International Scholarly Research Network
ISRN Mathematical Analysis
Volume 2011, Article ID 529407, 17 pages
doi:10.5402/2011/529407

Research Article

Shrinking Projection Method of Common Fixed Point Problems for Discrete Asymptotically Strictly Pseudocontractive Semigroups and Mixed Equilibrium Problems in Hilbert Spaces

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Received 13 July 2011; Accepted 18 August 2011

Academic Editor: A. Levy

This paper is concerned with a common element of the set of common fixed point for a discrete asymptotically strictly pseudocontractive semigroup and the set of solutions of the mixed equilibrium problems in Hilbert spaces. The strong convergence theorem for the above two sets is obtained by a general iterative scheme based on the shrinking projection method which extends and improves the corresponding ones due to Kim [Proceedings of the Asian Conference on Nonlinear Analysis and Optimization (Matsue, Japan, 2008), 139–162].

1. Introduction

Throughout this paper, we always assume that C is a nonempty closed convex subset of a real Hilbert space H with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The *domain* of the function $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ is the set

$$\operatorname{dom} \varphi = \{x \in C : \varphi(x) < +\infty\}. \tag{1.1}$$

Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function, and let Φ be a bifunction from $C \times C$ into $\mathbb{R}$ such that $C \cap \operatorname{dom} \varphi \neq \emptyset$, where $\mathbb{R}$ is the set of real numbers. The so-called *mixed equilibrium problem* is to find $x \in C$ such that

$$\Phi(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of problem (1.2) is denoted by $MEP(\Phi, \varphi)$; that is,

$$MEP(\Phi, \varphi) = \{x \in C : \Phi(x, y) + \varphi(y) - \varphi(x) \geq, \ \forall y \in C\}. \tag{1.3}$$

It is obvious that if x is a solution of problem (1.2) then $x \in \operatorname{dom} \varphi$. As special cases of problem (1.2), we have the following.

(i) If $\varphi = 0$, then problem (1.2) is reduced to find $x \in C$ such that

$$\Phi(x, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

We denote by $EP(\Phi)$ the set of solutions of *equilibrium problem*, problem (1.4) which was studied by Blum and Oettli [1].

(ii) If $\Phi(x, y) = \langle Bx, y - x \rangle$ for all $x, y \in C$ where a mapping $B : C \to H$, then problem (1.4) is reduced to find $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

We denote by $VI(C, B)$ the set of solutions of *variational inequality problem*, problem (1.5) which was studied by Hartman and Stampacchia [2].

(iii) If $\Phi = 0$, then problem (1.2) is reduced to find $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.6}$$

We denote by $\operatorname{Argmin}(\varphi)$ the set of solutions of *minimize problem*.

Recall that P_C is the *metric projection* of H onto C; that is, for each $x \in H$, there exists the unique point in $P_C x \in C$ such that $\|x - P_C x\| = \min_{y \in C} \|x - y\|$. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and a mapping $f : C \to C$ is called a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a *fixed point* of T provided $Tx = x$. We denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. If C is a nonempty bounded closed convex subset of H and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty (see [3]).

Iterative methods are often used to solve the fixed point equation $Tx = x$. The most well-known method is perhaps the Picard successive iteration method when T is a contraction. Picard's method generates a sequence $\{x_n\}$ successively as $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ with $x_1 = x$ chosen arbitrarily, and this sequence converges in norm to the unique fixed point of T. However, if T is not a contraction (for instance, if T is a nonexpansive), then Picard's successive iteration fails, in general, to converge. Instead, Mann's iteration method for a nonexpansive mapping T (see [4]) prevails and generates a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N}, \tag{1.7}$$

where $x_1 = x \in C$ chosen arbitrarily and the sequence $\{\alpha_n\}$ lies in the interval $[0, 1]$. Recall that a mapping $T : C \to C$ is said to be as follows.

(i) *κ-strictly pseudocontractive* (see [5]) if there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C, \tag{1.8}$$

in brief, we use κ-SPC to denote the κ-strictly pseudocontractive, it is obvious that T is a nonexpansive if and only if T is a 0-SPC.

(ii) *Asymptotically κ-SPC* (see [6]) if there exists a constant $\kappa \in [0,1)$ and a sequence $\{\gamma_n\}$ of nonnegative real numbers with $\lim_{n\to\infty}\gamma_n = 0$ such that

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + \kappa\|(I - T^n)x - (I - T^n)y\|^2, \quad \forall x,\ y \in C, \tag{1.9}$$

for all $n \in \mathbb{N}$, if $\kappa = 0$ then T is an *asymptotically nonexpansive* with $k_n = \sqrt{1 + \gamma_n}$ for all $n \in \mathbb{N}$; that is, T is an asymptotically nonexpansive (see [7]) if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty}k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \quad \forall x,\ y \in C, \tag{1.10}$$

for all $n \in \mathbb{N}$, it is known that the class of κ-SPC mappings, and the classes of asymptotically κ-SPC mappings are independent (see [8]).

