International Scholarly Research Network
ISRN Applied Mathematics
Volume 2011, Article ID 491941, 19 pages
doi:10.5402/2011/491941

Research Article

Optimality and Duality for Nondifferentiable Minimax Fractional Programming with Generalized Convexity

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Received 15 March 2011; Accepted 2 May 2011

Academic Editors: P. J. García Nieto, G. Liu, and H. C. So

We establish several sufficient optimality conditions for a class of nondifferentiable minimax fractional programming problems from a view point of generalized convexity. Subsequently, these optimality criteria are utilized as a basis for constructing dual models, and certain duality results have been derived in the framework of generalized convex functions. Our results extend and unify some known results on minimax fractional programming problems.

1. Introduction

Several authors have been interested in the optimality conditions and duality results for minimax programming problems. Necessary and sufficient conditions for generalized minimax programming were developed first by Schmitendorf [1]. Tanimoto [2] defined a dual problem and derived duality theorems for convex minimax programming problems using Schmitendorf's results.

Yadav and Mukherjee [3] also employed the optimality conditions of Schmitendorf [1] to construct the two dual problems and derived duality theorems for differentiable fractional minimax programming problems. Chandra and Kumar [4] pointed out that the formulation of Yadav and Mukherjee [3] has some omissions and inconsistencies, and they constructed two new dual problems and proved duality theorems for differentiable fractional minimax programming. Liu et al. [5, 6], Liang and Shi [7], and Yang and Hou [8] paid much attention on minimax fractional programming problem and established sufficient optimality conditions and duality results.

Lai et al. [9] derived necessary and sufficient conditions for nondifferentiable minimax fractional problem with generalized convexity and applied these optimality conditions to construct one parametric dual model and also discussed duality theorems. Lai and Lee [10]

obtained duality theorems for two parameter-free dual models of a nondifferentiable minimax fractional programming problem, involving generalized convexity assumptions. Ahmad and Husain [11, 12] established sufficient optimality conditions and duality theorems for nondifferentiable minimax fractional programming problem under (F, α, ρ, d) convexity assumptions, thus extending the results of Lai et al. [9] and Lai and Lee [10]. Jayswal [13] discussed the optimality conditions and duality results for nondifferentiable minimax fractional programming under α-univexity. Yuan et al. [14] introduced the concept of generalized (C, α, ρ, d)-convexity and focused their study on a nondifferentiable minimax fractional programming problems. Recently, Jayswal and Kumar [15] established sufficient optimality conditions and duality theorems for a class of nondifferentiable minimax fractional programming involving (C, α, ρ, d)-convexity.

In the present paper, we discuss the sufficient optimality conditions for a nondifferentiable minimax fractional programming problem from a view point of generalized convexity. Subsequently, we apply the optimality conditions to formulate a dual problem, and we prove weak, strong and strict converse duality theorems involving generalized convexity.

The paper is organized as follows. In Section 2, we present a few definitions and notations and recall a set of necessary optimality conditions for a nondifferentiable minimax fractional programming problem which will be needed in the sequel. In Section 3, we discussed sufficient optimality conditions with somewhat limited structures of generalized convexity. Furthermore, a dual problem is formulated and duality results are presented in Section 4. Finally, in Section 5, we summarize our main results and also point out some additional research opportunities arising from certain modifications of the principal problem model considered in this paper.

2. Notations and Preliminaries

Let R^n denote the n-dimensional Euclidean space and let R^n_+ be its nonnegative orthant.

In this paper, we consider the following nondifferentiable minimax fractional programming problem:

$$\begin{aligned} \min_{x\in R^n} \quad & \sup_{y\in Y} \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{h(x,y) - \langle x, Bx\rangle^{1/2}} \\ \text{subject to} \quad & g(x) \le 0, \end{aligned} \tag{P}$$

where $f, h : R^n \times R^m \to R$ and $g : R^n \to R^p$ are continuous differentiable functions, Y is a compact subset of R^m, and A and B are $n \times n$ positive semidefinite matrices. The problem (P) is nondifferentiable programming problem if either A or B is nonzero. If A and B are null matrices, then the problem (P) is a usual minimax fractional programming problem which was studied by Liang and Shi [7] and Yang and Hou [8].

Let $\Im_P = \{x \in R^n : g(x) \le 0\}$ be the set of all feasible solutions of (P). For each $(x, y) \in R^n \times R^m$, we define

$$\phi(x,y) = \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{h(x,y) - \langle x, Bx\rangle^{1/2}}. \tag{2.1}$$

Assume that for each $(x, y) \in R^n \times Y$, $f(x,y) + \langle x, Ax\rangle^{1/2} \ge 0$, and $h(x,y) - \langle x, Bx\rangle^{1/2} > 0$.

Denote

$$\overline{Y}(x) = \left\{ \overline{y} \in Y : \frac{f(x,\overline{y}) + \langle x, Ax\rangle^{1/2}}{h(x,\overline{y}) - \langle x, Bx\rangle^{1/2}} = \sup_{z\in Y} \frac{f(x,z) + \langle x, Ax\rangle^{1/2}}{h(x,z) - \langle x, Bx\rangle^{1/2}} \right\},$$

$$J = \{1,2,\ldots,p\},$$

$$J(x) = \{j \in J : g_j(x) = 0\}, \tag{2.2}$$

$$K(x) = \left\{ (s,t,\tilde{y}) \in N \times R_+^s \times R^{ms} : 1 \leqq s \leqq n+1,\ t = (t_1,t_2,\ldots,t_s) \in R_+^s \text{ with } \sum_{i=1}^{s} t_i = 1,\ \tilde{y} = (\overline{y}_1,\overline{y}_2\ldots,\overline{y}_s),\ \overline{y}_i \in \overline{Y}(x),\ i = 1,2,\ldots,s \right\}.$$

Since f and h are continuously differentiable and Y is a compact subset of R^m, it follows that for each $x^* \in \Im_P$, $\overline{Y}(x^*) \neq \phi$. Thus, for any $\overline{y}_i \in \overline{Y}(x^*)$, we have a positive constant $v^* = \phi(x^*,\overline{y}_i)$.

Definition 2.1. A functional $F : X \times X \times R^n \to R$ (where $X \subseteq R^n$) is said to be sublinear in its third argument, if for all $(x,x_0) \in X \times X$,

$$\begin{gathered} F(x,x_0;a_1+a_2) \leq F(x,x_0;a_1) + F(x,x_0;a_2), \quad \forall a_1,a_2 \in R^n, \\ F(x,x_0;\alpha a) = \alpha F(x,x_0;a), \quad \forall \alpha \in R,\ \alpha \geq 0,\ \forall a \in R^n. \end{gathered} \tag{2.3}$$

The following result from Lai and Lee [10] is needed in the sequel.

