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Research Article **The** L**-Total Graph of an** L**-Module**

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Let *L* be a complete lattice. We introduce and investigate the *L*-total graph of an *L*-module over an *L*-commutative ring. The main purpose of this paper is to extend the definition and results given in (Anderson and Badawi, 2008) to more generalize the *L*-total graph of an *L*-module case.

1. Introduction

It was Beck (see [1]) who first introduced the notion of a zero-divisor graph for commutative rings. This notion was later redefined by Anderson and Livingston in [2]. Since then, there has been a lot of interest in this subject, and various papers were published establishing different properties of these graphs as well as relations between graphs of various extensions (see [2–5]). Let *R* be a commutative ring with *Z*(*R*)being its set of zero-divisors elements. The total graph of *R*, denoted by *T*($\Gamma(R)$), is the (undirected) graph with all elements of *R* as vertices, and, for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. The total graph of a commutative ring has been introduced and studied by Anderson and Badawi in [3]. In [6], the notion of the total torsion element graph of a module over a commutative ring is introduced.

In [7], Zadeh introduced the concept of fuzzy set, which is a very useful tool to describe the situation in which the data is imprecise or vague. Many researchers used this concept to generalize some notions of algebra. Goguen in [8] generalized the notion of fuzzy subset of X to that of an *L*-subset, namely, a function from X to a lattice *L*. In [9], Rosenfeld considered the fuzzification of algebraic structures. Liu [10] introduced and examined the notion of a fuzzy ideal of a ring. Since then several authors have obtained interesting results on *L*-ideals of a ring *R* and *L*-modules (see [11, 12]). Also, *L*-zero-divisor graph of an *L*-commutative ring has been introduced and studied in [13].

In the present paper we introduce a new class of graphs, called the *L*-total torsion element graph of a L-module (see Definition 2.2), and we completely characterize the structure of this graph. The total torsion element graph of a module over a commutative ring and the L-total torsion element graph of a L-module over a L-commutative ring are different concepts. Some of our results are analogous to the results given in [6]. The corresponding results are obtained by modification, and here we give a complete description of the L-total torsion element graph of an *L*-module.

For the sake of completeness, we state some definitions and notation used throughout. For a graph Γ , by $E(\Gamma)$ and $V(\Gamma)$, we denote the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices a and b, denoted by d(a, b), is the length of the shortest path connecting them (if such a path does not exist, then d(a, a) = 0 and $d(a, b) = \infty$). The diameter of a graph Γ , denoted by diam(Γ), is equal to sup{ $d(a, b) : a, b \in V(\Gamma)$ }. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ , denoted gr(Γ), is the length of the shortest cycle in Γ , provided Γ contains a cycle; otherwise, $gr(\Gamma) = \infty$. We denote the complete graph on *n* vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). We will sometimes call a $K^{1,m}$ a star graph. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertex of Γ_1 (resp., Γ_2) is adjacent (in Γ) to any vertex not in Γ_1 (resp., Γ_2).

Let *R* be a commutative ring, and *L* stands for a complete lattice with least element 0 and greatest element 1. By an *L*-subset μ of a nonempty set *X*, we mean a function μ from *X* to L. If L = [0, 1], then μ is called a fuzzy subset of X. L^X denotes the set of all L-subsets of X. We recall some definitions and lemmas from the book [12], which we need for development of our paper.

Definition 1.1. An L-ring is a function $\mu : R \to L$, where (R, +, .) is a ring, which satisfies the following.

(1) $\mu \neq 0$. (2) $\mu(x - y) \ge \mu(x) \land \mu(y)$ for every x, y in R. (3) $\mu(xy) \ge \mu(x) \lor \mu(y)$ for every *x*, *y* in *R*.

Definition 1.2. Let $\mu \in L^R$. Then μ is called an L-ideal of R if for every $x, y \in R$ the following conditions are satisfied.

- (1) $\mu(x-y) \ge \mu(x) \land \mu(y)$.
- (2) $\mu(xy) \ge \mu(x) \lor \mu(y)$.

The set of all *L*-ideals of *R* is denoted by LI(R).

The set of all *L*-fuzzy *R*-modules of *M* is denoted by L(M).

