c_1 - c_2}{2} \le x(t) \le 1,\ t \ge t_0 \right\}. \tag{2.5}$$

Define two maps S_1 and $S_2 : \Omega \to C([t_0,\infty),R)$ as follows:

$$(S_1x)(t) = \begin{cases} \dfrac{3+c_1-3c_2}{4} - P(t)x(t-\tau), & t \geq T, \\ (S_1x)(T), & t_0 \leq t \leq T, \end{cases}$$
$$(S_2x)(t) = \begin{cases} \dfrac{(-1)^{n-1}}{(n-2)!}\displaystyle\int_t^\infty (s-t)^{n-2}\int_s^\infty \frac{1}{r(s)}\left(\sum_{i=1}^m Q_i(u)f_i(x(u-\sigma_i))\right)du\,ds, & t \geq T, \\ (S_2x)(T), & t_0 \leq t \leq T. \end{cases} \tag{2.6}$$

(i) We shall show that for any $x, y \in \Omega$, $S_1x + S_2y \in \Omega$.

In fact, $x, y \in \Omega$, and $t \geq T$, we get

$$\begin{aligned} (S_1x)(t) + (S_2y)(t) &\leq \frac{3+c_1-3c_2}{4} - P(t)x(t-\tau) \\ &\quad + \frac{1}{(n-2)!}\int_t^\infty (s-t)^{n-2}\int_s^\infty \frac{1}{r(s)}\left(\sum_{i=1}^m \left|Q_i(u)f_i(y(u-\sigma_i))\right|\right)du\,ds \\ &\leq \frac{3+c_1-3c_2}{4} + c_2 + \frac{1}{(n-2)!}\int_T^\infty s^{n-2}\int_s^\infty \frac{M}{r(s)}\sum_{i=1}^m |Q_i(u)|du\,ds \\ &\leq \frac{3+c_1-3c_2}{4} + c_2 + \frac{1-c_1-c_2}{4} = 1. \end{aligned} \tag{2.7}$$

Furthermore, we have

$$\begin{aligned} (S_1x)(t) + (S_2y)(t) &\geq \frac{3+c_1-3c_2}{4} - P(t)x(t-\tau) \\ &\quad - \frac{1}{(n-2)!}\int_t^\infty s^{n-2}\int_s^\infty \frac{1}{r(s)}\left(\sum_{i=1}^m \left|Q_i(u)f_i(y(u-\sigma_i))\right|\right)du\,ds \\ &\geq \frac{3+c_1-3c_2}{4} - c_1 - \frac{1}{(n-2)!}\int_T^\infty s^{n-2}\int_s^\infty \frac{M}{r(s)}\sum_{i=1}^m |Q_i(u)|du\,ds \\ &\geq \frac{3+c_1-3c_2}{4} - c_1 - \frac{1-c_1-c_2}{4} = \frac{1-c_1-c_2}{2}. \end{aligned} \tag{2.8}$$

Hence,

$$\frac{1-c_1-c_2}{2} \leq (S_1x)(t) + (S_2y)(t) \leq 1, \quad \text{for } t \geq t_0. \tag{2.9}$$

Thus, we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

(ii) We shall show that S_1 is a contraction mapping on Ω.

In fact, for $x, y \in \Omega$ and $t \geq T$, we have

$$\left|(S_1x)(t) - (S_1y)(t)\right| \leq |P(t)|\left|x(t-\tau) - y(t-\tau)\right| \leq c_0\|x-y\|, \tag{2.10}$$

where $c_0 = \max\{c_1, c_2\}$. This implies that

$$\|S_1 x - S_1 y\| \le c_0 \|x - y\|. \tag{2.11}$$

Since $0 < c_0 < 1$, we conclude that S_1 is a contraction mapping on Ω.

(iii) We now show that S_2 is completely continuous.