Lemma 2.2. *Let x^* be an optimal solution for* (P) *satisfying $\langle x^*, Ax^*\rangle > 0, \langle x^*, Bx^*\rangle > 0$ and let $\nabla g_j(x^*)$, $j \in J(x^*)$ be linearly independent, then there exist $(s,t^*,\tilde{y}) \in K(x^*)$, $v^* \in R_+$, $u,v \in R^n$ and $\mu^* \in R_+^p$ such that*

$$\sum_{i=1}^{s} t_i^* \left(\nabla f(x^*,\overline{y}_i) + Au - v^*\left(\nabla h(x^*,\overline{y}_i) - Bv\right)\right) + \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*) = 0, \tag{2.4}$$

$$f(x^*,\overline{y}_i) + \langle x^*, Ax^*\rangle^{1/2} - v^*\left(h(x^*,\overline{y}_i) - \langle x^*, Bx^*\rangle^{1/2}\right) = 0, \quad i = 1,2,\ldots,s, \tag{2.5}$$

$$\sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0, \tag{2.6}$$

$$t_i^* \in R_+^s, \quad \sum_{i=1}^{s} t_i^* = 1, \quad \overline{y}_i \in Y(x^*), \quad i = 1,2,\ldots,s, \tag{2.7}$$

$$\langle u, Au\rangle \leq 1, \qquad \langle v, Bv\rangle \leq 1, \qquad \langle x^*, Au\rangle = \langle x^*, Ax^*\rangle^{1/2}, \qquad \langle x^*, Bv\rangle = \langle x^*, Bx^*\rangle^{1/2}. \tag{2.8}$$

It should be noted that both the matrices A and B are positive definite at the solution x^* in the above Lemma. If one of $\langle Ax^*, x^*\rangle$ and $\langle Bx^*, x^*\rangle$ is zero, or both A and B are singular

at x^*, then for $(s, t^*, \tilde{y}) \in K(x^*)$, we can take $Z_{\tilde{y}}(x^*)$ defined in Lai and Lee [10] by

$Z_{\tilde{y}}(x^*) = \{z \in R^n : \langle \nabla g_j(x^*), z\rangle \le 0,\ j \in J(x^*)\}$ with any one of the following (i)–(iii) holds

$$\begin{aligned}
&\text{(i)}\quad \langle Ax^*, x^*\rangle > 0, \qquad \langle Bx^*, x^*\rangle = 0\\
&\qquad \Longrightarrow \left\langle \sum_{i=1}^{s} t_i^* \nabla f(x^*, \overline{y}_i) + \frac{Ax^*}{\langle Ax^*, x^*\rangle^{1/2}} - v^* \nabla h(x^*, \overline{y}_i), z \right\rangle + \left\langle \left(v^{*2} B\right) z, z\right\rangle^{1/2} < 0,\\
&\text{(ii)}\quad \langle Ax^*, x^*\rangle = 0, \qquad \langle Bx^*, x^*\rangle > 0\\
&\qquad \Longrightarrow \left\langle \sum_{i=1}^{s} t_i^* \left(\nabla f(x^*, \overline{y}_i) - v^* \left(\nabla h(x^*, \overline{y}_i) - \frac{Bx^*}{\langle Bx^*, x^*\rangle^{1/2}} \right)\right), z \right\rangle + \langle Bz, z\rangle^{1/2} < 0,\\
&\text{(iii)}\quad \langle Ax^*, x^*\rangle = 0, \qquad \langle Bx^*, x^*\rangle = 0\\
&\qquad \Longrightarrow \left\langle \sum_{i=1}^{s} t_i^* (\nabla f(x^*, \overline{y}_i) - v^* \nabla h(x^*, \overline{y}_i)), z \right\rangle + \langle (v^* B) z, z\rangle^{1/2} + \langle Bz, z\rangle^{1/2} < 0.
\end{aligned} \tag{2.9}$$

If we take the condition $Z_{\tilde{y}}(x^*) = \phi$ in Lemma 2.2, then the result of Lemma 2.2 still holds.

3. Sufficient Optimality Conditions

In this section, we present three sets of sufficient optimality conditions for (P) in the framework of generalized convexity.

Let $F : X \times X \times R^n \to R$ be sublinear functional, $\phi_0, \phi_1 : R \to R$, $\theta : R^n \times R^n \to R^n$, and $b_0, b_1 : X \times X \to R_+$. Let ρ_0, ρ_1 be real numbers.

Theorem 3.1. *Let $x^* \in \mathfrak{I}_P$ be a feasible solution for* (P)*, and there exist $v^* \in R_+$, $(s, t^*, \tilde{y}) \in K(x^*)$, $u, v \in R^n$, and $\mu^* \in R_+^p$ satisfying* (2.4)–(2.8)*. Suppose that there exist $F, \theta, \phi_0, b_0, \rho_0$ and ϕ_1, b_1, ρ_1 such that*

$$\begin{aligned}
&F\left(x, x^*; \sum_{i=1}^{s} t_i^* \left((\nabla f(x^*, \overline{y}_i) + Au) - v^* (\nabla h(x^*, \overline{y}_i) - Bv) \right)\right) \ge -\rho_0 \|\theta(x, x^*)\|^2\\
&\Longrightarrow b_0(x, x^*) \phi_0 \left[\sum_{i=1}^{s} t_i^* \left(f(x, \overline{y}_i) + \langle x, Au\rangle - v^* (h(x, \overline{y}_i) - \langle x, Bv\rangle) \right) \right.\\
&\qquad \left. - \sum_{i=1}^{s} t_i^* \left(f(x^*, \overline{y}_i) + \langle x^*, Au\rangle - v^* (h(x^*, \overline{y}_i) - \langle x^*, Bv\rangle) \right) \right] \ge 0,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&-b_1(x, x^*) \phi_1 \left(\sum_{j=1}^{p} \mu_j^* g_j(x^*) \right) \le 0\\
&\Longrightarrow F\left(x, x^*; \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right) \le -\rho_1 \|\theta(x, x^*)\|^2.
\end{aligned} \tag{3.2}$$

Further, assume that

$$a \geq 0 \Longrightarrow \phi_1(a) \geq 0, \tag{3.3}$$

$$\phi_0(a) \geq 0 \Longrightarrow a \geq 0, \tag{3.4}$$

$$b_0(x, x^*) \geq 0, \qquad b_1(x, x^*) > 0, \tag{3.5}$$

$$\rho_0 + \rho_1 \geq 0, \tag{3.6}$$

then x^* is an optimal solution of (P).