Definition 1.3. Assume that M is an R-module, and let $\mu \in L^M$. Then μ is called an L-fuzzy *R*-module of *M* if for all $x, y \in M$ and for all $r \in R$ the following conditions are satisfied.

(1) $\mu(x-y) \ge \mu(x) \land \mu(y)$. (2) $\mu(rx) \ge \mu(x)$.

(3) $\mu(0_M) = \mu(1)$.

Lemma 1.4. Let M be a module over a ring R, and $\mu \in L(M)$. Then $\mu(m) \leq \mu(0_M)$ for every $m \in M$.

2. $T(\mu)$ Is a Submodule of M

Let *M* be a module over a commutative ring *R*, and let $\mu \in L(M)$. The structure of the *L*-total torsion element graph $T(\Gamma(\mu))$ may be completely described in those cases when μ -torsion elements form a submodule of *M*. We begin with the key definition of this paper.

Definition 2.1. Let *M* be a module over a commutative ring *R*, and let $\mu \in L(M)$. A μ -torsion element is an element $m \in M$ with $\mu(m) \neq \mu(0_M)$ for which there exists a nonzero element *r* of *R* such that $\mu(rm) = \mu(0_M)$.

The set of μ -torsion elements in *M* will be denoted by $T(\mu)$.

Definition 2.2. Let *M* be a module over a ring *R*, and let $\mu \in L(M)$. We define the *L*-total torsion element graph of an *L*-module $T(\Gamma(\mu))$ as follows: $V(T(\Gamma(\mu))) = M$, $E(T(\Gamma(\mu))) = \{\{x, y\} : x + y \in T(\mu)\}$.

Notation 1. For the μ -torsion element graph $T(\Gamma(\mu))$, we denote the diameter, the girth, and the distance between two distinct vertices *a* and *b*, by diam $(T(\Gamma(\mu)))$, gr $(T(\Gamma(\mu)))$, and $d_{\mu}(a, b)$, respectively.

Remark 2.3. Let M be a module over a ring R, and let $\mu \in L(M)$. Clearly, if μ is a nonzero constant, then $T(\Gamma(\mu)) = \emptyset$. So throughout this paper, we will assume, unless otherwise stated, that μ is not a nonzero constant. Thus, there is a nonzero element y of M such that $\mu(y) \neq \mu(0_M)$.

We will use $\text{Tof}(\mu)$ to denote the set of elements of M that are not μ -torsion elements. Let $\text{Tof}(\Gamma(\mu))$ be the (induced) subgraph of $T(\Gamma(\mu))$ with vertices $\text{Tof}(\mu)$, and let $\text{Tor}(\Gamma(\mu))$ be the (induced) subgraph of $T(\Gamma(\mu))$ with vertices $T(\mu)$.

Definition 2.4. Let *M* be a module over a ring *R*, and $\mu \in L(M)$. One defines the set $ann_{\mu}(M)$ by $ann_{\mu}(M) = \{r \in R : \mu(rM) = \{\mu(0_M)\}\}$, the μ -annihilator of *M*.

Lemma 2.5. Let M be a module over a ring R, and let $\mu \in L(M)$. Then $ann_{\mu}(M)$ is an L-ideal of R.

Proof. Let $r, s \in \operatorname{ann}_{\mu}(M)$ and $t \in R$. If $m \in M$, then we have $\mu((r-s)m) \ge \mu(rm) \land \mu(-sm) = \mu(0_M) \land \mu(0_M) = \mu(0_M)$ and $\mu(trm) = \mu(t(rm)) \ge \mu(rm) = \mu(0_M)$. It then follows from Lemma 1.4 that $\mu((r-s)m) = \mu(0_M)$; hence $r - s \in \operatorname{ann}_{\mu}(M)$. Similarly, $rt \in \operatorname{ann}_{\mu}(M)$.

Theorem 2.6. Let M be a module over a ring R and let $\mu \in L(M)$. Then the L-torsion element graph $T(\Gamma(\mu))$ is complete if and only if $T(\mu) = M$.

Proof. If $T(\mu) = M$, then for any vertices $m, m' \in M$, one has $m + m' \in T(\mu)$; hence they are adjacent in $T(\Gamma(\mu))$. On the other hand, if $T(\Gamma(\mu))$ is complete, then every vertex is adjacent to 0. Thus, $m = m + 0 \in T(\mu)$ for every $m \in M$. This completes the proof.

