## Research Article

# The $L$-Total Graph of an $L$-Module 

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Let $L$ be a complete lattice. We introduce and investigate the $L$-total graph of an $L$-module over an $L$-commutative ring. The main purpose of this paper is to extend the definition and results given in (Anderson and Badawi, 2008) to more generalize the $L$-total graph of an $L$-module case.

## 1. Introduction

It was Beck (see [1]) who first introduced the notion of a zero-divisor graph for commutative rings. This notion was later redefined by Anderson and Livingston in [2]. Since then, there has been a lot of interest in this subject, and various papers were published establishing different properties of these graphs as well as relations between graphs of various extensions (see [2$5])$. Let $R$ be a commutative ring with $Z(R)$ being its set of zero-divisors elements. The total graph of $R$, denoted by $T(\Gamma(R)$ ), is the (undirected) graph with all elements of $R$ as vertices, and, for distinct $x, y \in R$, the vertices $x$ and $y$ are adjacent if and only if $x+y \in Z(R)$. The total graph of a commutative ring has been introduced and studied by Anderson and Badawi in [3]. In [6], the notion of the total torsion element graph of a module over a commutative ring is introduced.

In [7], Zadeh introduced the concept of fuzzy set, which is a very useful tool to describe the situation in which the data is imprecise or vague. Many researchers used this concept to generalize some notions of algebra. Goguen in [8] generalized the notion of fuzzy subset of $X$ to that of an $L$-subset, namely, a function from $X$ to a lattice $L$. In [9], Rosenfeld considered the fuzzification of algebraic structures. Liu [10] introduced and examined the notion of a fuzzy ideal of a ring. Since then several authors have obtained interesting results on $L$-ideals of a ring $R$ and $L$-modules (see [11, 12]). Also, $L$-zero-divisor graph of an $L$ commutative ring has been introduced and studied in [13].

In the present paper we introduce a new class of graphs, called the $L$-total torsion element graph of a $L$-module (see Definition 2.2), and we completely characterize the structure of this graph. The total torsion element graph of a module over a commutative ring and the $L$-total torsion element graph of a $L$-module over a $L$-commutative ring are different concepts. Some of our results are analogous to the results given in [6]. The corresponding results are obtained by modification, and here we give a complete description of the $L$-total torsion element graph of an $L$-module.

For the sake of completeness, we state some definitions and notation used throughout. For a graph $\Gamma$, by $E(\Gamma)$ and $V(\Gamma)$, we denote the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices $a$ and $b$, denoted by $d(a, b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a, a)=0$ and $d(a, b)=\infty)$. The diameter of a graph $\Gamma$, denoted by $\operatorname{diam}(\Gamma)$, is equal to $\sup \{d(a, b): a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph $\Gamma$, denoted $\operatorname{gr}(\Gamma)$, is the length of the shortest cycle in $\Gamma$, provided $\Gamma$ contains a cycle; otherwise, $\operatorname{gr}(\Gamma)=\infty$. We denote the complete graph on $n$ vertices by $K^{n}$ and the complete bipartite graph on $m$ and $n$ vertices by $K^{m, n}$ (we allow $m$ and $n$ to be infinite cardinals). We will sometimes call a $K^{1, m}$ a star graph. We say that two (induced) subgraphs $\Gamma_{1}$ and $\Gamma_{2}$ of $\Gamma$ are disjoint if $\Gamma_{1}$ and $\Gamma_{2}$ have no common vertices and no vertex of $\Gamma_{1}$ (resp., $\Gamma_{2}$ ) is adjacent (in $\Gamma$ ) to any vertex not in $\Gamma_{1}$ (resp., $\Gamma_{2}$ ).

Let $R$ be a commutative ring, and $L$ stands for a complete lattice with least element 0 and greatest element 1 . By an $L$-subset $\mu$ of a nonempty set $X$, we mean a function $\mu$ from $X$ to $L$. If $L=[0,1]$, then $\mu$ is called a fuzzy subset of $X . L^{X}$ denotes the set of all $L$-subsets of $X$. We recall some definitions and lemmas from the book [12], which we need for development of our paper.

Definition 1.1. An $L$-ring is a function $\mu: R \rightarrow L$, where $(R,+,$.$) is a ring, which satisfies the$ following.
(1) $\mu \neq 0$.
(2) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$ for every $x, y$ in $R$.
(3) $\mu(x y) \geq \mu(x) \vee \mu(y)$ for every $x, y$ in $R$.

Definition 1.2. Let $\mu \in L^{R}$. Then $\mu$ is called an L-ideal of $R$ if for every $x, y \in R$ the following conditions are satisfied.
(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$.
(2) $\mu(x y) \geq \mu(x) \vee \mu(y)$.