First, we will show that S_2 is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \ge T$, we have

$$\begin{aligned}
&|(S_2 x_k)(t) - (S_2 x)(t)| \\
&\quad \le \frac{1}{(n-2)!} \int_t^\infty s^{n-2} \int_s^\infty \frac{1}{r(s)} \left(\sum_{i=1}^m |Q_i(u)| \left| f_i(x_k(u-\sigma_i)) - f_i(x(u-\sigma_i)) \right| \right) du\, ds \\
&\quad \le \frac{1}{(n-2)!} \int_T^\infty s^{n-2} \int_s^\infty \frac{1}{r(s)} \left(\sum_{i=1}^m |Q_i(u)| \left| f_i(x_k(u-\sigma_i)) - f_i(x(u-\sigma_i)) \right| \right) du\, ds.
\end{aligned} \tag{2.12}$$

Since $|f_i(x_k(t-\sigma_i)) - f_i(x(t-\sigma_i))| \to 0$ as $k \to \infty$ for $i = 1, 2, \ldots, m$, by applying the Lebesgue dominated convergence theorem, we conclude that $\lim_{k\to\infty} \|(S_2 x_k)\ (t) - (S_2 x)(t)\| = 0$. This means that S_2 is continuous.

Next, we show that $S_2\Omega$ is relatively compact. It suffices to show that the family of functions $\{S_2 x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniform boundedness is obvious. For the equicontinuity, according to Levitan's result [4], we only need to show that, for any given $\varepsilon > 0$, $[T, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ε. By (2.3), for any $\varepsilon > 0$, take $T^* \ge T$ large enough so that

$$\frac{1}{(n-2)!} \int_{T^*}^\infty s^{n-2} \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du\, ds < \frac{\varepsilon}{2}. \tag{2.13}$$

Then, for $x \in \Omega$, $t_2 \ge t_1 \ge T^*$,

$$\begin{aligned}
|(S_2 x)(t_2) - (S_2 x)(t_1)| &\le \frac{1}{(n-2)!} \int_{t_2}^\infty s^{n-2} \int_s^\infty \frac{1}{r(s)} \left(\sum_{i=1}^m |Q_i(u)| \left| f_i(x(u-\sigma_i)) \right| \right) du\, ds \\
&\quad + \frac{1}{(n-2)!} \int_{t_1}^\infty s^{n-2} \int_s^\infty \frac{1}{r(s)} \left(\sum_{i=1}^m |Q_i(u)| \left| f_i(x(u-\sigma_i)) \right| \right) du\, ds \\
&\le \frac{1}{(n-2)!} \int_{t_2}^\infty s^{n-2} \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du\, ds \\
&\quad + \frac{1}{(n-2)!} \int_{t_1}^\infty s^{n-2} \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du\, ds \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{2.14}$$

For $x \in \Omega$, $T \le t_1 < t_2 \le T^* + 1$,

$$\begin{aligned}
&|(S_2x)(t_2) - (S_2x)(t_1)| \\
&\quad \le \frac{1}{(n-2)!}\left|\int_{t_1}^{t_2}(s-t_1)^{n-2}\int_s^\infty \frac{1}{r(s)}\left(\sum_{i=1}^m Q_i(u)f_i(x(u-\sigma_i))\right)du\,ds\right| \\
&\qquad + \frac{1}{(n-2)!}\left|\int_{t_2}^{\infty}\left[(s-t_2)^{n-2}-(s-t_1)^{n-2}\right]\int_s^\infty \frac{1}{r(s)}\left(\sum_{i=1}^m Q_i(u)f_i(x(u-\sigma_i))\right)du\,ds\right| \\
&\quad \le \frac{1}{(n-2)!}\int_{t_1}^{t_2} s^{n-2}\int_s^\infty \frac{M}{r(s)}\sum_{i=1}^m |Q_i(u)|du\,ds \\
&\qquad + \frac{1}{(n-3)!}(t_2-t_1)\int_{t_2}^\infty (s-\xi)^{n-3}\int_s^\infty \frac{M}{r(s)}\sum_{i=1}^m |Q_i(u)|du\,ds \\
&\quad \le \frac{1}{(n-2)!}\int_{t_1}^{t_2} s^{n-2}\int_s^\infty \frac{M}{r(s)}\sum_{i=1}^m |Q_i(u)|du\,ds \\
&\qquad + \frac{1}{(n-3)!}(t_2-t_1)\int_{T}^\infty s^{n-2}\int_s^\infty \frac{M}{r(s)}\sum_{i=1}^m |Q_i(u)|du\,ds,
\end{aligned} \tag{2.15}$$

where $t_1 < \xi < t_2$.