Proof. Suppose to the contrary that x^* is not an optimal solution of (P), then there exists $x \in \mathfrak{I}_P$ such that

$$\sup_{y \in Y} \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{h(x,y) - \langle x, Bx\rangle^{1/2}} < \sup_{y \in Y} \frac{f(x^*,y) + \langle x^*, Ax^*\rangle^{1/2}}{h(x^*,y) - \langle x^*, Bx^*\rangle^{1/2}}. \tag{3.7}$$

We note that

$$\sup_{y \in Y} \frac{f(x^*,y) + \langle x^*, Ax^*\rangle^{1/2}}{h(x^*,y) - \langle x^*, Bx^*\rangle^{1/2}} = \frac{f(x^*,\overline{y}_i) + \langle x^*, Ax^*\rangle^{1/2}}{h(x^*,\overline{y}_i) - \langle x^*, Bx^*\rangle^{1/2}} = v^*, \tag{3.8}$$

for $\overline{y}_i \in \overline{Y}(x^*)$, $i = 1, 2, \ldots, s$,

$$\frac{f(x,\overline{y}_i) + \langle x, Ax\rangle^{1/2}}{h(x,\overline{y}_i) - \langle x, Bx\rangle^{1/2}} \leq \sup_{y \in Y} \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{h(x,y) - \langle x, Bx\rangle^{1/2}}. \tag{3.9}$$

Thus, we have

$$\frac{f(x,\overline{y}_i) + \langle x, Ax\rangle^{1/2}}{h(x,\overline{y}_i) - \langle x, Bx\rangle^{1/2}} < v^* \quad \text{for } i = 1, 2, \ldots, s. \tag{3.10}$$

It follows that

$$f(x,\overline{y}_i) + \langle x, Ax\rangle^{1/2} - v^*\left(h(x,\overline{y}_i) - \langle x, Bx\rangle^{1/2}\right) < 0, \quad \text{for } i = 1, 2, \ldots, s. \tag{3.11}$$

From (2.5), (2.7), (2.8), and (3.11), we get

$$\begin{aligned} &\sum_{i=1}^{s} t_i^* \left(f(x,\overline{y}_i) + \langle x, Au\rangle - v^*\left(h(x,\overline{y}_i) - \langle x, Bv\rangle\right)\right) \\ &\quad < \sum_{i=1}^{s} t_i^* \left(f(x^*,\overline{y}_i) + \langle x^*, Au\rangle - v^*\left(h(x^*,\overline{y}_i) - \langle x^*, Bv\rangle\right)\right). \end{aligned} \tag{3.12}$$

On the other hand, from (2.6), (3.3), and (3.5), we have

$$-b_1(x,x^*)\phi_1\left(\sum_{j=1}^{p}\mu_j^* g_j(x^*)\right)\leq 0. \tag{3.13}$$

It follows from (3.2) that

$$F\left(x,x^*;\sum_{j=1}^{p}\mu_j^*\nabla g_j(x^*)\right)\leq -\rho_1\|\theta(x,x^*)\|^2. \tag{3.14}$$

From (2.4), the sublinearity of F, and (3.6), we get

$$F\left(x,x^*;\sum_{i=1}^{s}t_i^*\left(\nabla f(x^*,\overline{y}_i)+Au-v^*\left(\nabla h(x^*,\overline{y}_i)-Bv\right)\right)\right)\geq -\rho_0\|\theta(x,x^*)\|^2. \tag{3.15}$$

Then by (3.1), we have

$$\begin{aligned} b_0(x,x^*)\phi_0\Bigg[&\sum_{i=1}^{s}t_i^*\left(f(x,\overline{y}_i)+\langle x,Au\rangle-v^*\left(h(x,\overline{y}_i)-\langle x,Bv\rangle\right)\right)\\ &-\sum_{i=1}^{s}t_i^*\left(f(x^*,\overline{y}_i)+\langle x^*,Au\rangle-v^*\left(h(x^*,\overline{y}_i)-\langle x^*,Bv\rangle\right)\right)\Bigg]\geq 0. \end{aligned} \tag{3.16}$$

From (3.4), (3.5), and the above inequality, we obtain

$$\begin{aligned} &\sum_{i=1}^{s}t_i^*\left(f(x,\overline{y}_i)+\langle x,Au\rangle-v^*\left(h(x,\overline{y}_i)-\langle x,Bv\rangle\right)\right)\\ &\quad-\sum_{i=1}^{s}t_i^*\left(f(x^*,\overline{y}_i)+\langle x^*,Au\rangle-v^*\left(h(x^*,\overline{y}_i)-\langle x^*,Bv\rangle\right)\right)\geq 0, \end{aligned} \tag{3.17}$$

which contradicts (3.12). Therefore, x^* is an optimal solution for (P). This completes the proof. □

Remark 3.2. If both A and B are zero matrices, then Theorem 3.1 above reduces to Theorem 3.1 given in Yang and Hou [8].

Theorem 3.3. *Let $x^*\in\mathfrak{I}_P$ be a feasible solution for* (P), *and there exist $v^*\in R_+$, $(s,t^*,\widetilde{y})\in K(x^*)$, $u,v\in R^n$, and $\mu^*\in R_+^p$ satisfying* (2.4)–(2.8). *Suppose that there exist $F,\theta,\phi_0,b_0,\rho_0$ and ϕ_1,b_1,ρ_1*

such that

$$F\left(x,x^*;\sum_{i=1}^{s}t_i^*\left(\nabla f\left(x^*,\overline{y}_i\right)+Au-v^*\left(\nabla h\left(x^*,\overline{y}_i\right)-Bv\right)\right)\right)\geq-\rho_0\|\theta(x,x^*)\|^2$$
$$\Longrightarrow b_0(x,x^*)\phi_0\left[\sum_{i=1}^{s}t_i^*\left(f\left(x,\overline{y}_i\right)+\langle x,Au\rangle-v^*\left(h\left(x,\overline{y}_i\right)-\langle x,Bv\rangle\right)\right)\right.$$
$$\left.-\sum_{i=1}^{s}t_i^*\left(f\left(x^*,\overline{y}_i\right)+\langle x^*,Au\rangle-v^*\left(h\left(x^*,\overline{y}_i\right)-\langle x^*,Bv\rangle\right)\right)\right]>0, \tag{3.18}$$