The Mann's algorithm for nonexpansive mappings has been extensively investigated (see [5, 9, 10] and the references therein). One of the well-known results is proven by Reich [10] for a nonexpansive mapping T on C, which asserts the weak convergence of the sequence $\{x_n\}$ generated by (1.7) in a uniformly convex Banach space with a Frechet differentiable norm under the control condition $\sum_{n=1}^{\infty}\alpha_n(1 - \alpha_n) = \infty$. Recently, Marino and Xu [11] devoloped and extended Reich's result to SPC mapping in Hilbert space setting. More precisely, they proved the weak convergence of the Mann's iteration process (1.7) for a κ-SPC mapping T on C, and, subsequently, this result was improved and carried over the class of asymptotically κ-SPC mappings by Kim and Xu [12].

It is known that the Mann's iteration (1.7) is in general not strongly convergent (see [13]). The strong convergence is guaranteed and has been proposed by Nakajo and Takahashi [14], they modified the Mann's iteration method (1.7) which is to find a fixed point of a nonexpansive mapping by a hybrid method, which called the shrinking projection method (or the CQ method) as the following theorem.

Theorem NT. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose that $x_1 = x \in C$ chosen arbitrarily, and let $\{x_n\}$ be the sequence defined by*

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n\cap Q_n}(x_1), \quad \forall n \in \mathbb{N}, \end{aligned} \tag{1.11}$$

where $0 \leq \alpha_n \leq \alpha < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}(x_1)$.

Subsequently, Marino and Xu [15] introduced an iterative scheme for finding a fixed point of a κ-SPC mapping as the following theorem.

Theorem MX. *Let C be a closed convex subset of a Hilbert space H, and, $T : C \to C$ be a κ-SPC mapping for some $0 \le \kappa < 1$. Assume that $F(T) \neq \emptyset$. Suppose that $x_1 = x \in C$ chosen arbitrarily, and let $\{x_n\}$ be the sequence defined by*

$$\begin{gathered} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \left\{ z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|x_n - Tx_n\|^2 \right\}, \\ Q_n = \{ z \in C : \langle x_n - z, x_1 - x_n \rangle \ge 0 \}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1), \quad \forall n \in \mathbb{N}, \end{gathered} \tag{1.12}$$

where $0 \le \alpha_n < 1$. Then the sequence $\{x_n\}$ converges strongly to $P_{F(T)}(x_1)$.

Quite recently, Kim and Xu [12] has improved and carried Theorem MX over the more wider class of asymptotically κ-SPC mappings as the following theorem.

Theorem KX. *Let C be a closed convex subset of a Hilbert space H, and, $T : C \to C$ be an asymptotically κ-SPC mapping for some $0 \le \kappa < 1$ and a bounded sequence $\{\gamma_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty}\gamma_n = 0$. Assume that $F(T)$ is a nonempty bounded subset of C. Suppose that $x_1 = x \in C$ chosen arbitrarily, and let $\{x_n\}$ be the sequence defined by*

$$\begin{gathered} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \left\{ z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (\kappa - \alpha_n(1-\alpha_n))\|x_n - T^n x_n\|^2 + \theta_n \right\}, \\ Q_n = \{ z \in C : \langle x_n - z, x_1 - x_n \rangle \ge 0 \}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1), \quad \forall n \in \mathbb{N}, \end{gathered} \tag{1.13}$$

where $\theta_n = \Delta_n^2(1-\alpha_n)\gamma_n \to 0$ as $n \to \infty$, $\Delta_n = \sup\{\|x_n - z\| : z \in F(T)\} < \infty$, and $0 \le \alpha_n < 1$ such that $\limsup_{n\to\infty}\alpha_n < 1 - \kappa$. Then the sequence $\{x_n\}$ converges strongly to $P_{F(T)}(x_1)$.

Recall that a discrete family $\mathcal{S} = \{T_n : n \ge 0\}$ of self-mappings of C is said to be a *Lipschitzian semigroup* on C if the following conditions are satisfied.

(1) $T_0 = I$ where I denotes the identity operator on C.

(2) $T_{n+m}x = T_n T_m x, \forall n, m \ge 0, \forall x \in C$.

(3) There exists a sequence $\{L_n\}$ of nonnegative real numbers such that

$$\|T_n x - T_n y\| \le L_n \|x - y\|, \quad \forall x, y \in C, \ \forall n \ge 0. \tag{1.14}$$

A discrete Lipschitzian semigroup $\mathcal{S}$ is called *nonexpansive semigroup* if $L_n = 1$ for all $n \ge 0$, *contraction semigroup* if $0 < L_n < 1$ for all $n \ge 0$ and, *asymptotically nonexpansive semigroup* if $\limsup_{n\to\infty} L_n \le 1$, respectively. We use $F(\mathcal{S})$ to denote the common fixed point set of the semigroup $\mathcal{S}$; that is, $F(\mathcal{S}) = \{x \in C : T_n x = x, \forall n \ge 0\}$.

Very recently, Kim [16] introduced *asymptotically κ-SPC semigroup*, a discrete family $\mathcal{S} = \{T_n : n \geq 0\}$ of self-mappings of C which is said to be asymptotically κ-SPC semigroup on C if, in addition to (1), (2) and the following condition (3′) are satisfied.