Theorem 2.7. Let *M* be a module over a ring *R*, and let $\mu \in L(M)$ such that $T(\mu)$ is a submodule of *M*. Then one has the following.

- (i) Tor(Γ(μ)) is a complete (induced) subgraph of T(Γ(μ)) and Tor(Γ(μ)) is disjoint from Tof(Γ(μ)).
- (ii) If $\operatorname{ann}_{\mu}(M) \neq 0$, then $T(\Gamma(\mu))$ is a complete graph.

Proof. (i) Tor($\Gamma(\mu)$) is complete directly from the definition. Finally, if $m \in T(\mu)$ and $m' \in Tof(\mu)$ were adjacent, then $m + m' \in T(\mu)$; so this, since $T(\mu)$ is a submodule, would lead to the contradiction $m' \in T(\mu)$.

(ii) Let $m \in M$. we may assume that $\mu(m) \neq \mu(0_M)$. By assumption, there exists $0 \neq s \in R$ with $\mu(sM) = \mu(0_M)$, so $\mu(sm) = \mu(0_M)$. Thus $m \in T(\mu)$, and; therefore, $T(\Gamma(\mu))$ is a complete graph by Theorem 2.6.

Theorem 2.8. Let *M* be a module over a ring *R*, and let $\mu \in L(M)$. Then $T(\Gamma(\mu))$ is totally disconnected if and only if *R* has characteristic 2 and $T(\mu) = \{0_M\}$.

Proof. If $T(\mu) = \{0_M\}$, then the vertices m_1 and m_2 are adjacent if and only if $m_1 = -m_2$. Then $T(\Gamma(\mu))$ is a disconnected graph, and its only edges are those that connect vertices m_i and $-m_i$ (we do not need a priori assumption that R has characteristic 2). Conversely, assume that $T(\Gamma(\mu))$ is totally disconnected. Then $0 + m \notin T(\mu)$ for every nonzero element m of M. Thus, $T(\mu) = \{0_M\}$. Further, since m + (-m) = 0, we have m = -m (so $\mu(2m) = \mu(0_M)$) for every $m \in M$ with $\mu(m) \neq \mu(0_M)$ by the total disconnectedness of the graph $T(\Gamma(\mu))$. As $T(\mu) = \{0_M\}$, it follows that $2 = 1_R + 1_R = 0$. Thus, char(R) = 2.

Proposition 2.9. Let *M* be a module over a ring *R*, and let $\mu \in L(M)$ such that $T(\mu)$ is a submodule of *M*. If $m \in Tof(\mu)$, then $2m \in T(\mu)$ if and only if $2 \in Z(R)$.

Proof. First suppose that $2m \in T(\mu)$. Since $m \notin T(\mu)$, we get that $\mu(m) \neq \mu(0_M)$, and, for all $r \in R$, $\mu(rm) = \mu(0_M)$ implies that r = 0. Since $2m \in T(\mu)$, there is a nonzero element $c \in R$ such that $\mu(c(2m)) = \mu((2c)m) = \mu(0_M)$, and, since $m \notin T(\mu)$, one must have 2c = 0; hence, $2 \in Z(R)$. Conversely, assume that $2 \in Z(R)$. Then there exists $0 \neq d \in R$ with 2d = 0. Since $\mu(0_M) = \mu((2d)m) = \mu(d(2m))$, we have $2m \in T(\mu)$.

Theorem 2.10. Let *M* be a module over a ring *R*, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of *M*. Then $T(\Gamma(\mu))$ is disconnected.

Proof. If $T(\mu) = \{0_M\}$, then $T(\Gamma(\mu))$ is disconnected by Theorem 2.8. If $T(\mu) \neq \{0_M\}$, then the subgraphs of $\text{Tor}(\Gamma(\mu))$ and $\text{Tof}(\Gamma(\mu))$ are disjoint by Theorem 2.7 (i), as required.

Theorem 2.11. Let *M* be a module over a ring *R*, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of *M*. Suppose $|T(\mu)| = \alpha$ and $|M/T(\mu)| = \beta$. Then one has the following.