or equivalently,

$$b_0(x,x^*)\phi_0\left[\sum_{i=1}^{s}t_i^*\left(f\left(x,\overline{y}_i\right)+\langle x,Au\rangle-v^*\left(h\left(x,\overline{y}_i\right)-\langle x,Bv\rangle\right)\right)\right.$$
$$\left.-\sum_{i=1}^{s}t_i^*\left(f\left(x^*,\overline{y}_i\right)+\langle x^*,Au\rangle-v^*\left(h\left(x^*,\overline{y}_i\right)-\langle x^*,Bv\rangle\right)\right)\right]\leq 0 \tag{3.19}$$

$$\Longrightarrow F\left(x,x^*;\sum_{i=1}^{s}t_i^*\left(\nabla f\left(x^*,\overline{y}_i\right)+Au-v^*\left(\nabla h\left(x^*,\overline{y}_i\right)-Bv\right)\right)\right)<-\rho_0\|\theta(x,x^*)\|^2,$$
$$-b_1(x,x^*)\phi_1\left(\sum_{j=1}^{p}\mu_j^*g_j(x^*)\right)\leq 0$$
$$\Longrightarrow F\left(x,x^*;\sum_{j=1}^{p}\mu_j^*\nabla g_j(x^*)\right)\leq-\rho_1\|\theta(x,x^*)\|^2. \tag{3.20}$$

Further, assume that (3.3), (3.5), (3.6), *and*

$$a\leq 0\Longrightarrow\phi_0(a)\leq 0, \tag{3.21}$$

are satisfied, then x^* is an optimal solution of (P).

Proof. Suppose to the contrary that x^* is not an optimal solution of (P). Following the proof of Theorem 3.1, we get

$$\sum_{i=1}^{s}t_i^*\left(f\left(x,\overline{y}_i\right)+\langle x,Au\rangle-v^*\left(h\left(x,\overline{y}_i\right)-\langle x,Bv\rangle\right)\right)$$
$$<\sum_{i=1}^{s}t_i^*\left(f\left(x^*,\overline{y}_i\right)+\langle x^*,Au\rangle-v^*\left(h\left(x^*,\overline{y}_i\right)-\langle x^*,Bv\rangle\right)\right). \tag{3.22}$$

Using (3.5), (3.19), (3.21), and (3.22), we have

$$F\left(x,x^*;\sum_{i=1}^{s}t_i^*\left(\nabla f\left(x^*,\overline{y}_i\right)+Au-v^*\left(\nabla h\left(x^*,\overline{y}_i\right)-Bv\right)\right)\right)<-\rho_0\|\theta(x,x^*)\|^2. \tag{3.23}$$

By (2.6), (3.3), (3.5), and (3.20), it follows that

$$F\left(x, x^*; \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right) \leq -\rho_1 \|\theta(x, x^*)\|^2. \tag{3.24}$$

On adding (3.23) and (3.24), and making use of the sublinearity of F and (3.6), we have

$$\begin{aligned} &F\left(x, x^*; \sum_{i=1}^{s} t_i^* \left(\nabla f(x^*, \overline{y}_i) + Au - v^*\left(\nabla h(x^*, \overline{y}_i) - Bv\right)\right)\right) + F\left(x, x^*; \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right) \\ &\quad \leq -(\rho_0 + \rho_1)\|\theta(x, x^*)\|^2 < 0 \\ &\Longrightarrow F\left(x, x^*; \sum_{i=1}^{s} t_i^* \left(\nabla f(x^*, \overline{y}_i) + Au - v^*\left(\nabla h(x^*, \overline{y}_i) - Bv\right) + \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right)\right) < 0. \end{aligned} \tag{3.25}$$

On the other hand, (2.4) implies

$$F\left(x, x^*; \sum_{i=1}^{s} t_i^* \left(\nabla f(x^*, \overline{y}_i) + Au - v^*\left(\nabla h(x^*, \overline{y}_i) - Bv\right) + \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right)\right) = 0. \tag{3.26}$$

Hence we have a contradiction to inequality (3.25). Therefore, x^* is an optimal solution for (P). This completes the proof. □

Theorem 3.4. *Let $x^* \in \mathfrak{I}_P$ be a feasible solution for* (P) *and there exist $v^* \in R_+$, $(s, t^*, \widetilde{y}) \in K(x^*)$, $u, v \in R^n$, and $\mu^* \in R_+^p$ satisfying* (2.4)–(2.8). *Suppose that there exist $F, \theta, \phi_0, b_0, \rho_0$ and ϕ_1, b_1, ρ_1 such that*

$$\begin{aligned} &b_0(x, x^*)\phi_0\left[\sum_{i=1}^{s} t_i^* \left(f(x, \overline{y}_i) + \langle x, Au\rangle - v^*\left(h(x, \overline{y}_i) - \langle x, Bv\rangle\right)\right)\right. \\ &\qquad \left. - \sum_{i=1}^{s} t_i^* \left(f(x^*, \overline{y}_i) + \langle x^*, Au\rangle - v^*\left(h(x^*, \overline{y}_i) - \langle x^*, Bv\rangle\right)\right)\right] \leq 0 \\ &\Longrightarrow F\left(x, x^*; \sum_{i=1}^{s} t_i^* \left(\nabla f(x^*, \overline{y}_i) + Au - v^*\left(\nabla h(x^*, \overline{y}_i) - Bv\right)\right)\right) \leq -\rho_0 \|\theta(x, x^*)\|^2, \\ &F\left(x, x^*; \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right) \geq -\rho_1 \|\theta(x, x^*)\|^2 \\ &\Longrightarrow -b_1(x, x^*)\phi_1\left(\sum_{j=1}^{p} \mu_j^* g_j(x^*)\right) > 0. \end{aligned} \tag{3.27}$$

Further, assume that (3.3), (3.5), (3.6), *and* (3.21) *are satisfied, then x^* is an optimal solution of* (P).

Proof. The proof is similar to that of Theorem 3.3 and hence omitted. □

Remark 3.5. (i) If both A and B are zero matrices, then Theorems 3.3 and 3.4 above reduce to Theorems 3.3 given in Yang and Hou [8].

(ii) If $F(x,u;a) = \langle \eta(x,u), a\rangle$ where η is a function from $X \times X \to R^n$, $-\phi_1(\sum_{j=1}^{p} \mu_j^* g_j(x^*)) = \phi_1(\sum_{j=1}^{p} \mu_j^* g_j(x) - \mu_j^* g_j(x^*))$, and $\rho_0 = \rho_1 = 0$, then Theorems 3.3 and 3.4 above reduce to Theorems 1(b) and 1(c) given by Mishra et al. [16].

4. Duality

In this section, we present a dual model to (P) and establish weak, strong, and strict converse duality results.