(3′) There exists a constant $\kappa \in [0,1)$ and a bounded sequence $\{L_n\}$ of nonnegative real numbers with $\limsup_{n\to\infty} L_n \leq 1$ such that

$$\|T_n x - T_n y\|^2 \leq L_n \|x - y\|^2 + \kappa \|(I - T_n)x - (I - T_n)y\|^2, \ \forall x, y \in C, \ \forall n \geq 0. \tag{1.15}$$

Note that for both discrete asymptotically nonexpansive semigroups and discrete asymptotically κ-SPC semigroups, we can always assume that the Lipschitzian constants $\{L_n\}_{n\geq 0}$ are such that $L_n \geq 1$ for all $n \geq 0$ and $\lim_{n\to\infty} L_n = 1$; otherwise, we replace L_n for all $n \geq 0$ with $L_n' = \max\{\sup_{m\geq n} L_m, 1\}$. Therefore, for a single asymptotically κ-SPC mapping $T : C \to C$ note that (1.15) immediately reduces to (1.9) by taking $T_n \equiv T^n$ and $\gamma_n = L_n - 1$ such that $L_n \geq 1$ for all $n \geq 0$ and $\lim_{n\to\infty} L_n = 1$.

To be more precise, Kim also showed in the framework of Hilbert spaces for the asymptotically κ-SPC semigroups that T_n is continuous on C for all $n \geq 0$ and that $F(\mathcal{S})$ is closed and convex (see Lemma 3.2 in [16]), and the demiclosedness principle (see Theorem 3.3 in [16]) holds in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T_m x_n\| = 0$ then, $z \in F(\mathcal{S}) = \bigcap_{n=0}^{\infty} F(T_n)$, and he also introduced an iterative scheme to find a common fixed point of a discrete asymptotically κ-SPC semigroup and a bounded sequence $\{L_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} L_n = 1$ as follows:

$$\begin{gathered}
x_0 = x \in C \text{ chosen arbitrarily,} \\
y_n = \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\
C_n = \left\{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1 - \alpha_n)\left(\theta_n + (\kappa - \alpha_n)\|x_n - T_n x_n\|^2\right) \right\}, \\
Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\},
\end{gathered} \tag{1.16}$$

where $\theta_n = (L_n - 1) \cdot \sup\{\|x_n - z\|^2 : z \in F(\mathcal{S})\} < \infty$. He proved that under the parameter $0 \leq \alpha_n < 1$ for all $n \in \mathbb{N} \cup \{0\}$, if $F(\mathcal{S})$ is a nonempty bounded subset of C, then the sequence $\{x_n\}$ generated by (1.16) converges strongly to $P_{F(\mathcal{S})}(x_0)$.

Inspired and motivated by the works mentioned above, in this paper, we introduce a general iterative scheme (3.1) below to find a common element of the set of common fixed point for a discrete asymptotically κ-SPC semigroup and the set of solutions of the mixed equilibrium problems in Hilbert spaces. The strong convergence theorem for the above two sets is obtained based on the shrinking projection method which extend and improve the corresponding ones due to Kim [16].

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. For a sequence $\{x_n\}$ in H, we denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively, and the weak ω-limit set of $\{x_n\}$ by $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$.

For solving the mixed equilibrium problem, let us assume that the bifunction $\Phi : C \times C \to \mathbb{R}$, the function $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ and the set C satisfy the following conditions.

(A1) $\Phi(x, x) = 0$ for all $x \in C$.

(A2) Φ is monotone; that is, $\Phi(x, y) + \Phi(y, x) \leq 0$ for all $x, y \in C$.

(A3) For each $x, y, z \in C$,

$$\lim_{t \downarrow 0} \Phi(tz + (1-t)x, y) \leq \Phi(x, y). \tag{2.1}$$

(A4) For each $x \in C, y \mapsto \Phi(x, y)$ is convex and lower semicontinuous.

(A5) For each $y \in C, x \mapsto \Phi(x, y)$ is weakly upper semicontinuous.

(B1) For each $x \in C$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Phi(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0. \tag{2.2}$$

(B2) C is a bounded set.

Lemma 2.1 (see [17]). *Let H be a Hilbert space. For any $x, y \in H$ and $\lambda \in \mathbb{R}$, one has*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.3}$$

Lemma 2.2 (see [3]). *Let C be a nonempty closed convex subset of a Hilbert space H. Then the following inequality holds:*

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \quad \forall x \in H,\ y \in C. \tag{2.4}$$

Lemma 2.3 (see [18]). *Let C be a nonempty closed convex subset of a Hilbert space H, $\Phi : C \times C \to \mathbb{R}$ satisfying the conditions $(A1)$–$(A5)$, and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either $(B1)$ or $(B2)$ holds. For $r > 0$, define a mapping $S_r : C \to C$ as follows:*

$$S_r(x) = \left\{ z \in C : \Phi(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}, \tag{2.5}$$

for all $x \in C$. Then, the following statement hold.

(1) *For each $x \in C$, $S_r(x) \neq \emptyset$.*

(2) *S_r is single-valued.*

(3) *S_r is firmly nonexpansive; that is, for any $x, y \in C$,*

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y \rangle. \tag{2.6}$$

(4) $F(S_r) = MEP(\Phi, \varphi)$.