- (i) If $2 \in Z(R)$, then $T(\Gamma(\mu))$ is a union of β disjoint complete graphs K^{α} .
- (ii) If $2 \notin Z(R)$, then $T(\Gamma(\mu))$ is a union of $(\beta 1)/2$ disjoint bipartite graphs $K^{\alpha,\alpha}$ and one complete graph K^{α} .

Proof. (i) Assume that $2 \in Z(R)$ and let $m, m' \in \text{Tof}(\mu)$ be such that $m + T(\mu) \neq m' + T(\mu)$. The elements m + t, m + t' from the same coset $m + T(\mu)$ are adjacent if and only if $2m \in T(\mu)$, so $2 \in Z(R)$, according to the Proposition 2.9. Then m + t and m' + t' are not adjacent (otherwise,

we would have $m - m' = m + m' - 2m' \in T(\mu)$, and; therefore, $m + T(\mu) = m' + T(\mu)$. Since every coset has cardinality α , we conclude that $T(\Gamma(\mu))$ is the disjoint union of β complete graph K^{α} .

(ii) If $2 \notin Z(R)$, then the elements m+t, m+t' from $m+T(\mu)$ are obviously not adjacent. The elements m + t, m' + t' from different cosets are adjacent if and only if $m + m' \in T(\mu)$ or $m+T(\mu) = (-m)+T(\mu)$. In this way we obtain that the subgraph spanned by the vertices from Tof(μ) is a disjoint union of $(\beta - 1)/2$ (= β if β is infinite) disjoint bipartite graph $K^{\alpha,\alpha}$.

Proposition 2.12. Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of M. Then one has the following.

- (i) Tof($\Gamma(\mu)$) is complete if and only if either $|M/T(\mu)| = 2$ or $|M/T(\mu)| = |M| = 3$.
- (ii) Tof($\Gamma(\mu)$) is connected if and only if either $|M/T(\mu)| = 2$ or $|M/T(\mu)| = 3$.
- (iii) $\operatorname{Tof}(\Gamma(\mu))$ and, hence; $(\operatorname{Tor}(\Gamma(\mu)))$ and $T(\Gamma(\mu)))$ is totally disconnected if and only if $T(\mu) = \{0_M\}$ and $2 \in Z(R)$.

Proof. Let $|M/T(\mu)| = \beta$ and $|T(\mu)| = \alpha$.

- (i) Let $\operatorname{Tof}(\Gamma(\mu))$ be complete. Then, by Theorem 2.11, $\operatorname{Tof}(\Gamma(\mu))$ is complete if and only if $\operatorname{Tof}(\Gamma(\mu))$ is a single K^{α} or $K^{1,1}$. If $2 \in Z(R)$, then $\beta 1 = 1$. Thus, $\beta = 2$, and hence $|M/T(\mu)| = 2$. If $2 \notin Z(R)$, then $\alpha = 1$ and $(\beta 1)/2 = 1$. Thus, $T(\mu) = \{0\}$ and $\beta = 3$; hence, $|M| = |M/T(\mu)| = 3$. The reverse implication may be proved in a similar way as in [6, Theorem 2.6 (1)].
- (ii) By theorem 2.11, Tof(Γ(μ)) is connected if and only if Tof(Γ(μ)) is a single K^α or K^{α,α}. Thus, either β − 1 = 1 if 2 ∈ Z(R) or (β − 1)/2 = 1 if 2 ∉ Z(R); hence, β = 2 or β = 3, respectively, as needed. The reverse implication may be proved in a similar way as in [3, Theorem 2.6 (2)].
- (iii) Tof($\Gamma(\mu)$) is totally disconnected if and only if it is a disjoint union of K^{1} 's. So by Theorem 2.11, $|T(\mu)| = 1$ and $|M/T(\mu)| = 1$, and the proof is complete.

By the proof of the Proposition 2.12, the next theorem gives a more explicit description of the diameter of Tof($\Gamma(\mu)$).

Theorem 2.13. Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of M. Then one has the following.

- (i) diam(Tof($\Gamma(\mu)$)) = 0 if and only if $T(\mu) = \{0\}$ and |M| = 2.
- (ii) diam(Tof($\Gamma(\mu)$)) = 1 if and only if either $T(\mu) \neq \{0_M\}$ and $|M/T(\mu)| = 2$ or $T(\mu) = \{0\}$ and |M| = 3.
- (iii) diam(Tof($\Gamma(\mu)$)) = 2 if and only if $T(\mu) \neq \{0_M\}$ and $|M/T(\mu)| = 3$.
- (iv) *Otherwise*, diam(Tof($\Gamma(\mu)$)) = ∞ .