The set of all $L$-ideals of $R$ is denoted by $L I(R)$.
Definition 1.3. Assume that $M$ is an $R$-module, and let $\mu \in L^{M}$. Then $\mu$ is called an $L$-fuzzy $R$-module of $M$ if for all $x, y \in M$ and for all $r \in R$ the following conditions are satisfied.
(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$.
(2) $\mu(r x) \geq \mu(x)$.
(3) $\mu\left(0_{M}\right)=\mu(1)$.

The set of all $L$-fuzzy $R$-modules of $M$ is denoted by $L(M)$.

Lemma 1.4. Let $M$ be a module over a ring $R$, and $\mu \in L(M)$. Then $\mu(m) \leq \mu\left(0_{M}\right)$ for every $m \in M$.

## 2. $T(\mu)$ Is a Submodule of $M$

Let $M$ be a module over a commutative ring $R$, and let $\mu \in L(M)$. The structure of the $L$-total torsion element graph $T(\Gamma(\mu))$ may be completely described in those cases when $\mu$-torsion elements form a submodule of $M$. We begin with the key definition of this paper.

Definition 2.1. Let $M$ be a module over a commutative ring $R$, and let $\mu \in L(M)$. A $\mu$-torsion element is an element $m \in M$ with $\mu(m) \neq \mu\left(0_{M}\right)$ for which there exists a nonzero element $r$ of $R$ such that $\mu(r m)=\mu\left(0_{M}\right)$.

The set of $\mu$-torsion elements in $M$ will be denoted by $T(\mu)$.
Definition 2.2. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$. We define the $L$-total torsion element graph of an L-module $T(\Gamma(\mu))$ as follows: $V(T(\Gamma(\mu)))=M, E(T(\Gamma(\mu)))=$ $\{\{x, y\}: x+y \in T(\mu)\}$.

Notation 1. For the $\mu$-torsion element graph $T(\Gamma(\mu))$, we denote the diameter, the girth, and the distance between two distinct vertices $a$ and $b$, by diam $(T(\Gamma(\mu))), \operatorname{gr}(T(\Gamma(\mu)))$, and $d_{\mu}(a, b)$, respectively.

Remark 2.3. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$. Clearly, if $\mu$ is a nonzero constant, then $T(\Gamma(\mu))=\emptyset$. So throughout this paper, we will assume, unless otherwise stated, that $\mu$ is not a nonzero constant. Thus, there is a nonzero element $y$ of $M$ such that $\mu(y) \neq \mu\left(0_{M}\right)$.

We will use $\operatorname{Tof}(\mu)$ to denote the set of elements of $M$ that are not $\mu$-torsion elements. Let $\operatorname{Tof}(\Gamma(\mu))$ be the (induced) subgraph of $T(\Gamma(\mu))$ with vertices $\operatorname{Tof}(\mu)$, and let $\operatorname{Tor}(\Gamma(\mu))$ be the (induced) subgraph of $T(\Gamma(\mu))$ with vertices $T(\mu)$.

Definition 2.4. Let $M$ be a module over a ring $R$, and $\mu \in L(M)$. One defines the set $\operatorname{ann}_{\mu}(M)$ by $\operatorname{ann}_{\mu}(M)=\left\{r \in R: \mu(r M)=\left\{\mu\left(0_{M}\right)\right\}\right\}$, the $\mu$-annihilator of $M$.

Lemma 2.5. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$. Then ann $\mu_{\mu}(M)$ is an L-ideal of $R$.
Proof. Let $r, s \in \operatorname{ann}_{\mu}(M)$ and $t \in R$. If $m \in M$, then we have $\mu((r-s) m) \geq \mu(r m) \wedge \mu(-s m)=$ $\mu\left(0_{M}\right) \wedge \mu\left(0_{M}\right)=\mu\left(0_{M}\right)$ and $\mu(\operatorname{trm})=\mu(t(r m)) \geq \mu(r m)=\mu\left(0_{M}\right)$. It then follows from Lemma 1.4 that $\mu((r-s) m)=\mu\left(0_{M}\right)$; hence $r-s \in \operatorname{ann}_{\mu}(M)$. Similarly, $r t \in \operatorname{ann}_{\mu}(M)$.

Theorem 2.6. Let $M$ be a module over a ring $R$ and let $\mu \in L(M)$. Then the $L$-torsion element graph $T(\Gamma(\mu))$ is complete if and only if $T(\mu)=M$.

Proof. If $T(\mu)=M$, then for any vertices $m, m^{\prime} \in M$, one has $m+m^{\prime} \in T(\mu)$; hence they are adjacent in $T(\Gamma(\mu))$. On the other hand, if $T(\Gamma(\mu))$ is complete, then every vertex is adjacent to 0 . Thus, $m=m+0 \in T(\mu)$ for every $m \in M$. This completes the proof.