To unify and extend the dual models, we need to divide $\{1,2,\ldots,p\}$ into several parts. Let $J_\alpha (0 \le \alpha \le r)$ be a partition of $\{1,2,\ldots p\}$, that is,

$$J_\alpha \cap J_\beta = \phi, \quad \text{for } \alpha \neq \beta, \quad \bigcup_{\alpha=0}^{r} J_\alpha = \{1,2,\ldots,p\}. \tag{4.1}$$

We note that for (P)-optimal x^*, (2.6) implies

$$\sum_{j\in J_\alpha} \mu_j^* g_j(x^*) = 0, \quad \alpha = 0,1,\ldots,r. \tag{4.2}$$

We now recast the necessary condition in Lemma 2.2 in the following form.

Lemma 4.1. *Let x^* be an optimal solution for* (P) *satisfying $\langle x^*, Ax^*\rangle > 0$, $\langle x^*, Bx^*\rangle > 0$ and let $\nabla g_j(x^*)$, $j \in J(x^*)$ be linearly independent, then there exist $(s,t^*,\tilde{y}) \in K(x^*)$, $u,v \in R^n$ and $\mu^* \in R_+^p$ such that*

$$\begin{aligned}&\left(\sum_{i=1}^{s} t_i^*\left(h(x^*,\overline{y}_i) - \langle x^*, Bx^*\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s} t_i^*(f(x^*,\overline{y}_i) + Au) + \sum_{j=1}^{p}\mu_j^* g_j(x^*)\right)\\ &\quad -\left(\sum_{i=1}^{s} t_i^*\left(f(x^*,\overline{y}_i) + \langle x^*, Ax^*\rangle^{1/2}\right) + \sum_{j\in J_0}\mu_j^* g_j(x^*)\right)\nabla\left(\sum_{i=1}^{s} t_i^*(h(x^*,\overline{y}_i) - Bv)\right) = 0,\end{aligned} \tag{4.3}$$

$$\sum_{j\in J_\alpha} \mu_j^* g_j(x^*) = 0, \quad \alpha = 1,2,\ldots,r, \tag{4.4}$$

$$\mu^* \in R_+^p t_i^* \ge 0, \quad \sum_{i=1}^{s} t_i^* = 1, \quad \overline{y}_i \in Y(x^*), \quad i = 1,2,\ldots,s, \tag{4.5}$$

where J_α $(0 \le \alpha \le r)$ is a partition of $\{1,2,\ldots p\}$.

Proof. It suffices to establish (4.3). From (2.4) and (2.5),

$$\begin{aligned}&\left(\nabla\sum_{i=1}^{s} t_i^*(f(x^*,\overline{y}_i) + Au)\right) - \frac{f(x^*,\overline{y}_i) + \langle x^*, Ax^*\rangle^{1/2}}{h(x^*,\overline{y}_i) - \langle x^*, Bx^*\rangle^{1/2}}\left(\nabla\sum_{i=1}^{s} t_i^*(h(x^*,\overline{y}_i) - Bv)\right)\\ &\quad + \nabla\sum_{j=1}^{p}\mu_j^* g_j(x^*) = 0, \quad i = 1,2,\ldots,s.\end{aligned} \tag{4.6}$$

Multiply the respective equation above by $t_i^*(h(x^*,\overline{y}_i)-\langle x^*,Bx^*\rangle^{1/2})$, $i=1,2,\ldots,s$ and add them together, we have

$$\begin{aligned}&\left(\sum_{i=1}^{s}t_i^*\left(h(x^*,\overline{y}_i)-\langle x^*,Bx^*\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s}t_i^*(f(x^*,\overline{y}_i)+Au)+\sum_{j=1}^{p}\mu_j^*g_j(x^*)\right)\\&\quad-\left(\sum_{i=1}^{s}t_i^*\left(f(x^*,\overline{y}_i)+\langle x^*,Ax^*\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s}t_i^*(h(x^*,\overline{y}_i)-Bv)\right)=0.\end{aligned}\tag{4.7}$$

The above equation together with (2.6) implies that

$$\begin{aligned}&\left(\sum_{i=1}^{s}t_i^*\left(h(x^*,\overline{y}_i)-\langle x^*,Bx^*\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s}t_i^*(f(x^*,\overline{y}_i)+Au)+\sum_{j=1}^{p}\mu_j^*g_j(x^*)\right)\\&\quad-\left(\sum_{i=1}^{s}t_i^*\left(f(x^*,\overline{y}_i)+\langle x^*,Ax^*\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j^*g_j(x^*)\right)\nabla\left(\sum_{i=1}^{s}t_i^*(h(x^*,\overline{y}_i)-Bv)\right)=0.\end{aligned}\tag{4.8}$$

Hence, the lemma is established. □

Our dual model is as follows:

$$\max_{(s,t,\tilde{y})\in K(z)}\ \sup_{(z,\mu,u,v)\in H(s,t,\tilde{y})}\left(\frac{\sum_{i=1}^{s}t_if(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}+\sum_{j\in J_0}\mu_jg_j(z)}{\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Az\rangle^{1/2}\right)}\right),\tag{D}$$

where $H(s,t,\tilde{y})$ denotes the set of all $(z,\mu,u,v)\in R^n\times R_+^n\times R^n\times R^n$ satisfying

$$\begin{aligned}&\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s}t_i(f(z,\overline{y}_i)+Au)+\sum_{j=1}^{p}\mu_jg_j(z)\right)\\&\quad-\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_jg_j(z)\right)\nabla\left(\sum_{i=1}^{s}t_i(h(z,\overline{y}_i)-Bv)\right)=0,\end{aligned}\tag{4.9}$$

$$\begin{aligned}&\sum_{j\in J_\alpha}\mu_jg_j(z)\geq 0,\quad \alpha=1,2,\ldots,r,\\&J_\alpha\cap J_\beta=\phi,\quad \text{for } \alpha\neq\beta,\quad \bigcup_{\alpha=0}^{r}J_\alpha=\{1,2,\ldots,p\}.\end{aligned}\tag{4.10}$$