Proposition 2.14. Let *M* be a module over a ring *R*, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of *M*. Then $gr(Tof(\Gamma(\mu))) = 3, 4$ or ∞ . In particular, $gr(Tof(\Gamma(\mu))) \leq 4$ if $Tof(\Gamma(\mu))$ contains a cycle.

Proof. Let $\text{Tof}(\Gamma(\mu))$ contain a cycle. Then since $\text{Tof}(\Gamma(\mu))$ is disjoint union of either complete or complete bipartite graphs by Theorem 2.11, it must contain either a 3 cycles or a 4 cycles. Thus $\text{gr}(\text{Tof}(\Gamma(\mu))) \leq 4$.

Theorem 2.15. Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of M. Then one has the following.

- (i) (a) gr(Tof(Γ(μ))) = 3 *if and only if* 2 ∈ Z(R) *and* |T(μ)| ≥ 3.
 (b) gr(Tof(Γ(μ))) = 4 *if and only if* 2 ∉ Z(R) *and* |T(μ)| ≥ 2.
 (c) *Otherwise*, gr(Tof(Γ(μ))) = ∞.
- (ii) (a) gr(T(Γ(μ))) = 3 *if and only if* |T(μ)| ≥ 3.
 (b) gr(T(Γ(μ))) = 4 *if and only if* 2 ∉ Z(R) *and* |T(μ)| = 2.
 (c) Otherwise, gr(T(Γ(μ))) = ∞.

Proof. Apply Theorem 2.11, Proposition 2.14, and Theorem 2.7 (i).

The previous theorems give a complete description of the structure of the *L*-total torsion element graph of an *L*-module *M* when $T(\mu)$ is a submodule. The question under what conditions $T(\mu)$ is a submodule of *M* and how is this related to the condition that Z(R) is an ideal in *R* naturally arises. We prove that the following results holds.

Theorem 2.16. Let M be a module over a ring R, and let $\mu \in L(M)$. Then one has the following.

- (i) If $Z(R) = \{0_R\}$, then $T(\mu)$ is a submodule of M.
- (ii) If Z(R) = Rc is a principal ideal of R with c a nilpotent element of R, then $T(\mu)$ is a submodule of M.

Proof. (i) Let $m, m' \in T(\mu)$ and $r \in R$. There are nonzero elements $a, b \in R$ such that $\mu(m) \neq \mu(0_M), \mu(m') \neq \mu(0_M)$, and $\mu(am) = \mu(bm') = \mu(0_M)$ with $ab \neq 0$ (since R is an integral domain). It follows that $\mu(ab(m + m')) \geq \mu(abm) \wedge \mu(abm') = \mu(0_M) \wedge \mu(0_M) = \mu(0_M)$; hence, $\mu(ab(m + m')) = \mu(0_M)$ by Lemma 1.4. Thus, $m + m' \in T(\mu)$. Similarly, $rm \in T(\mu)$, and this completes the proof.

(ii) Assume that $T(\mu)$ is not a submodule of M. Then there are elements $m, m' \in T(\mu)$ such that $m + m' \notin T(\mu)$. By assumption, there exist nonzero elements $r, s \in R$ such that $\mu(rm) = \mu(0_M) = \mu(sm') = \mu(0_M)$, where $\mu(m) \neq \mu(0_M)$ and $\mu(m') \neq \mu(0_M)$. Then $\mu(rs(m + m')) = \mu(0_M)$ and $m + m' \notin T(\mu)$, so we must have rs = 0, and; thus, $r, s \in Z(R)$. Since c is nilpotent, we have $r = r_1c^t$ and $s = s_1c^u$, for some $r_1, s_1 \notin Z(R)$. We may assume that $t \geq u$. Then for the nonzero element s_1r of R we have $\mu(s_1r(m + m')) = \mu(0_M)$ which is contrary to the assumption that $m + m' \notin T(\mu)$.