Theorem 2.7. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is a submodule of M. Then one has the following.
(i) $\operatorname{Tor}(\Gamma(\mu))$ is a complete (induced) subgraph of $T(\Gamma(\mu))$ and $\operatorname{Tor}(\Gamma(\mu))$ is disjoint from $\operatorname{Tof}(\Gamma(\mu))$.
(ii) If $\operatorname{ann}_{\mu}(M) \neq 0$, then $T(\Gamma(\mu))$ is a complete graph.

Proof. (i) $\operatorname{Tor}(\Gamma(\mu))$ is complete directly from the definition. Finally, if $m \in T(\mu)$ and $m^{\prime} \in$ $\operatorname{Tof}(\mu)$ were adjacent, then $m+m^{\prime} \in T(\mu)$; so this, since $T(\mu)$ is a submodule, would lead to the contradiction $m^{\prime} \in T(\mu)$.
(ii) Let $m \in M$. we may assume that $\mu(m) \neq \mu\left(0_{M}\right)$. By assumption, there exists $0 \neq s \in$ $R$ with $\mu(s M)=\mu\left(0_{M}\right)$, so $\mu(s m)=\mu\left(0_{M}\right)$. Thus $m \in T(\mu)$, and; therefore, $T(\Gamma(\mu))$ is a complete graph by Theorem 2.6.

Theorem 2.8. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$. Then $T(\Gamma(\mu))$ is totally disconnected if and only if $R$ has characteristic 2 and $T(\mu)=\left\{0_{M}\right\}$.

Proof. If $T(\mu)=\left\{0_{M}\right\}$, then the vertices $m_{1}$ and $m_{2}$ are adjacent if and only if $m_{1}=-m_{2}$. Then $T(\Gamma(\mu))$ is a disconnected graph, and its only edges are those that connect vertices $m_{i}$ and $-m_{i}$ (we do not need a priori assumption that $R$ has characteristic 2 ). Conversely, assume that $T(\Gamma(\mu))$ is totally disconnected. Then $0+m \notin T(\mu)$ for every nonzero element $m$ of $M$. Thus, $T(\mu)=\left\{0_{M}\right\}$. Further, since $m+(-m)=0$, we have $m=-m\left(\right.$ so $\left.\mu(2 m)=\mu\left(0_{M}\right)\right)$ for every $m \in M$ with $\mu(m) \neq \mu\left(0_{M}\right)$ by the total disconnectedness of the graph $T(\Gamma(\mu))$. As $T(\mu)=\left\{0_{M}\right\}$, it follows that $2=1_{R}+1_{R}=0$. Thus, $\operatorname{char}(R)=2$.

Proposition 2.9. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is a submodule of $M$. If $m \in \operatorname{Tof}(\mu)$, then $2 m \in T(\mu)$ if and only if $2 \in Z(R)$.

Proof. First suppose that $2 m \in T(\mu)$. Since $m \notin T(\mu)$, we get that $\mu(m) \neq \mu\left(0_{M}\right)$, and, for all $r \in R, \mu(r m)=\mu\left(0_{M}\right)$ implies that $r=0$. Since $2 m \in T(\mu)$, there is a nonzero element $c \in R$ such that $\mu(c(2 m))=\mu((2 c) m)=\mu\left(0_{M}\right)$, and, since $m \notin T(\mu)$, one must have $2 c=0$; hence, $2 \in Z(R)$. Conversely, assume that $2 \in Z(R)$. Then there exists $0 \neq d \in R$ with $2 d=0$. Since $\mu\left(0_{M}\right)=\mu((2 d) m)=\mu(d(2 m))$, we have $2 m \in T(\mu)$.

Theorem 2.10. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of $M$. Then $T(\Gamma(\mu))$ is disconnected.

Proof. If $T(\mu)=\left\{0_{M}\right\}$, then $T(\Gamma(\mu))$ is disconnected by Theorem 2.8. If $T(\mu) \neq\left\{0_{M}\right\}$, then the subgraphs of $\operatorname{Tor}(\Gamma(\mu))$ and $\operatorname{Tof}(\Gamma(\mu))$ are disjoint by Theorem 2.7 (i), as required.

Theorem 2.11. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of $M$. Suppose $|T(\mu)|=\alpha$ and $|M / T(\mu)|=\beta$. Then one has the following.
(i) If $2 \in Z(R)$, then $T(\Gamma(\mu))$ is a union of $\beta$ disjoint complete graphs $K^{\alpha}$.
(ii) If $2 \notin Z(R)$, then $T(\Gamma(\mu))$ is a union of $(\beta-1) / 2$ disjoint bipartite graphs $K^{\alpha, \alpha}$ and one complete graph $K^{\alpha}$.