Theorem 4.2 (weak duality). *Let x be a feasible solution for* (P), *and let $(z,\mu,u,v,s,t,\tilde{y})$ be a feasible solution for* (4.18). *Suppose that there exist $F,\theta,\phi_0,b_0,\rho_0$ and $\phi_\alpha,b_\alpha,\rho_\alpha$, $\alpha=1,2,\ldots,r$ such*

that

$$
\begin{aligned}
&F\Bigg(x,z;\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s}t_i(f(z,\overline{y}_i)+Au)+\sum_{j\in J_0}\mu_j g_j(z)\right)\\
&\quad-\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s}t_i(h(z,\overline{y}_i)-Bv)\right)\Bigg)\\
&\quad\geq-\rho_0\|\theta(x,z)\|^2\\
&\Longrightarrow b_0(x,z)\phi_0\Bigg(\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{i=1}^{s}t_i\left(f(x,\overline{y}_i)+\langle x,Ax\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(x)\right)\\
&\qquad-\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)g\\
&\qquad\times\left(\sum_{i=1}^{s}t_i\left(h(x,\overline{y}_i)-\langle x,Bx\rangle^{1/2}\right)\right)\Bigg)\geq 0,
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
&-b_\alpha(x,z)\phi_\alpha\left(\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{j\in J_\alpha}\mu_j g_j(z)\right)\right)\leq 0\\
&\Longrightarrow F\left(x,z;\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\left(\sum_{j\in J_\alpha}\mu_j\nabla g_j(z)\right)\right)\\
&\quad\leq-\rho_\alpha\|\theta(x,z)\|^2,\quad \alpha=1,2,\ldots,r.
\end{aligned}
\tag{4.12}
$$

Further, assume that

$$a\geq 0\Longrightarrow\phi_\alpha(a)\geq 0,\quad \alpha=1,2,\ldots,r, \tag{4.13}$$

$$\phi_0(a)\geq 0\Longrightarrow a\geq 0, \tag{4.14}$$

$$b_0(x,z)>0,\quad b_\alpha(x,z)\geq 0,\quad \alpha=1,2,\ldots,r, \tag{4.15}$$

$$\rho_0+\sum_{\alpha=1}^{r}\rho_\alpha\geq 0, \tag{4.16}$$

then

$$\sup_{y\in Y}\frac{f(x,y)+\langle x,Ax\rangle^{1/2}}{h(x,y)-\langle x,Bx\rangle^{1/2}}\geq\left(\frac{\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)}{\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)}\right). \tag{4.17}$$

Proof. Suppose to contrary that

$$\sup_{y\in Y}\frac{f(x,y)+\langle x,Ax\rangle^{1/2}}{h(x,y)-\langle x,Bx\rangle^{1/2}}<\left(\frac{\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)}{\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)}\right), \tag{4.18}$$

then, we get

$$\begin{aligned}&\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\left(f(x,y)+\langle x,Ax\rangle^{1/2}\right)\\&\quad<\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\left(h(x,y)-\langle x,Bx\rangle^{1/2}\right),\quad \forall y\in Y.\end{aligned} \tag{4.19}$$

Further, this implies

$$\begin{aligned}&\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{i=1}^{s}t_i\left(f(x,\overline{y}_i)+\langle x,Ax\rangle^{1/2}\right)\right)\\&\quad<\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s}t_i\left(h(x,\overline{y}_i)-\langle x,Bx\rangle^{1/2}\right)\right).\end{aligned} \tag{4.20}$$

Hence, we have

$$\begin{aligned}&\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{i=1}^{s}t_i\left(f(x,\overline{y}_i)+\langle x,Ax\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(x)\right)\\&\quad-\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s}t_i\left(h(x,\overline{y}_i)-\langle x,Bx\rangle^{1/2}\right)\right)\\&\quad<\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{j\in J_0}\mu_j g_j(x)\right).\end{aligned} \tag{4.21}$$

Using the fact that $(\sum_{i=1}^{s}t_i(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}))>0$ and $\sum_{j\in J_0}\mu_j g_j(x)\leq 0$ and the last inequality, we have

$$\begin{aligned}&\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{i=1}^{s}t_i\left(f(x,\overline{y}_i)+\langle x,Ax\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(x)\right)\\&\quad-\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s}t_i\left(h(x,\overline{y}_i)-\langle x,Bx\rangle^{1/2}\right)\right)<0.\end{aligned} \tag{4.22}$$

From (4.11),(4.14),(4.15), and (4.22), we get

$$\begin{aligned} &F\left(x,z;\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s}t_i(f(z,\overline{y}_i)+Au)+\sum_{j\in J_0}\mu_j g_j(z)\right)\right.\\ &\quad\left.-\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s}t_i(h(z,\overline{y}_i)-Bv)\right)\right)\\ &\quad<-\rho_0\|\theta(x,z)\|^2. \end{aligned} \tag{4.23}$$

Using $(\sum_{i=1}^{s}t_i(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}))>0$, (4.10), (4.13), and (4.15), we get

$$-b_\alpha(x,z)\phi_\alpha\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\left(\sum_{j\in J_\alpha}\mu_j g_j(z)\right)\right)\le 0,\quad \alpha=1,2,\ldots,r. \tag{4.24}$$

From (4.12), we have

$$\begin{aligned} &F\left(x,z;\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\left(\sum_{j\in J_\alpha}\mu_j\nabla g_j(z)\right)\right)\\ &\quad\le-\rho_\alpha\|\theta(x,z)\|^2,\quad \alpha=1,2,\ldots,r. \end{aligned} \tag{4.25}$$

On adding (4.23) and (4.25) and making use of sublinearity of F and (4.16), we have

$$\begin{aligned} &F\left(x,z;\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s}t_i(f(z,\overline{y}_i)+Au)+\sum_{j=1}^{p}\mu_j g_j(z)\right)\right.\\ &\quad\left.-\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s}t_i(h(z,\overline{y}_i)-Bv)\right)\right)<0, \end{aligned} \tag{4.26}$$

which contradicts (4.9). This completes the proof. □

Theorem 4.3 (weak duality). *Let x be a feasible solution for* (P) *and let $(z,\mu,u,v,s,t,\tilde{y})$ be a feasible solution for* (4.18). *Suppose that there exist $F,\theta,\phi_0,b_0,\rho_0$ and $\phi_\alpha,b_\alpha,\rho_\alpha$, $\alpha=1,2,\ldots,r$ such that*

$$\begin{aligned} &b_0(x,z)\phi_0\left(\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{i=1}^{s}t_i\left(f(x,\overline{y}_i)+\langle x,Ax\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(x)\right)\right.\\ &\quad-\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\\ &\quad\left.\times\left(\sum_{i=1}^{s}t_i\left(h(x,\overline{y}_i)-\langle x,Bx\rangle^{1/2}\right)\right)\right)<0 \end{aligned}$$

$$
\begin{aligned}
&\Longrightarrow F\left(x,z;\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s}t_i(f(z,\overline{y}_i)+Au)+\sum_{j\in J_0}\mu_j g_j(z)\right)\right.\\
&\qquad\left.-\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s}t_i(h(z,\overline{y}_i)-Bv)\right)\right)\\
&\qquad\le -\rho_0\|\theta(x,z)\|^2,\\
&\qquad -b_\alpha(x,z)\phi_\alpha\left(\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{j\in J_\alpha}\mu_j g_j(z)\right)\right)\le 0\\
&\Longrightarrow F\left(x,z;\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\left(\sum_{j\in J_\alpha}\mu_j\nabla g_j(z)\right)\right)\\
&\qquad<-\rho_\alpha\|\theta(x,z)\|^2,\alpha=1,2,\ldots,r.
\end{aligned}
\tag{4.27}
$$