Example 2.17. Assume that $R = \mathbb{Z}$ is the ring integers, and let M = R. We define the mapping $\mu : M \rightarrow [0,1]$ by

$$\mu(m) = \begin{cases} \frac{1}{2} & \text{if } x \in 2\mathbb{Z}, \\ \frac{1}{5} & \text{otherwise.} \end{cases}$$
(2.1)

Then $\mu \in L(M)$ and $T(\mu) = M$. Thus, $T(\Gamma(\mu))$ is a complete graph by Theorem 2.6.

Example 2.18. Let $M_1 = R_1 = Z_8$ denote the ring of integers modulo 8 and $M_2 = R_2 = Z_{25}$ the ring of integers modulo 25. We define the mappings $\mu_1 : M_1 \rightarrow [0, 1]$ by

$$\mu_1(x) = \begin{cases} 1 & \text{if } x = \overline{0}, \\ \frac{1}{2} & \text{otherwise} \end{cases}$$
(2.2)

and $\mu_2: M_2 \to [0,1]$ by

$$\mu_2(m) = \begin{cases} 1 & \text{if } x = \overline{0}, \\ \frac{1}{3} & \text{otherwise.} \end{cases}$$
(2.3)

Then, for each i $(1 \le i \le 2)$, $\mu_i \in L(M_i)$, $T(\mu_1) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$, and $T(\mu_2) = \{\overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}\}$. An inspection will show that $T(\mu_1)$ and $T(\mu_2)$ are submodules of M_1 and M_2 , respectively. Therefore, by Theorem 2.11, we have the following results.

- (1) Since $2 \in Z(R_1)$, we conclude that $T(\Gamma(\mu_1))$ is a union of 2 disjoint K^4 .
- (2) Since $2 \notin Z(R_2)$, we conclude that $T(\Gamma(\mu_2))$ is a disjoint union of 2 complete graph K^5 and 5 bipartite $K^{5,5}$.

3. $T(\mu)$ Is Not a Submodule of M

We continue to use the notation already established, so *M* is a module over a commutative ring *R* and $\mu \in L(M)$. In this section, we study the *L*-torsion element graph $T(\Gamma(\mu))$ when $T(\mu)$ is not a submodule of *M*.

Lemma 3.1. Let *M* be a module over a ring *R*, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of *M*. Then there are distinct $m, m' \in T(\mu)^*$ such that $m + m' \in \text{Tof}(\mu)$.

Proof. It suffices to show that $T(\mu)$ is always closed under scalar multiplication of its elements by elements of R. Let $m \in T(\mu)$ and $r \in R$. There is a nonzero element $s \in R$ with $\mu(sm) = \mu(0_M)$ such that $\mu(m) \neq \mu(0_M)$, so $\mu(s(rm)) = \mu(r(sm)) \geq \mu(sm) = \mu(0_M)$; hence, $\mu(s(rm)) = \mu(0_M)$ by Lemma 1.4, as required.

Theorem 3.2. Let *M* be a module over a ring *R*, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of *M*. Then one has the following.

- (i) $\operatorname{Tor}(\Gamma(\mu))$ is connected with diam $(\operatorname{Tor}(\Gamma(\mu))) = 2$.
- (ii) Some vertex of $\text{Tor}(\Gamma(\mu))$ is adjacent to a vertex of $\text{Tof}(\Gamma(\mu))$. In particular, the subgraphs $\text{Tor}(\Gamma(\mu))$ and $\text{Tof}(\Gamma(\mu))$ of $T(\Gamma(\mu))$ are not disjoint.
- (iii) If $\operatorname{Tof}(\Gamma(\mu))$ is connected, then $T(\Gamma(\mu))$ is connected.

Proof. (i) Let $x \in T(\mu)^*$. Then x is adjacent to 0. Thus, x - 0 - y is a path in $\text{Tor}(\Gamma(\mu))$ of length two between any two distinct $x, y \in T(\mu)^*$. Moreover, there exist nonadjacent $x, y \in T(\mu)^*$ by Lemma 3.1; thus, diam $(\text{Tor}(\Gamma(\mu))) = 2$.

(ii) By Lemma 3.1, there exist distinct $x, y \in T(\mu)^*$ such that $x + y \in \text{Tof}(\mu)$. Then $-x \in T(\mu)$ and $x + y \in \text{Tof}(\mu)$ are adjacent vertices in $T(\Gamma(\mu))$ since $-x + (x + y) = y \in T(\mu)$. Finally, the "in particular" statement follows from Lemma 3.1.