Proof. (i) Assume that $2 \in Z(R)$ and let $m, m^{\prime} \in \operatorname{Tof}(\mu)$ be such that $m+T(\mu) \neq m^{\prime}+T(\mu)$. The elements $m+t, m+t^{\prime}$ from the same coset $m+T(\mu)$ are adjacent if and only if $2 m \in T(\mu)$, so $2 \in Z(R)$, according to the Proposition 2.9. Then $m+t$ and $m^{\prime}+t^{\prime}$ are not adjacent (otherwise,
we would have $\left.m-m^{\prime}=m+m^{\prime}-2 m^{\prime} \in T(\mu)\right)$, and; therefore, $m+T(\mu)=m^{\prime}+T(\mu)$. Since every coset has cardinality $\alpha$, we conclude that $T(\Gamma(\mu))$ is the disjoint union of $\beta$ complete graph $K^{\alpha}$.
(ii) If $2 \notin Z(R)$, then the elements $m+t, m+t^{\prime}$ from $m+T(\mu)$ are obviously not adjacent. The elements $m+t, m^{\prime}+t^{\prime}$ from different cosets are adjacent if and only if $m+m^{\prime} \in T(\mu)$ or $m+T(\mu)=(-m)+T(\mu)$. In this way we obtain that the subgraph spanned by the vertices from $\operatorname{Tof}(\mu)$ is a disjoint union of $(\beta-1) / 2$ (= $\beta$ if $\beta$ is infinite) disjoint bipartite graph $K^{\alpha, \alpha}$.

Proposition 2.12. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of $M$. Then one has the following.
(i) $\operatorname{Tof}(\Gamma(\mu))$ is complete if and only if either $|M / T(\mu)|=2$ or $|M / T(\mu)|=|M|=3$.
(ii) $\operatorname{Tof}(\Gamma(\mu))$ is connected if and only if either $|M / T(\mu)|=2$ or $|M / T(\mu)|=3$.
(iii) $\operatorname{Tof}(\Gamma(\mu))$ and, hence; $(\operatorname{Tor}(\Gamma(\mu))$ and $T(\Gamma(\mu)))$ is totally disconnected if and only if $T(\mu)=\left\{0_{M}\right\}$ and $2 \in Z(R)$.

Proof. Let $|M / T(\mu)|=\beta$ and $|T(\mu)|=\alpha$.
(i) Let $\operatorname{Tof}(\Gamma(\mu))$ be complete. Then, by Theorem 2.11, $\operatorname{Tof}(\Gamma(\mu))$ is complete if and only if $\operatorname{Tof}(\Gamma(\mu))$ is a single $K^{\alpha}$ or $K^{1,1}$. If $2 \in Z(R)$, then $\beta-1=1$. Thus, $\beta=2$, and hence $|M / T(\mu)|=2$. If $2 \notin Z(R)$, then $\alpha=1$ and $(\beta-1) / 2=1$. Thus, $T(\mu)=\{0\}$ and $\beta=3$; hence, $|M|=|M / T(\mu)|=3$. The reverse implication may be proved in a similar way as in [6, Theorem 2.6 (1)].
(ii) By theorem 2.11, $\operatorname{Tof}(\Gamma(\mu))$ is connected if and only if $\operatorname{Tof}(\Gamma(\mu))$ is a single $K^{\alpha}$ or $K^{\alpha, \alpha}$. Thus, either $\beta-1=1$ if $2 \in Z(R)$ or $(\beta-1) / 2=1$ if $2 \notin Z(R)$; hence, $\beta=2$ or $\beta=3$, respectively, as needed. The reverse implication may be proved in a similar way as in [3, Theorem 2.6 (2)].
(iii) $\operatorname{Tof}(\Gamma(\mu))$ is totally disconnected if and only if it is a disjoint union of $K^{1}$ s. So by Theorem 2.11, $|T(\mu)|=1$ and $|M / T(\mu)|=1$, and the proof is complete.

By the proof of the Proposition 2.12, the next theorem gives a more explicit description of the diameter of $\operatorname{Tof}(\Gamma(\mu))$.

Theorem 2.13. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of $M$. Then one has the following.
(i) $\operatorname{diam}(\operatorname{Tof}(\Gamma(\mu)))=0$ if and only if $T(\mu)=\{0\}$ and $|M|=2$.
(ii) $\operatorname{diam}(\operatorname{Tof}(\Gamma(\mu)))=1$ if and only if either $T(\mu) \neq\left\{0_{M}\right\}$ and $|M / T(\mu)|=2$ or $T(\mu)=\{0\}$ and $|M|=3$.
(iii) $\operatorname{diam}(\operatorname{Tof}(\Gamma(\mu)))=2$ if and only if $T(\mu) \neq\left\{0_{M}\right\}$ and $|M / T(\mu)|=3$.
(iv) Otherwise, $\operatorname{diam}(\operatorname{Tof}(\Gamma(\mu)))=\infty$.