Further, assume that (4.14), (4.15), *and* (4.16) *are satisfied, then*

$$
\sup_{y\in Y}\frac{f(x,y)+\langle x,Ax\rangle^{1/2}}{h(x,y)-\langle x,Bx\rangle^{1/2}}\ge\left(\frac{\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)}{\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)}\right).
\tag{4.28}
$$

Proof. The proof is similar to that of the above theorem and hence omitted. □

Theorem 4.4 (strong duality). *Assume that x^* is an optimal solution for* (P) *and $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent. Then there exist $(s^*,t^*,\tilde{y}^*)\in K(x^*)$ and $(x^*,\mu^*,u^*,v^*)\in H(s^*,t^*,\tilde{y}^*)$such that $(x^*,\mu^*,u^*,v^*,s^*,t^*,\tilde{y}^*)$ is an optimal solution for* (4.18). *If, in addition, the hypotheses of any of the weak duality (Theorem 4.2 or Theorem 4.3) holds for a feasible point $(z,\mu,u,v,s,t,\tilde{y})$, then the problems* (P) *and* (4.18) *have the same optimal values.*

Proof. By Lemma 4.1, there exist $(s^*,t^*,\tilde{y}^*)\in K(x^*)$ and $(x^*,\mu^*,u^*,v^*)\in H(s^*,t^*,\tilde{y}^*)$ such that $(x^*,\mu^*,u^*,v^*,s^*,t^*,\tilde{y}^*)$ is a feasible for (4.18), optimality of this feasible solution for (4.18) follows from Theorems 4.2 or 4.3 accordingly. □

Theorem 4.5 (strict converse duality). *Let x^* and $(z,\mu,u,v,s,t,\tilde{y})$ be optimal solutions for* (P) *and* (4.18), *respectively. Suppose that $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent and there exist $F,\theta,\phi_0,b_0,\rho_0$, and $\phi_\alpha,b_\alpha,\rho_\alpha,\alpha=1,2,\ldots,r$ such that*

$$
\begin{aligned}
&F\left(x^*,z;\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s}t_i(f(z,\overline{y}_i)+Au)+\sum_{j\in J_0}\mu_j g_j(z)\right)\right.\\
&\qquad\left.\times\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s}t_i(h(z,\overline{y}_i)-Bv)\right)\right)\\
&\qquad\ge-\rho_0\|\theta(x^*,z)\|^2
\end{aligned}
$$

$$\begin{aligned}
\Longrightarrow b_0(x^*,z)\phi_0\Bigg(&\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\\
&\times\left(\sum_{i=1}^{s}t_i\left(f(x^*,\overline{y}_i)+\langle x^*,Ax^*\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(x^*)\right)\\
&-\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\\
&\left(\sum_{i=1}^{s}t_i\left(h(x^*,\overline{y}_i)-\langle x^*,Bx^*\rangle^{1/2}\right)\right)\Bigg)\geq 0,
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
&-b_\alpha(x^*,z)\phi_\alpha\left(\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{j\in J_\alpha}\mu_j g_j(z)\right)\right)\leq 0\\
&\Longrightarrow F\left(x^*,z;\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\left(\sum_{j\in J_\alpha}\mu_j\nabla g_j(z)\right)\right)\\
&\quad\leq -\rho_\alpha\|\theta(x^*,z)\|^2,\quad \alpha=1,2,\ldots,r.
\end{aligned} \tag{4.30}$$

Further, assume (4.13), (4.15), (4.16),

$$\phi_0(a)\geq 0\Longrightarrow a>0, \tag{4.31}$$

then $x^*=z$, that is, z is an optimal solution for (P).

Proof. Suppose to contrary that $x^*\neq z$. From the strong duality Theorem 4.4, we know that

$$\sup_{y\in Y}\frac{f(x^*,y)+\langle x^*,Ax^*\rangle^{1/2}}{h(x^*,y)-\langle x^*,Bx^*\rangle^{1/2}}=\left(\frac{\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)}{\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)}\right). \tag{4.32}$$

Then, we get

$$\begin{aligned}
&\left(\sum_{i=1}^{s}t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(f(x^*,y)+\langle x^*,Ax^*\rangle^{1/2}\right)\\
&\quad\leq\left(\sum_{i=1}^{s}t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\left(h(x^*,y)-\langle x^*,Bx^*\rangle^{1/2}\right),\quad \forall y\in Y.
\end{aligned} \tag{4.33}$$

Further, this implies

$$\begin{aligned}
&\left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{i=1}^{s} t_i\left(f(x^*,\overline{y}_i)+\langle x^*,Ax^*\rangle^{1/2}\right)\right)\\
&\quad\le\left(\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\\
&\qquad\times\left(\sum_{i=1}^{s} t_i\left(h(x^*,\overline{y}_i)-\langle x^*,Bx^*\rangle^{1/2}\right)\right),\quad \forall y\in Y.
\end{aligned}\tag{4.34}$$

Hence, we have

$$\begin{aligned}
&\left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{i=1}^{s} t_i\left(f(x^*,\overline{y}_i)+\langle x^*,Ax^*\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(x^*)\right)\\
&\quad-\left(\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s} t_i\left(h(x^*,\overline{y}_i)-\langle x^*,Bx^*\rangle^{1/2}\right)\right)\\
&\quad\le\left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{j\in J_0}\mu_j g_j(x^*)\right).
\end{aligned}\tag{4.35}$$

Using the fact that $(\sum_{i=1}^{s} t_i(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}))>0$ and $\sum_{j\in J_0}\mu_j g_j(x^*)\le 0$ and the last inequality, we have

$$\begin{aligned}
&\left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\left(\sum_{i=1}^{s} t_i\left(f(x^*,\overline{y}_i)+\langle x^*,Ax^*\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(x^*)\right)\\
&\quad-\left(\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s} t_i\left(h(x^*,\overline{y}_i)-\langle x^*,Bx^*\rangle^{1/2}\right)\right)\le 0.
\end{aligned}\tag{4.36}$$