(iii) By part (i) above, it suffices to show that there is a path from *x* to *y* in $T(\Gamma(\mu))$ for any $x \in T(\mu)$ and $y \in \text{Tof}(\mu)$. By part (ii) above, there exist adjacent vertices *c* and *d* in $\text{Tor}(\Gamma(\mu))$ and $\text{Tof}(\Gamma(\mu))$, respectively. Since $\text{Tor}(\Gamma(\mu))$ is connected, there is a path from *x* to *c* in $\text{Tor}(\Gamma(\mu))$, and, since $\text{Tof}(\Gamma(\mu))$ is connected, there is a path from *d* to *y* in $\text{Tof}(\Gamma(\mu))$. Then there is a path from *x* to *y* in $T(\Gamma(\mu))$ since *c* and *d* are adjacent in $T(\Gamma(\mu))$. Thus, $T(\Gamma(\mu))$ is connected.

Proposition 3.3. Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of M. If the identity of the ring R is a sum of n zero divisors, then every element of the M is the sum of at most $n \mu$ -torsion elements.

Proof. Let $x \in M$ and $r \in Z(R)$. We may assume that $\mu(x) \neq \mu(0_M)$. Then there is a nonzero element $b \in R$ such that rb = 0, so $\mu(b(rx)) = \mu((rb)x) = \mu(0_M)$ with $\mu(rx) \neq \mu(0_M)$. Therefore, if $x \in M$ and $r \in R$, then $rx \in T(\mu)$, so, for all $x \in M$, $1 = c_1 + \cdots + c_n$ implies that $x = c_1x + \cdots + c_nx$, as needed.

Theorem 3.4. Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of M. Then $T(\Gamma(\mu))$ is connected if and only if M is generated by its μ -torsion elements.

Proof. Let us first prove that the connectedness of the graph $T(\Gamma(\mu))$ implies that the module M is generated by its μ -torsion elements. Suppose that this is not true. Then there exists $x \in M$ which does not have a representation of the form $x = x_1 + \cdots + x_n$, where $x_i \in T(\mu)$. Moreover, $x \neq 0$ since $0 \in T(\mu)$. We show that there does not exist a path from 0 to x in $T(\Gamma(\mu))$. If $0 - y_1 - y_2 - \cdots - y_m - x$ is a path in $T(\Gamma(\mu))$, $y_1, y_1 + y_2, \ldots, y_{m-1} + y_m, y_m + x$ are μ -torsion elements and x may be represented as $x = (y_m + x) - (y_{m-1} + y_m) + \cdots + (-1)^{m-1}(y_1 + y_2) + (-1)^m y_1$. This contradicts the assumption that x is not a sum of μ -torsion elements. The reverse implication may be proved in a similar way as in [6, Theorem 3.2].

We give here with an interesting result linking the *L*-torsion element graph $T(\Gamma(\mu))$ to the total graph of a commutative ring $T(\Gamma(R))$.

Theorem 3.5. Let *M* be a module over a ring *R*, and let $\mu \in L(M)$. If $T(\Gamma(R))$ is connected, then $T(\Gamma(\mu))$ is a connected graph. In particular, $d_{\mu}(0, x) \leq d(0, 1)$ for every $x \in M$.

Proof. Note that, if $x \in M$ and $r \in Z(R)$, then $rm \in T(\mu)$ (see Proposition 3.3). Now suppose that $T(\Gamma(R))$ is connected, and let $x \in M$. Let $0 - s_1 - s_2 - \cdots - s_n - 1$ be a path from 0 to 1 in $T(\Gamma(R))$. Then $s_1, s_1 + s_2, \ldots, s_n + 1 \in Z(R)$; hence, $0_M - s_1x - \cdots - s_nx - x$ is a path from 0_M to x. As all vertices may be connected via $0_M, T(\Gamma(\mu))$ is connected.

Theorem 3.6. Let M be a module over a ring R, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of M. If every element of M is a sum of at most n μ -torsion elements, then diam $(T(\Gamma(\mu))) \leq n$. If n is the smallest such number, then diam $(T(\Gamma(\mu))) = n$.