(5) $MEP(\Phi, \varphi)$ *is closed and convex.*

Lemma 2.4 (see [3]). *Every Hilbert space H has Radon-Riesz property or Kadec-Klee property; that is, for a sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ then $x_n \to x$.*

Lemma 2.5 (see [16]). *Let C be a nonempty closed convex subset of a Hilbert space H, and let $\mathcal{S} = \{T_n : n \geq 0\}$ be an asymptotically κ-strictly pseudocontractive semigroup on C. Let $\{x_n\}$ be a sequence in C such that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ and $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. Then $\omega_w(x_n) \subset F(\mathcal{S})$.*

3. Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let Φ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying the conditions (A1)–(A5), and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2) holds. Let $\mathcal{S} = \{T_n : n \geq 0\}$ be an asymptotically κ-SPC semigroup on C for some $\kappa \in [0,1)$ and a bounded sequence $\{L_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} L_n = 1$. Assume that $\Omega := F(\mathcal{S}) \cap MEP(\Phi, \varphi)$ is a nonempty bounded subset of C. For $x_0 = x \in C$ chosen arbitrarily, suppose that $\{x_n\}, \{y_n\}$, and $\{u_n\}$ are generated iteratively by*

$$\begin{gathered}
u_n \in C \text{ such that } \Phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
y_n = \alpha_n u_n + (1 - \alpha_n) T_n u_n, \\
C_{n+1} = \left\{ z \in C_n \cap Q_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1 - \alpha_n)\left(\theta_n + (\kappa - \alpha_n)\|u_n - T_n u_n\|^2\right) \right\}, \\
Q_{n+1} = \{ z \in C_n \cap Q_n : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\
C_0 = Q_0 = C, \\
x_{n+1} = P_{C_{n+1} \cap Q_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\},
\end{gathered} \tag{3.1}$$

where $\theta_n = (L_n - 1) \cdot \sup\{\|x_n - z\|^2 : z \in \Omega\} < \infty$ satisfying the following conditions:

(C1) $\{\alpha_n\} \subset [a, b]$ *such that* $\kappa < a < b < 1$,

(C2) $\{r_n\} \subset [r, \infty)$ *for some* $r > 0$,

(C3) $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequences $\{x_n\}, \{y_n\}$, and $\{u_n\}$ converge strongly to $w = P_\Omega(x_0)$.

Proof. Pick $p \in \Omega$. Therefore, by (3.1) and the definition of S_{r_n} in Lemma 2.3, we have

$$u_n = S_{r_n} x_n \in \operatorname{dom} \varphi, \tag{3.2}$$

and, by $F(\mathcal{S}) = \bigcap_{n=0}^{\infty} F(T_n)$ and Lemma 2.3(4), we have

$$T_n p = p = S_{r_n} p. \tag{3.3}$$

By (3.2), (3.3), and the nonexpansiveness of S_{r_n}, we have

$$\|u_n - p\| = \|S_{r_n} x_n - S_{r_n} p\| \leq \|x_n - p\|. \tag{3.4}$$

By (3.3), (3.4), Lemma 2.1, and the asymptotically κ-SPC semigroupness of $\mathcal{S}$, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(u_n - p) + (1-\alpha_n)(T_n u_n - p)\|^2 \\
&= \alpha_n \|u_n - p\|^2 + (1-\alpha_n)\|T_n u_n - p\|^2 - \alpha_n(1-\alpha_n)\|u_n - T_n u_n\|^2 \\
&\leq \alpha_n \|u_n - p\|^2 + (1-\alpha_n)\left(L_n \|u_n - p\|^2 + \kappa \|u_n - T_n u_n\|^2\right) - \alpha_n(1-\alpha_n)\|u_n - T_n u_n\|^2 \\
&= (1 + (1-\alpha_n)(L_n - 1))\|u_n - p\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|u_n - T_n u_n\|^2 \\
&\leq \|x_n - p\|^2 + (1-\alpha_n)\left(\theta_n + (\kappa - \alpha_n)\|u_n - T_n u_n\|^2\right),
\end{aligned} \tag{3.5}$$

where $\theta_n := (L_n - 1) \cdot \sup\{\|x_n - z\|^2 : z \in \Omega\}$ for all $n \in \mathbb{N} \cup \{0\}$.

Firstly, we prove that $C_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. It is obvious that $C_0 \cap Q_0$ is closed and by mathematical induction that $C_n \cap Q_n$ is closed for all $n \geq 1$; that is $C_n \cap Q_n$ is closed for all $n \in \mathbb{N} \cup \{0\}$. Let $\epsilon_n = (1-\alpha_n)(\theta_n + (k - \alpha_n)\|u_n - T_n u_n\|^2)$ since, for any $z \in C$, $\|y_n - z\|^2 \leq \|x_n - z\|^2 + \epsilon_n$ is equivalent to

$$\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle - \epsilon_n \leq 0, \tag{3.6}$$

for all $n \in \mathbb{N} \cup \{0\}$. Therefore, for any $z_1, z_2 \in C_{n+1} \cap Q_{n+1} \subset C_n \cap Q_n$ and $\epsilon \in (0,1)$, we have

$$\begin{aligned}
\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - (\epsilon z_1 + (1-\epsilon) z_2)\rangle - \epsilon_n &= \epsilon\left(\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z_1\rangle - \epsilon_n\right) + (1-\epsilon) \\
&\quad \times \left(\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z_2\rangle - \epsilon_n\right) \\
&\leq 0,
\end{aligned} \tag{3.7}$$