Proposition 2.14. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of $M$. Then $\operatorname{gr}(\operatorname{Tof}(\Gamma(\mu)))=3,4$ or $\infty$. In particular, $\operatorname{gr}(\operatorname{Tof}(\Gamma(\mu))) \leq 4$ if $\operatorname{Tof}(\Gamma(\mu))$ contains a cycle.

Proof. Let $\operatorname{Tof}(\Gamma(\mu))$ contain a cycle. Then since $\operatorname{Tof}(\Gamma(\mu))$ is disjoint union of either complete or complete bipartite graphs by Theorem 2.11, it must contain either a 3 cycles or a 4 cycles. Thus $\operatorname{gr}(\operatorname{Tof}(\Gamma(\mu))) \leq 4$.

Theorem 2.15. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is a proper submodule of $M$. Then one has the following.
(i) (a) $\operatorname{gr}(\operatorname{Tof}(\Gamma(\mu)))=3$ if and only if $2 \in Z(R)$ and $|T(\mu)| \geq 3$.
(b) $\operatorname{gr}(\operatorname{Tof}(\Gamma(\mu)))=4$ if and only if $2 \notin Z(R)$ and $|T(\mu)| \geq 2$.
(c) Otherwise, $\operatorname{gr}(\operatorname{Tof}(\Gamma(\mu)))=\infty$.
(ii) (a) $\operatorname{gr}(T(\Gamma(\mu)))=3$ if and only if $|T(\mu)| \geq 3$.
(b) $\operatorname{gr}(T(\Gamma(\mu)))=4$ if and only if $2 \notin Z(R)$ and $|T(\mu)|=2$.
(c) Otherwise, $\operatorname{gr}(T(\Gamma(\mu)))=\infty$.

Proof. Apply Theorem 2.11, Proposition 2.14, and Theorem 2.7 (i).
The previous theorems give a complete description of the structure of the $L$-total torsion element graph of an $L$-module $M$ when $T(\mu)$ is a submodule. The question under what conditions $T(\mu)$ is a submodule of $M$ and how is this related to the condition that $Z(R)$ is an ideal in $R$ naturally arises. We prove that the following results holds.

Theorem 2.16. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$. Then one has the following.
(i) If $Z(R)=\left\{0_{R}\right\}$, then $T(\mu)$ is a submodule of $M$.
(ii) If $Z(R)=R c$ is a principal ideal of $R$ with $c$ a nilpotent element of $R$, then $T(\mu)$ is a submodule of $M$.

Proof. (i) Let $m, m^{\prime} \in T(\mu)$ and $r \in R$. There are nonzero elements $a, b \in R$ such that $\mu(m) \neq \mu\left(0_{M}\right), \mu\left(m^{\prime}\right) \neq \mu\left(0_{M}\right)$, and $\mu(a m)=\mu\left(b m^{\prime}\right)=\mu\left(0_{M}\right)$ with $a b \neq 0$ (since $R$ is an integral domain). It follows that $\mu\left(a b\left(m+m^{\prime}\right)\right) \geq \mu(a b m) \wedge \mu\left(a b m^{\prime}\right)=\mu\left(0_{M}\right) \wedge \mu\left(0_{M}\right)=\mu\left(0_{M}\right)$; hence, $\mu\left(a b\left(m+m^{\prime}\right)\right)=\mu\left(0_{M}\right)$ by Lemma 1.4. Thus, $m+m^{\prime} \in T(\mu)$. Similarly, $r m \in T(\mu)$, and this completes the proof.
(ii) Assume that $T(\mu)$ is not a submodule of $M$. Then there are elements $m, m^{\prime} \in T(\mu)$ such that $m+m^{\prime} \notin T(\mu)$. By assumption, there exist nonzero elements $r, s \in R$ such that $\mu(r m)=\mu\left(0_{M}\right)=\mu\left(s m^{\prime}\right)=\mu\left(0_{M}\right)$, where $\mu(m) \neq \mu\left(0_{M}\right)$ and $\mu\left(m^{\prime}\right) \neq \mu\left(0_{M}\right)$. Then $\mu(r s(m+$ $\left.\left.m^{\prime}\right)\right)=\mu\left(0_{M}\right)$ and $m+m^{\prime} \notin T(\mu)$, so we must have $r s=0$, and; thus, $r, s \in Z(R)$. Since $c$ is nilpotent, we have $r=r_{1} c^{t}$ and $s=s_{1} c^{u}$, for some $r_{1}, s_{1} \notin Z(R)$. We may assume that $t \geq u$. Then for the nonzero element $s_{1} r$ of $R$ we have $\mu\left(s_{1} r\left(m+m^{\prime}\right)\right)=\mu\left(0_{M}\right)$ which is contrary to the assumption that $m+m^{\prime} \notin T(\mu)$.