Then there exists $\delta > 0$ such that

$$|(S_2x)(t_2) - (S_2x)(t_1)| < \varepsilon, \quad \text{if } 0 < t_2 - t_1 < \delta. \tag{2.16}$$

For any $x \in \Omega$, $t_0 \le t_1 < t_2 \le T$, it is easy to see that

$$|(S_2x)(t_2) - (S_2x)(t_1)| = 0 < \varepsilon. \tag{2.17}$$

Therefore, $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S_2\Omega$ is relatively compact. By Lemma 2.1, there is $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. It is easy to see that $x_0(t)$ is a nonoscillatory solution of (1.1). The proof is complete. □

Theorem 2.3. *Assume that* $-\infty < c_1 \le P(t) \le c_2 < -1$ *and* (2.2) *holds. Then* (1.1) *has a bounded nonoscillatory solution.*

Proof. We choose positive constants M_1, M_2, α such that $-c_1M_1 < \alpha < (-c_2 - 1)M_2$. $c = \min\{(\alpha + M_1c_1)c_2/c_1, ((-c_2-1)M_2) - \alpha\}$. Choosing $T > t_0$ sufficiently large such that

$$\frac{1}{(n-2)!}\int_T^\infty s^{n-2}\int_s^\infty \frac{M'}{r(s)}\sum_{i=1}^m |Q_i(u)|du\,ds < c, \tag{2.18}$$

where $M' = \max_{M_1 \le x \le M_2}\{|f_i(x)| : 1 \le i \le m\}$.

Let $C([t_0,\infty),R)$ be the set as in the proof of Theorem 2.2. We define a bounded, closed, and convex subset Ω of $C([t_0,\infty),R)$ as follows:

$$\Omega=\{x=x(t)\in C([t_0,\infty),R):M_1\le x(t)\le M_2,\ t\ge t_0\}. \tag{2.19}$$

Define two maps S_1 and $S_2:\Omega\to C([t_0,\infty),R)$ as follows:

$$(S_1x)(t)=\begin{cases}\dfrac{-\alpha}{P(t+\tau)}-\dfrac{x(t+\tau)}{P(t+\tau)}, & t\ge T,\\ (S_1x)(T), & t_0\le t\le T,\end{cases}$$

$$(S_2x)(t)=\begin{cases}\dfrac{(-1)^{n-1}}{(n-2)!}\dfrac{1}{P(t+\tau)}\displaystyle\int_{t+\tau}^{\infty}(s-t-\tau)^{n-2}\int_s^{\infty}\frac{1}{r(s)}\left(\sum_{i=1}^{m}Q_i(u)f_i(x(u-\sigma_i))\right)du\,ds, & t\ge T,\\ (S_2x)(T), & t_0\le t\le T.\end{cases} \tag{2.20}$$

(i) We shall show that for any $x,y\in\Omega$, $S_1x+S_2y\in\Omega$.

In fact, for every $x,y\in\Omega$, and $t\ge T$, we get

$$\begin{aligned}(S_1x)(t)+(S_2y)(t)&\ge\frac{-\alpha}{c_1}+\frac{c}{c_2}\ge M_1,\\ (S_1x)(t)+(S_2y)(t)&\le\frac{-\alpha}{c_2}-\frac{M_2}{c_2}-\frac{c}{c_2}\le M_2.\end{aligned} \tag{2.21}$$

Thus, we have proved that $S_1x+S_2y\in\Omega$. Since $-\infty<c_1\le P(t)\le c_2<-1$, we get that S_1 is a contraction mapping. We also can prove that $\{S_2x:x\in\Omega\}$ is uniformly bounded and equicontinuous on $[t_0,\infty)$, and hence $S_2\Omega$ is relatively compact. So by Lemma 2.1, there is $x_0\in\Omega$ such that $S_1x_0+S_2x_0=x_0$. That is,

$$\begin{aligned}x_0(t)=&\frac{-\alpha}{P(t+\tau)}-\frac{x_0(t+\tau)}{P(t+\tau)}+\frac{(-1)^{n-1}}{(n-2)!}\frac{1}{P(t+\tau)}\\ &\times\int_{t+\tau}^{\infty}(s-t-\tau)^{n-2}\int_s^{\infty}\frac{1}{r(s)}\left(\sum_{i=1}^{m}Q_i(u)f_i(x_0(u-\sigma_i))\right)du\,ds.\end{aligned} \tag{2.22}$$

It is easy to see that $x_0(t)$ is a bounded nonoscillatory solution of (1.1).