for all $n \in \mathbb{N} \cup \{0\}$, and we have

$$\begin{aligned}
\langle x_n - (\epsilon z_1 + (1-\epsilon) z_2), x_0 - x_n\rangle &= \epsilon\langle x_n - z_1, x_0 - x_n\rangle + (1-\epsilon)\langle x_n - z_2, x_0 - x_n\rangle \\
&\geq 0,
\end{aligned} \tag{3.8}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since $C_0 \cap Q_0$ is convex and by putting $n = 0$ in (3.6), (3.7), and (3.8), we have that $C_1 \cap Q_1$ is convex. Suppose that x_k is given and $C_k \cap Q_k$ is convex for some $k \geq 1$. It follows by putting $n = k$ in (3.6), (3.7), and (3.8) that $C_{k+1} \cap Q_{k+1}$ is convex. Therefore, by mathematical induction, we have that $C_n \cap Q_n$ is convex for all $n \geq 1$; that is, $C_n \cap Q_n$ is convex for all $n \in \mathbb{N} \cup \{0\}$. Hence, we obtain that $C_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N} \cup \{0\}$.

Next, we prove that $\Omega \subset C_n \cap Q_n$ for all $n \in \mathbb{N} \cup \{0\}$. It is obvious that $p \in \Omega \subset C = C_0 \cap Q_0$. Therefore, by (3.1) and (3.5), we have $p \in C_1$ and note that $p \in C = Q_1$, and so $p \in C_1 \cap Q_1$. Hence, we have $\Omega \subset C_1 \cap Q_1$. Since $C_1 \cap Q_1$ is a nonempty closed convex subset of C, there exists a unique element $x_1 \in C_1 \cap Q_1$ such that $x_1 = P_{C_1 \cap Q_1}(x_0)$. Suppose that $x_k \in C_k \cap Q_k$ is given such that $x_k = P_{C_k \cap Q_k}(x_0)$ and $p \in \Omega \subset C_k \cap Q_k$ for some $k \geq 1$. Therefore, by (3.1) and (3.5), we have $p \in C_{k+1}$. Since $x_k = P_{C_k \cap Q_k}(x_0)$; therefore, by Lemma 2.2, we have

$$\langle x_k - z, x_0 - x_k \rangle \geq 0, \tag{3.9}$$

for all $z \in C_k \cap Q_k$. Thus, by (3.1), we have $p \in Q_{k+1}$, and so $p \in C_{k+1} \cap Q_{k+1}$. Hence, we have $\Omega \subset C_{k+1} \cap Q_{k+1}$. Since $C_{k+1} \cap Q_{k+1}$ is a nonempty closed convex subset of C, there exists a unique element $x_{k+1} \in C_{k+1} \cap Q_{k+1}$ such that $x_{k+1} = P_{C_{k+1} \cap Q_{k+1}}(x_0)$. Therefore, by mathematical induction, we obtain $\Omega \subset C_n \cap Q_n$ for all $n \geq 1$, and so $\Omega \subset C_n \cap Q_n$ for all $n \in \mathbb{N} \cup \{0\}$, and we can define $x_{n+1} = P_{C_{n+1} \cap Q_{n+1}}(x_0)$ for all $n \in \mathbb{N} \cup \{0\}$. Hence, we obtain that the iteration (3.1) is well defined.

Next, we prove that $\{x_n\}$ is bounded. Since $x_n = P_{C_n \cap Q_n}(x_0)$ for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\|x_n - x_0\| \leq \|z - x_0\|, \tag{3.10}$$

for all $z \in C_n \cap Q_n$. It follows by $p \in \Omega \subset C_n \cap Q_n$ that $\|x_n - x_0\| \leq \|p - x_0\|$ for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{u_n\}$.

Next, we prove that $\|x_n - x_{n+1}\| \to 0$ and $\|u_n - u_{n+1}\| \to 0$ as $n \to \infty$. Since $x_{n+1} = P_{C_{n+1} \cap Q_{n+1}}(x_0) \in C_{n+1} \cap Q_{n+1} \subset C_n \cap Q_n$; therefore, by (3.10), we have $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$ for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{\|x_n - x_0\|\}$ is a bounded nondecreasing sequence; there exists the limit of $\|x_n - x_0\|$; that is,

$$\lim_{n \to \infty} \|x_n - x_0\| = m, \tag{3.11}$$

for some $m \geq 0$. Since $x_{n+1} \in Q_{n+1}$; therefore, by (3.1), we have

$$\langle x_n - x_{n+1}, x_0 - x_n \rangle \geq 0. \tag{3.12}$$

It follows by (3.12) that

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n \rangle \\ &\quad + 2\langle x_n - x_0, x_n - x_{n+1} \rangle + \|x_{n+1} - x_0\|^2 \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned} \tag{3.13}$$

Therefore, by (3.11), we obtain

$$\|x_n - x_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.14}$$

Indeed, from (3.1), we have

$$\Phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \tag{3.15}$$

$$\Phi(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall y \in C. \tag{3.16}$$

Substituting $y = u_{n+1}$ into (3.15) and $y = u_n$ into (3.16), we have

$$\begin{gathered}\Phi(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0, \\ \Phi(u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.\end{gathered} \tag{3.17}$$

Therefore, by the condition (A2), we get

$$\begin{aligned}0 &\leq \Phi(u_n, u_{n+1}) + \Phi(u_{n+1}, u_n) + \left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \\ &\leq \left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle.\end{aligned} \tag{3.18}$$