Example 2.17. Assume that $R=\mathbb{Z}$ is the ring integers, and let $M=R$. We define the mapping $\mu: M \rightarrow[0,1]$ by

$$
\mu(m)= \begin{cases}\frac{1}{2} & \text { if } x \in 2 \mathbb{Z}  \tag{2.1}\\ \frac{1}{5} & \text { otherwise }\end{cases}
$$

Then $\mu \in L(M)$ and $T(\mu)=M$. Thus, $T(\Gamma(\mu))$ is a complete graph by Theorem 2.6.

Example 2.18. Let $M_{1}=R_{1}=Z_{8}$ denote the ring of integers modulo 8 and $M_{2}=R_{2}=Z_{25}$ the ring of integers modulo 25 . We define the mappings $\mu_{1}: M_{1} \rightarrow[0,1]$ by

$$
\mu_{1}(x)= \begin{cases}1 & \text { if } x=\overline{0}  \tag{2.2}\\ \frac{1}{2} & \text { otherwise }\end{cases}
$$

and $\mu_{2}: M_{2} \rightarrow[0,1]$ by

$$
\mu_{2}(m)= \begin{cases}1 & \text { if } x=\overline{0},  \tag{2.3}\\ \frac{1}{3} & \text { otherwise }\end{cases}
$$

Then, for each $i(1 \leq i \leq 2), \mu_{i} \in L\left(M_{i}\right), T\left(\mu_{1}\right)=\{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$, and $T\left(\mu_{2}\right)=\{\overline{0}, \overline{5}, \overline{1} 0, \overline{1} 5, \overline{2} 0\}$. An inspection will show that $T\left(\mu_{1}\right)$ and $T\left(\mu_{2}\right)$ are submodules of $M_{1}$ and $M_{2}$, respectively. Therefore, by Theorem 2.11, we have the following results.
(1) Since $2 \in Z\left(R_{1}\right)$, we conclude that $T\left(\Gamma\left(\mu_{1}\right)\right)$ is a union of 2 disjoint $K^{4}$.
(2) Since $2 \notin Z\left(R_{2}\right)$, we conclude that $T\left(\Gamma\left(\mu_{2}\right)\right)$ is a disjoint union of 2 complete graph $K^{5}$ and 5 bipartite $K^{5,5}$.

## 3. $T(\mu)$ Is Not a Submodule of $M$

We continue to use the notation already established, so $M$ is a module over a commutative ring $R$ and $\mu \in L(M)$. In this section, we study the $L$-torsion element graph $T(\Gamma(\mu))$ when $T(\mu)$ is not a submodule of $M$.

Lemma 3.1. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of $M$. Then there are distinct $m, m^{\prime} \in T(\mu)^{*}$ such that $m+m^{\prime} \in \operatorname{Tof}(\mu)$.

Proof. It suffices to show that $T(\mu)$ is always closed under scalar multiplication of its elements by elements of $R$. Let $m \in T(\mu)$ and $r \in R$. There is a nonzero element $s \in R$ with $\mu(s m)=$ $\mu\left(0_{M}\right)$ such that $\mu(m) \neq \mu\left(0_{M}\right)$, so $\mu(s(r m))=\mu(r(s m)) \geq \mu(s m)=\mu\left(0_{M}\right)$; hence, $\mu(s(r m))=$ $\mu\left(0_{M}\right)$ by Lemma 1.4 , as required.

Theorem 3.2. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of $M$. Then one has the following.
(i) $\operatorname{Tor}(\Gamma(\mu))$ is connected with diam $(\operatorname{Tor}(\Gamma(\mu)))=2$.
(ii) Some vertex of $\operatorname{Tor}(\Gamma(\mu))$ is adjacent to a vertex of $\operatorname{Tof}(\Gamma(\mu))$. In particular, the subgraphs $\operatorname{Tor}(\Gamma(\mu))$ and $\operatorname{Tof}(\Gamma(\mu))$ of $T(\Gamma(\mu))$ are not disjoint.
(iii) If $\operatorname{Tof}(\Gamma(\mu))$ is connected, then $T(\Gamma(\mu))$ is connected.