The proof is complete. □

Theorem 2.4. *Assume that* $1<c_1\le P(t)\le c_2<+\infty$ *and* (2.2) *holds. Then* (1.1) *has a bounded nonoscillatory solution.*

Proof. We choose positive constants $M_3,\ M_4,\ \alpha$ such that $M_4+c_2M_3<\alpha<c_1M_4$. $c=\min\{\alpha-M_4-c_2M_3,\ c_1M_4-\alpha\}$. Choosing $T>t_0$ sufficiently large such that

$$\frac{1}{(n-2)!}\int_T^\infty s^{n-2}\int_s^\infty \frac{M''}{r(s)}\sum_{i=1}^m |Q_i(u)|du\,ds < c, \tag{2.23}$$

where $M'' = \max_{M_3\le x\le M_4}\{|f_i(x)| : 1\le i\le m\}$.

Let $C([t_0,\infty),R)$ be the set as in the proof of Theorem 2.2. We define a bounded, closed, and convex subset Ω of $C([t_0,\infty),R)$ as follows:

$$\Omega = \{x = x(t)\in C([t_0,\infty),R) : M_3\le x(t)\le M_4, t\ge t_0\}. \tag{2.24}$$

Define two maps S_1 and $S_2 : \Omega \to C([t_0,\infty),R)$ as follows:

$$(S_1x)(t) = \begin{cases} \dfrac{\alpha}{P(t+\tau)} - \dfrac{x(t+\tau)}{P(t+\tau)}, & t\ge T,\\ (S_1x)(T), & t_0\le t\le T,\end{cases}$$

$$(S_2x)(t) = \begin{cases} \dfrac{(-1)^{n-1}}{(n-2)!}\dfrac{1}{P(t+\tau)}\displaystyle\int_{t+\tau}^\infty (s-t-\tau)^{n-2}\int_s^\infty \frac{1}{r(s)}\left(\sum_{i=1}^m Q_i(u)f_i(x(u-\sigma_i))\right)du\,ds, & t\ge T,\\ (S_2x)(T), & t_0\le t\le T.\end{cases} \tag{2.25}$$

(i) We shall show that for any $x,y\in\Omega$, $S_1x+S_2y\in\Omega$.

In fact, for every $x,y\in\Omega$ and $t\ge T$, we get

$$(S_1x)(t)+(S_2y)(t)\ge \frac{1}{c_2}(\alpha - M_4 - c)\ge M_3,\quad (S_1x)(t)+(S_2y)(t)\le \frac{\alpha}{c_1}+\frac{c}{c_1}\le M_4. \tag{2.26}$$

Thus, we have proved that $S_1x+S_2y\in\Omega$. Since $1<c_1\le P(t)\le c_2<+\infty$, we get S_1 is a contraction mapping. We also can prove that $\{S_2x : x\in\Omega\}$ is uniformly bounded and equicontinuous on $[t_0,\infty)$, and, hence, $S_2\Omega$ is relatively compact. So by Lemma 2.1, there is $x_0\in\Omega$ such that $S_1x_0+S_2x_0 = x_0$. That is,

$$\begin{aligned} x_0(t) = {} & \frac{\alpha}{P(t+\tau)} - \frac{x_0(t+\tau)}{P(t+\tau)} + \frac{(-1)^{n-1}}{(n-2)!}\frac{1}{P(t+\tau)}\\ & \times\int_{t+\tau}^\infty (s-t-\tau)^{n-2}\int_s^\infty \frac{1}{r(s)}\left(\sum_{i=1}^m Q_i(u)f_i(x_0(u-\sigma_i))\right)du\,ds. \end{aligned} \tag{2.27}$$

It is easy to see that $x_0(t)$ is a bounded nonoscillatory solution of (1.1).

The proof is complete. □

Remark 2.5. If we let $n=2$ in Theorem 2.2, we get the Theorem 1 in [3]. In the case where $n=2$, $r(t)\equiv 1$, Theorem 2.2 improves essentially Theorem 2.2 in [5].

Remark 2.6. The conditions of Theorem 2.4 relaxing the hypotheses (H_4) of Theorem 3 in [2].

Remark 2.7. Theorems 2.3 and 2.4 improve essentially Theorem 3 in [2], we allow that $Q_i(t)$ $(i = 1, 2, \ldots, m)$ are oscillatory.

Acknowledgments

This research was supported by Natural Science Foundations of Shandong Province of China (ZR2009AM011 and ZR2009AQ010) and Doctor of Ministry of Education (20103705110003).

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