It follows that

$$\begin{aligned}0 &\leq \left\langle u_{n+1} - u_n, (u_n - u_{n+1}) + (u_{n+1} - x_n) - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \\ &= \langle u_{n+1} - u_n, u_n - u_{n+1}\rangle + \left\langle u_{n+1} - u_n, (u_{n+1} - x_{n+1}) + (x_{n+1} - x_n) - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle.\end{aligned} \tag{3.19}$$

Thus, we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, (x_{n+1} - x_n) + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle \\ &\leq \|u_{n+1} - u_n\|\left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|\right).\end{aligned} \tag{3.20}$$

It follows by the condition (C2) that

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \|x_{n+1}-x_n\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}\|u_{n+1}-x_{n+1}\| \\ &\le \|x_{n+1}-x_n\| + \left(\frac{L}{r}\right)|r_{n+1}-r_n|,\end{aligned} \tag{3.21}$$

where $L = \sup_{n\ge 0}\|u_n - x_n\| < \infty$. Therefore, by the condition (C3) and (3.14), we obtain

$$\|u_n - u_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.22}$$

Next, we prove that $\|u_n - T_n u_n\| \to 0$, $\|y_n - x_n\| \to 0$, and $\|u_n - x_n\| \to 0$ as $n \to \infty$. Since $x_{n+1} \in C_{n+1}$, by (3.1), we have

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + (1-\alpha_n)\left(\theta_n + (\kappa - \alpha_n)\|u_n - T_n u_n\|^2\right). \tag{3.23}$$

It follows that

$$\begin{aligned}(1-\alpha_n)(\alpha_n-\kappa)\|u_n - T_n u_n\|^2 &\le \|x_n - x_{n+1}\|^2 + (1-\alpha_n)\theta_n - \|y_n - x_{n+1}\|^2 \\ &\le \|x_n - x_{n+1}\|^2 + \theta_n.\end{aligned} \tag{3.24}$$

Therefore, by the condition (C1), (3.14), and $\lim_{n\to\infty}\theta_n = 0$, we obtain

$$\|u_n - T_n u_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.25}$$

From (3.23) and the condition (C1), we have

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n. \tag{3.26}$$

Therefore,

$$\begin{aligned}\|y_n - x_n\|^2 &= \|(y_n - x_{n+1}) + (x_{n+1} - x_n)\|^2 \\ &= \|y_n - x_{n+1}\|^2 + 2\langle y_n - x_{n+1}, x_{n+1} - x_n\rangle + \|x_{n+1} - x_n\|^2 \\ &\le \|y_n - x_{n+1}\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| + \|x_{n+1} - x_n\|^2 \\ &\le 2\|x_{n+1} - x_n\|\left(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|\right) + \theta_n.\end{aligned} \tag{3.27}$$

Hence, by (3.14) and $\lim_{n\to\infty}\theta_n = 0$, we obtain

$$\|y_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.28}$$

By (3.2), (3.3), and the firmly nonexpansiveness of S_{r_n}, we have

$$\begin{aligned}\|u_n - p\|^2 &\le \langle S_{r_n}x_n - S_{r_n}p, x_n - p\rangle = \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}\left(\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2\right).\end{aligned} \tag{3.29}$$

It follows that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - x_n\|^2. \tag{3.30}$$

Therefore, by the condition (C1) and (3.5), we have

$$\begin{aligned}\|y_n - p\|^2 &\le (1 + (1-\alpha_n)(L_n - 1))\|u_n - p\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|u_n - T_n u_n\|^2 \\ &\le (L_n - \alpha_n(L_n - 1))\|u_n - p\|^2 \le L_n\|u_n - p\|^2 \\ &\le L_n\left(\|x_n - p\|^2 - \|u_n - x_n\|^2\right).\end{aligned} \tag{3.31}$$

It follows that

$$\begin{aligned}L_n\|u_n - x_n\|^2 &\le L_n\|x_n - p\|^2 - \|y_n - p\|^2 \\ &= (L_n - 1)\|x_n - p\|^2 + \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\le \theta_n + \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|).\end{aligned} \tag{3.32}$$

Hence, by (3.28) and $\lim_{n\to\infty}\theta_n = 0$, we obtain

$$\|u_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.33}$$

Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ which converges weakly to $\overline{w}$. Next, we prove that $\overline{w} \in \Omega$. From (3.22) and (3.25), we have $\|u_{n_i} - u_{n_i+1}\| \to 0$, and $\|u_{n_i} - T_{n_i}u_{n_i}\| \to 0$ as $i \to \infty$; therefore, by Lemma 2.5, we obtain $\overline{w} \in F(\mathcal{S})$. From (3.1), we have

$$0 \le \Phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \quad \forall y \in C. \tag{3.34}$$

It follows by the condition (A2) that

$$\begin{aligned}\Phi(y, u_n) &\le \Phi(y, u_n) + \Phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \quad \forall y \in C \\ &\le \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \quad \forall y \in C.\end{aligned} \tag{3.35}$$

Hence,

$$\varphi(y) - \varphi(u_{n_i}) + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq \Phi(y, u_{n_i}), \quad \forall y \in C. \tag{3.36}$$