Proof. (i) Let $x \in T(\mu)^{*}$. Then $x$ is adjacent to 0 . Thus, $x-0-y$ is a path in $\operatorname{Tor}(\Gamma(\mu))$ of length two between any two distinct $x, y \in T(\mu)^{*}$. Moreover, there exist nonadjacent $x, y \in T(\mu)^{*}$ by Lemma 3.1; thus, $\operatorname{diam}(\operatorname{Tor}(\Gamma(\mu)))=2$.
(ii) By Lemma 3.1, there exist distinct $x, y \in T(\mu)^{*}$ such that $x+y \in \operatorname{Tof}(\mu)$. Then $-x \in T(\mu)$ and $x+y \in \operatorname{Tof}(\mu)$ are adjacent vertices in $T(\Gamma(\mu))$ since $-x+(x+y)=y \in T(\mu)$. Finally, the "in particular" statement follows from Lemma 3.1.
(iii) By part (i) above, it suffices to show that there is a path from $x$ to $y$ in $T(\Gamma(\mu))$ for any $x \in T(\mu)$ and $y \in \operatorname{Tof}(\mu)$. By part (ii) above, there exist adjacent vertices $c$ and $d$ in $\operatorname{Tor}(\Gamma(\mu))$ and $\operatorname{Tof}(\Gamma(\mu))$, respectively. Since $\operatorname{Tor}(\Gamma(\mu))$ is connected, there is a path from $x$ to $c$ in $\operatorname{Tor}(\Gamma(\mu))$, and, since $\operatorname{Tof}(\Gamma(\mu))$ is connected, there is a path from $d$ to $y$ in $\operatorname{Tof}(\Gamma(\mu))$. Then there is a path from $x$ to $y$ in $T(\Gamma(\mu))$ since $c$ and $d$ are adjacent in $T(\Gamma(\mu))$. Thus, $T(\Gamma(\mu))$ is connected.

Proposition 3.3. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of $M$. If the identity of the ring $R$ is a sum of $n$ zero divisors, then every element of the $M$ is the sum of at most $n \mu$-torsion elements.

Proof. Let $x \in M$ and $r \in Z(R)$. We may assume that $\mu(x) \neq \mu\left(0_{M}\right)$. Then there is a nonzero element $b \in R$ such that $r b=0$, so $\mu(b(r x))=\mu((r b) x)=\mu\left(0_{M}\right)$ with $\mu(r x) \neq \mu\left(0_{M}\right)$. Therefore, if $x \in M$ and $r \in R$, then $r x \in T(\mu)$, so, for all $x \in M, 1=c_{1}+\cdots+c_{n}$ implies that $x=c_{1} x+\cdots+c_{n} x$, as needed.

Theorem 3.4. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of $M$. Then $T(\Gamma(\mu))$ is connected if and only if $M$ is generated by its $\mu$-torsion elements.

Proof. Let us first prove that the connectedness of the graph $T(\Gamma(\mu))$ implies that the module $M$ is generated by its $\mu$-torsion elements. Suppose that this is not true. Then there exists $x \in M$ which does not have a representation of the form $x=x_{1}+\cdots+x_{n}$, where $x_{i} \in T(\mu)$. Moreover, $x \neq 0$ since $0 \in T(\mu)$. We show that there does not exist a path from 0 to $x$ in $T(\Gamma(\mu))$. If $0-y_{1}-y_{2}-\cdots-y_{m}-x$ is a path in $T(\Gamma(\mu)), y_{1}, y_{1}+y_{2}, \ldots, y_{m-1}+y_{m}, y_{m}+x$ are $\mu$-torsion elements and $x$ may be represented as $x=\left(y_{m}+x\right)-\left(y_{m-1}+y_{m}\right)+\cdots+(-1)^{m-1}\left(y_{1}+\right.$ $\left.y_{2}\right)+(-1)^{m} y_{1}$. This contradicts the assumption that $x$ is not a sum of $\mu$-torsion elements. The reverse implication may be proved in a similar way as in [6, Theorem 3.2].

We give here with an interesting result linking the $L$-torsion element graph $T(\Gamma(\mu))$ to the total graph of a commutative ring $T(\Gamma(R))$.

Theorem 3.5. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$. If $T(\Gamma(R))$ is connected, then $T(\Gamma(\mu))$ is a connected graph. In particular, $d_{\mu}(0, x) \leq d(0,1)$ for every $x \in M$.

Proof. Note that, if $x \in M$ and $r \in Z(R)$, then $r m \in T(\mu)$ (see Proposition 3.3). Now suppose that $T(\Gamma(R))$ is connected, and let $x \in M$. Let $0-s_{1}-s_{2}-\cdots-s_{n}-1$ be a path from 0 to 1 in $T(\Gamma(R))$. Then $s_{1}, s_{1}+s_{2}, \ldots, s_{n}+1 \in Z(R)$; hence, $0_{M}-s_{1} x-\cdots-s_{n} x-x$ is a path from $0_{M}$ to $x$. As all vertices may be connected via $0_{M}, T(\Gamma(\mu))$ is connected.

Theorem 3.6. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of $M$. If every element of $M$ is a sum of at most $n \mu$-torsion elements, then $\operatorname{diam}(T(\Gamma(\mu))) \leq n$. If $n$ is the smallest such number, then $\operatorname{diam}(T(\Gamma(\mu)))=n$.