From (4.15), (4.29), (4.31), and (4.36), we get

$$\begin{aligned}
&F\left(x^*,z;\left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i)-\langle z,Bz\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s} t_i(f(z,\overline{y}_i)+Au)+\sum_{j\in J_0}\mu_j g_j(z)\right)\right.\\
&\qquad\left.-\left(\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i)+\langle z,Az\rangle^{1/2}\right)+\sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s} t_i(h(z,\overline{y}_i)-Bv)\right)\right)\\
&\quad<-\rho_0\|\theta(x^*,z)\|^2.
\end{aligned}\tag{4.37}$$

Using $(\sum_{i=1}^{s} t_i(h(z,\overline{y}_i) - \langle z, Bz\rangle^{1/2})) > 0$, (4.10), (4.13), and (4.15), we get

$$-b_\alpha(x^*,z)\phi_\alpha\left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i) - \langle z, Bz\rangle^{1/2}\right)\left(\sum_{j\in J_\alpha}\mu_j g_j(z)\right)\right) \le 0, \quad \alpha = 1,2,\ldots,r. \tag{4.38}$$

From (4.30), we have

$$\begin{aligned} &F\left(x^*, z; \sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i) - \langle z, Bz\rangle^{1/2}\right)\left(\sum_{j\in J_\alpha}\mu_j \nabla g_j(z)\right)\right) \\ &\quad \le -\rho_\alpha \|\theta(x^*,z)\|^2, \quad \alpha = 1,2,\ldots,r. \end{aligned} \tag{4.39}$$

On adding (4.37) and (4.39) and making use of sublinearity of F and (4.16), we have

$$\begin{aligned} &F\left(x^*, z; \left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i) - \langle z, Bz\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s} t_i(f(z,\overline{y}_i) + Au) + \sum_{j=1}^{p}\mu_j g_j(z)\right)\right. \\ &\quad \left. -\left(\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) + \langle z, Az\rangle^{1/2}\right) + \sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s} t_i(h(z,\overline{y}_i) - Bv)\right)\right) < 0, \end{aligned} \tag{4.40}$$

which contradicts (4.9). This completes the proof. □

Theorem 4.6 (strict converse duality). *Let x^* and $(z,\mu,u,v,s,t,\widetilde{y})$ be optimal solutions for* (P) *and* (4.18)*, respectively. Suppose that $\nabla g_j(x^*)$, $j \in J(x^*)$ are linearly independent and there exist $F,\theta,\phi_0,b_0,\rho_0$, and $\phi_\alpha, b_\alpha, \rho_\alpha$, $\alpha = 1,2,\ldots,r$ such that*

$$\begin{aligned} &b_0(x^*,z)\phi_0\left(\left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i) - \langle z, Bz\rangle^{1/2}\right)\right)\left(\sum_{i=1}^{s} t_i\left(f(x^*,\overline{y}_i) + \langle x^*, Ax^*\rangle^{1/2}\right) + \sum_{j\in J_0}\mu_j g_j(x^*)\right)\right. \\ &\quad \left. -\left(\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) + \langle z, Az\rangle^{1/2}\right) + \sum_{j\in J_0}\mu_j g_j(z)\right)\left(\sum_{i=1}^{s} t_i\left(h(x^*,\overline{y}_i) - \langle x^*, Bx^*\rangle^{1/2}\right)\right)\right) < 0 \\ &\Longrightarrow F\left(x^*, z; \left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i) - \langle z, Bz\rangle^{1/2}\right)\right)\nabla\left(\sum_{i=1}^{s} t_i(f(z,\overline{y}_i) + Au) + \sum_{j\in J_0}\mu_j g_j(z)\right)\right. \\ &\quad \left. -\left(\sum_{i=1}^{s} t_i\left(f(z,\overline{y}_i) + \langle z, Az\rangle^{1/2}\right) + \sum_{j\in J_0}\mu_j g_j(z)\right)\nabla\left(\sum_{i=1}^{s} t_i(h(z,\overline{y}_i) - Bv)\right)\right) \\ &\quad < -\rho_0\|\theta(x^*,z)\|^2, \\ &\quad - b_\alpha(x^*,z)\phi_\alpha\left(\left(\sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i) - \langle z, Bz\rangle^{1/2}\right)\right)\left(\sum_{j\in J_\alpha}\mu_j g_j(z)\right)\right) \le 0 \end{aligned}$$

$$\Longrightarrow F\left(x^*, z; \sum_{i=1}^{s} t_i\left(h(z,\overline{y}_i) - \langle z, Bz\rangle^{1/2}\right)\left(\sum_{j\in J_\alpha}\mu_j \nabla g_j(z)\right)\right)$$
$$\leq -\rho_\alpha \|\theta(x^*, z)\|^2, \quad \alpha = 1, 2, \ldots, r. \tag{4.41}$$

Further, assume (4.13), (4.15), (4.16), *and*

$$\phi_0(a) \geq 0 \Longrightarrow a > 0, \tag{4.42}$$

then $x^* = z$, *that is,* z *is an optimal solution for* (P).

Proof. The proof is similar to that of the above theorem and hence omitted. □

Remark 4.7. If both A and B are zero matrices, then Theorems 4.2–4.6 above reduce to Theorems 4.1–4.5 given in Yang and Hou [8].

5. Conclusion

In this paper, we have discussed optimality conditions and duality results for nondifferentiable minimax fractional programming problems under the assumptions of generalized convexity. It may be noted that previously known results of Yang and Hou [8] and Mishra et al. [16] appear as special cases of our results. The duality results developed in this paper can be further extended for second order on the lines of Ahmad et al. [17]. It will be interesting to see whether or not the sufficiency and duality results developed in this paper still hold for the following nondifferentiable minimax fractional programming problem:

$$\min_{x\in R^n} \sup_{y\in W} \frac{f(\xi,\nu) + \left(x^H A x\right)^{1/2}}{h(\xi,\nu) - \left(x^H B x\right)^{1/2}} \tag{CP}$$
$$\text{subject to} \quad -g(\xi) \in S, \ \xi \in C^{2n},$$

where $\xi = (z, \overline{z})$, $\nu = (w, \overline{w})$ for $z \in C^n$, $w \in C^m$.

$f(\cdot,\cdot) : C^{2n} \times C^{2m} \to C$ and $h(\cdot,\cdot) : C^{2n} \times C^{2m} \to C$ are analytic with respect to ξ, W is a specified compact subset in C^{2m}, S is a polyhedral cone in C^P, and $g : C^{2n} \to C^P$ is analytic. Also $A, B \in C^{n\times n}$ are positive semidefinite Hermitian matrices.

Acknowledgment

The authors are very grateful to the referees for their valuable suggestions, which helped in improving the paper significantly.

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