Therefore, from (3.33) and by $u_{n_i} \rightharpoonup \overline{w}$ as $i \to \infty$, we obtain

$$\Phi(y, \overline{w}) + \varphi(\overline{w}) - \varphi(y) \leq 0, \quad \forall y \in C. \tag{3.37}$$

For a constant t with $0 < t < 1$ and $y \in C$, let $y_t = ty + (1-t)\overline{w}$. Since $y, \overline{w} \in C$, thus, $y_t \in C$. So, from (3.37), we have

$$\Phi(y_t, \overline{w}) + \varphi(\overline{w}) - \varphi(y_t) \leq 0. \tag{3.38}$$

By (3.38), the conditions (A1) and (A4), and the convexity of φ, we have

$$\begin{aligned} 0 &= \Phi(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq \left(t\Phi(y_t, y) + (1-t)\Phi(y_t, \overline{w})\right) + \left(t\varphi(y) + (1-t)\varphi(\overline{w})\right) - \varphi(y_t) \\ &= t\left(\Phi(y_t, y) + \varphi(y) - \varphi(y_t)\right) + (1-t)\left(\Phi(y_t, \overline{w}) + \varphi(\overline{w}) - \varphi(y_t)\right) \\ &\leq t\left(\Phi(y_t, y) + \varphi(y) - \varphi(y_t)\right). \end{aligned} \tag{3.39}$$

It follows that

$$\Phi(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0. \tag{3.40}$$

Therefore, by the condition (A3) and the weakly lower semicontinuity of φ, we have $\Phi(\overline{w}, y) + \varphi(y) - \varphi(\overline{w}) \geq 0$ as $t \to 0$ for all $y \in C$, and; hence, we obtain $\overline{w} \in MEP(\Phi, \varphi)$, and so $\overline{w} \in \Omega$.

Since Ω is a nonempty closed convex subset of C, there exists a unique $w \in \Omega$ such that $w = P_\Omega(x_0)$. Next, we prove that $x_n \to w$ as $n \to \infty$. Since $w = P_\Omega(x_0)$, we have $\|x_0 - w\| \leq \|x_0 - z\|$ for all $z \in \Omega$; it follows that

$$\|x_0 - w\| \leq \|x_0 - \overline{w}\|. \tag{3.41}$$

Since $w \in \Omega \subset C_n \cap Q_n$; therefore, by (3.10), we have

$$\|x_0 - x_n\| \leq \|x_0 - w\|. \tag{3.42}$$

Since $\|x_{n_i} - u_{n_i}\| \to 0$ by (3.33) and $u_{n_i} \rightharpoonup \overline{w}$, we have $x_{n_i} \rightharpoonup \overline{w}$ as $i \to \infty$. Therefore, by (3.41), (3.42), and the weak lower semicontinuity of norm, we have

$$\|x_0 - w\| \leq \|x_0 - \overline{w}\| \leq \liminf_{i \to \infty} \|x_0 - x_{n_i}\| \leq \limsup_{i \to \infty} \|x_0 - x_{n_i}\| \leq \|x_0 - w\|. \tag{3.43}$$

It follows that

$$\|x_0 - w\| = \lim_{i\to\infty}\|x_0 - x_{n_i}\| = \|x_0 - \overline{w}\|. \tag{3.44}$$

Since $x_{n_i} \rightharpoonup \overline{w}$ as $i \to \infty$; therefore, we have

$$(x_0 - x_{n_i}) \rightharpoonup (x_0 - \overline{w}) \quad \text{as } i \longrightarrow \infty. \tag{3.45}$$

Hence, from (3.44), (3.45), the Kadec-Klee property, and the uniqueness of $w = P_\Omega(x_0)$, we obtain

$$x_{n_i} \longrightarrow \overline{w} = w \quad \text{as } i \longrightarrow \infty. \tag{3.46}$$

It follows that $\{x_n\}$ converges strongly to w and so are $\{y_n\}$ and $\{u_n\}$. This completes the proof. □

Remark 3.2. The iteration (3.1) is the difference with the iterative scheme of Kim [16] as the following.

(1) The sequence $\{x_n\}$ is a projection sequence of x_0 onto $C_n \cap Q_n$ for all $n \in \mathbb{N} \cup \{0\}$ such that

$$C_0 \cap Q_0 \supset C_1 \cap Q_1 \supset \cdots \supset C_n \cap Q_n \supset \cdots \supset \Omega. \tag{3.47}$$

(2) A solving of a common element of the set of common fixed point for a discrete asymptotically κ-SPC semigroup and the set of solutions of the mixed equilibrium problems by iteration is obtained.