Proof. We first show that, by assumption, $d_{\mu}(0, x) \leq n$ for every nonzero element x of M. Assume that $x = x_1 + \cdots + x_n$, where $x_i \in T(\mu)$. Set $y_i = (-1)^{n+i}(x_1 + \cdots + x_n)$ for $i = 1, \ldots, n$. Then $0 - y_1 - y_2 - \cdots - y_n = x$ is a path from 0 to x of length n in $T(\Gamma(\mu))$. Let u and w be distinct elements in *M*. We show that $d_{\mu}(u, w) \leq n$. If $(u - w) - z_1 - z_2 - \cdots - z_{n-1}$ is a path from 0 to u - w and $u + w - s_1 - s_2 - \cdots - s_{n-1}$ is a path from 0 to u + w, then, from the previous discussion, the lengths of both paths are at most *n*. Depending on the fact whether *n* is even or odd, we obtain the paths

$$u - (z_1 - w) - (z_2 + w) - \dots - (z_{n-1} - w) - w$$
(3.1)

or $u - (s_1 + w) - (s_2 - w) - \cdots - (s_{n-1} - w) - w$ from u to w of length n. Assume that n is the smallest such number, and let $a = a_1 + a_2 + \cdots + a_n$ be the shortest representation of the elements x as a sum of μ -torsion elements. From the previous discussion, we have $d_{\mu}(0, x) \le n$. Suppose that $d_{\mu}(0, x) = k \le n$, and let $0 - t_1 - t_2 - \cdots - t_{k-1} - x$ be a path in $T(\Gamma(\mu))$. It means, a presentation of the element x as a sum of $k < n \mu$ -torsion elements (see the proof of Theorem 3.4), which is a contradiction. This completes the proof.

Corollary 3.7. Let *M* be a module over a ring *R*, and let $\mu \in L(M)$ such that Z(R) is not an ideal of *R* and < Z(R) >= R. If diam $T((\Gamma(R))) = n$, then diam $T((\Gamma(\mu))) \leq n$. In particular, if *R* is finite, then diam $T((\Gamma(\mu))) \leq 2$.

Proof. This follows from Proposition 3.3 and Theorem 3.6. Finally, if *R* is a finite ring such that Z(R) is not an ideal of *R*, then diam $T((\Gamma(R))) = 2$ by [3, Theorem 3.4], as required.

By Lemma 3.1, the following theorem may be proved in a similar way as in [6, Theorem 3.5].

Theorem 3.8. Let *M* be a module over a ring *R*, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of *M*. Then one has the following.

- (i) Either $\operatorname{gr}(\operatorname{Tor}(\Gamma(\mu))) = 3 \text{ or } \operatorname{gr}(\operatorname{Tor}(\Gamma(\mu))) = \infty$.
- (ii) $\operatorname{gr}(T(\Gamma(\mu))) = 3$ if and only if $\operatorname{gr}(\operatorname{Tor}(\Gamma(\mu))) = 3$.
- (iii) If $\operatorname{gr}(T(\Gamma(\mu))) = 4$, then $\operatorname{gr}(\operatorname{Tor}(\Gamma(\mu))) = \infty$.
- (iv) If $\operatorname{Char}(R) \neq 2$, then $\operatorname{gr}(\operatorname{Tof}(\Gamma(\mu))) = 3, 4 \text{ or } \infty$.

Example 3.9. Let $M = R = Z_6$ denote the ring of integers modulo 6. We define the mapping $\mu: M \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \overline{0}, \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$
(3.2)

Then $\mu \in L(M)$ and $T(\mu) = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}\}$. Now one can easily show that $T(\mu)$ is not a submodule of M and $\text{Tof}(\mu) = \{\overline{1}, \overline{5}\}$. Clearly, $\text{Tor}(\Gamma(\mu))$ is connected with diam $(\text{Tor}(\Gamma(\mu))) = 2$. Moreover, since $\overline{1} + \overline{3} \in T(\mu)$, we conclude that the subgraphs $\text{Tof}(\Gamma(\mu))$ and $\text{Tor}(\Gamma(\mu))$ of $T(\Gamma(\mu))$ are not disjoint. Furthermore, $T(\Gamma(\mu))$ is connected since $\text{Tof}(\Gamma(\mu))$ is connected.

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