Proof. We first show that, by assumption, $d_{\mu}(0, x) \leq n$ for every nonzero element $x$ of $M$. Assume that $x=x_{1}+\cdots+x_{n}$, where $x_{i} \in T(\mu)$. Set $y_{i}=(-1)^{n+i}\left(x_{1}+\cdots+x_{n}\right)$ for $i=1, \ldots, n$. Then $0-y_{1}-y_{2}-\cdots-y_{n}=x$ is a path from 0 to $x$ of length $n$ in $T(\Gamma(\mu))$. Let $u$ and $w$ be
distinct elements in $M$. We show that $d_{\mu}(u, w) \leq n$. If $(u-w)-z_{1}-z_{2}-\cdots-z_{n-1}$ is a path from 0 to $u-w$ and $u+w-s_{1}-s_{2}-\cdots-s_{n-1}$ is a path from 0 to $u+w$, then, from the previous discussion, the lengths of both paths are at most $n$. Depending on the fact whether $n$ is even or odd, we obtain the paths

$$
\begin{equation*}
u-\left(z_{1}-w\right)-\left(z_{2}+w\right)-\cdots-\left(z_{n-1}-w\right)-w \tag{3.1}
\end{equation*}
$$

or $u-\left(s_{1}+w\right)-\left(s_{2}-w\right)-\cdots-\left(s_{n-1}-w\right)-w$ from $u$ to $w$ of length $n$. Assume that $n$ is the smallest such number, and let $a=a_{1}+a_{2}+\cdots+a_{n}$ be the shortest representation of the elements $x$ as a sum of $\mu$-torsion elements. From the previous discussion, we have $d_{\mu}(0, x) \leq n$. Suppose that $d_{\mu}(0, x)=k \leq n$, and let $0-t_{1}-t_{2}-\cdots-t_{k-1}-x$ be a path in $T(\Gamma(\mu))$. It means, a presentation of the element $x$ as a sum of $k<n \mu$-torsion elements (see the proof of Theorem 3.4), which is a contradiction. This completes the proof.

Corollary 3.7. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $Z(R)$ is not an ideal of $R$ and $<Z(R)>=R$. If $\operatorname{diam} T((\Gamma(R)))=n$, then $\operatorname{diam} T((\Gamma(\mu))) \leq n$. In particular, if $R$ is finite, then $\operatorname{diam} T((\Gamma(\mu))) \leq 2$.

Proof. This follows from Proposition 3.3 and Theorem 3.6. Finally, if $R$ is a finite ring such that $Z(R)$ is not an ideal of $R$, then $\operatorname{diam} T((\Gamma(R)))=2$ by [3, Theorem 3.4], as required.

By Lemma 3.1, the following theorem may be proved in a similar way as in [6, Theorem 3.5].

Theorem 3.8. Let $M$ be a module over a ring $R$, and let $\mu \in L(M)$ such that $T(\mu)$ is not a submodule of $M$. Then one has the following.
(i) Either $\operatorname{gr}(\operatorname{Tor}(\Gamma(\mu)))=3$ or $\operatorname{gr}(\operatorname{Tor}(\Gamma(\mu)))=\infty$.
(ii) $\operatorname{gr}(T(\Gamma(\mu)))=3$ if and only if $\operatorname{gr}(\operatorname{Tor}(\Gamma(\mu)))=3$.
(iii) If $\operatorname{gr}(T(\Gamma(\mu)))=4$, then $\operatorname{gr}(\operatorname{Tor}(\Gamma(\mu)))=\infty$.
(iv) If $\operatorname{Char}(R) \neq 2$, then $\operatorname{gr}(\operatorname{Tof}(\Gamma(\mu)))=3,4$ or $\infty$.

Example 3.9. Let $M=R=Z_{6}$ denote the ring of integers modulo 6 . We define the mapping $\mu: M \rightarrow[0,1]$ by

$$
\mu(x)= \begin{cases}1 & \text { if } x=\overline{0}  \tag{3.2}\\ \frac{1}{4} & \text { otherwise } .\end{cases}
$$

Then $\mu \in L(M)$ and $T(\mu)=\{\overline{0}, \overline{2}, \overline{3}, \overline{4}\}$. Now one can easily show that $T(\mu)$ is not a submodule of $M$ and $\operatorname{Tof}(\mu)=\{\overline{1}, \overline{5}\}$. Clearly, $\operatorname{Tor}(\Gamma(\mu))$ is connected with diam $(\operatorname{Tor}(\Gamma(\mu)))=2$. Moreover, since $\overline{1}+\overline{3} \in T(\mu)$, we conclude that the subgraphs $\operatorname{Tof}(\Gamma(\mu))$ and $\operatorname{Tor}(\Gamma(\mu))$ of $T(\Gamma(\mu))$ are not disjoint. Furthermore, $T(\Gamma(\mu))$ is connected since $\operatorname{Tof}(\Gamma(\mu))$ is connected.

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