For solving the equilibrium problem, let us define the condition $(B3)$ as the condition $(B1)$ such that $\varphi = 0$. We have the following result.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let Φ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying the conditions (A1)–(A5) with either $(B2)$ or $(B3)$ holds. Let $\mathcal{S} = \{T_n : n \geq 0\}$ be an asymptotically κ-SPC semigroup on C for some $\kappa \in [0,1)$ and a bounded sequence $\{L_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} L_n = 1$. Assume that $\Omega := F(\mathcal{S}) \cap EP(\Phi)$ is a nonempty bounded subset of C. For $x_0 = x \in C$ chosen arbitrarily, suppose that $\{x_n\}, \{y_n\}$, and $\{u_n\}$ are generated iteratively by*

$$u_n \in C \text{ such that } \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$

$$y_n = \alpha_n u_n + (1-\alpha_n)T_n u_n,$$

$$C_{n+1} = \left\{z \in C_n \cap Q_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)\left(\theta_n + (\kappa - \alpha_n)\|u_n - T_n u_n\|^2\right)\right\},$$

$$Q_{n+1} = \{z \in C_n \cap Q_n : \langle x_n - z, x_0 - x_n\rangle \geq 0\},$$

$$C_0 = Q_0 = C,$$
$$x_{n+1} = P_{C_{n+1}\cap Q_{n+1}}(x_0), \quad \forall n \in \mathbb{N}\cup\{0\}, \tag{3.48}$$

where $\theta_n = (L_n - 1)\cdot \sup\{\|x_n - z\|^2 : z \in \Omega\} < \infty$ *satisfying the following conditions:*

(C1) $\{\alpha_n\} \subset [a,b]$ *such that* $\kappa < a < b < 1$,

(C2) $\{r_n\} \subset [r,\infty)$ *for some* $r > 0$,

(C3) $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$, *and* $\{u_n\}$ *converge strongly to* $w = P_\Omega(x_0)$.

Proof. It is concluded from Theorem 3.1 immediately, by putting $\varphi = 0$. □

Corollary 3.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\mathcal{S} = \{T_n : n \geq 0\}$ *be an asymptotically* κ*-SPC semigroup on* C *for some* $\kappa \in [0,1)$ *and a bounded sequence* $\{L_n\} \subset [1,\infty)$ *such that* $\lim_{n\to\infty} L_n = 1$. *Assume that* $F(\mathcal{S})$ *is a nonempty bounded subset of* C. *For* $x_0 = x \in C$ *chosen arbitrarily, suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are generated iteratively by*

$$y_n = \alpha_n x_n + (1-\alpha_n)T_n x_n,$$
$$C_{n+1} = \left\{z \in C_n \cap Q_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)\left(\theta_n + (\kappa - \alpha_n)\|x_n - T_n x_n\|^2\right)\right\},$$
$$Q_{n+1} = \{z \in C_n \cap Q_n : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \tag{3.49}$$
$$C_0 = Q_0 = C,$$
$$x_{n+1} = P_{C_{n+1}\cap Q_{n+1}}(x_0), \quad \forall n \in \mathbb{N}\cup\{0\},$$

where $\theta_n = (L_n - 1)\cdot \sup\{\|x_n - z\|^2 : z \in F(\mathcal{S})\} < \infty$ *and* $\{\alpha_n\} \subset [a,b]$ *such that* $\kappa < a < b < 1$. *Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $w = P_{F(\mathcal{S})}(x_0)$.

Proof. It is concluded from Corollary 3.3 immediately, by putting $\Phi = 0$. □

4. Applications

We introduce the equilibrium problem to the optimization problem:

$$\min_{x\in C} \zeta(x), \tag{4.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H and $\zeta : C \to \mathbb{R}\cup\{+\infty\}$ is a proper convex and lower semicontinuous. We denote by $\mathrm{Argmin}(\zeta)$ the set of solutions of problem (4.1). We define the condition $(B4)$ as the condition $(B3)$ such that $\Phi : C\times C \to \mathbb{R}$ is a bifunction defined by $\Phi(x,y) = \zeta(y) - \zeta(x)$ for all $x, y \in C$. Observe that $EP(\Phi) = \mathrm{Argmin}(\zeta)$. We give the interesting result as the following theorem.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $\zeta : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either $(B2)$ or $(B4)$ holds. Let $\mathcal{S} = \{T_n : n \geq 0\}$ be an asymptotically κ-SPC semigroup on C for some $\kappa \in [0,1)$ and a bounded sequence $\{L_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} L_n = 1$. Assume that $\Omega := F(\mathcal{S}) \cap Argmin(\zeta)$ is a nonempty bounded subset of C. For $x_0 = x \in C$ chosen arbitrarily, suppose that $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are generated iteratively by*

$$\begin{gathered}
u_n \in C \text{ such that } \zeta(y) - \zeta(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = \alpha_n u_n + (1-\alpha_n) T_n u_n, \\
C_{n+1} = \left\{ z \in C_n \cap Q_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)\left(\theta_n + (\kappa - \alpha_n)\|u_n - T_n u_n\|^2\right)\right\}, \\
Q_{n+1} = \{z \in C_n \cap Q_n : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\
C_0 = Q_0 = C, \\
x_{n+1} = P_{C_{n+1}\cap Q_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\},
\end{gathered} \tag{4.2}$$

where $\theta_n = (L_n - 1) \cdot \sup\{\|x_n - z\|^2 : z \in \Omega\} < \infty$ satisfying the following conditions:

(C1) $\{\alpha_n\} \subset [a,b]$ *such that* $\kappa < a < b < 1$,

(C2) $\{r_n\} \subset [r,\infty)$ *for some* $r > 0$,

(C3) $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequences $\{x_n\}, \{y_n\}$, and $\{u_n\}$ converge strongly to $w = P_\Omega(x_0)$.

Proof. It is concluded from Corollary 3.3 immediately, by defining $\Phi(x,y) = \zeta(y) - \zeta(x)$ for all $x, y \in C$. □

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