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Research Article

On Removable Sets of Solutions of Elliptic Equations

Tair S. Gadjiev, Mahammad-Rza I. Arazm, and Vafa A. Mamedova

Department of Nonlinear Analyses, Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9, F. Agaev street, Baku AZ1141, Azerbaijan

Correspondence should be addressed to Mahammad-Rza I. Arazm, shafa_mamedova@mail.ru

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We consider a nondivergent elliptic equation of second order whose leading coefficients are from some weight space. The sufficient condition of removability of a compact with respect to this equation in the weight space of Hölder functions was found.

Let D be a bounded domain situated in n-dimensional Euclidean space E_n of the points $x = (x_1, \ldots, x_n)$, $n \geq 3$, and let ∂D be its boundary. Consider in D the following elliptic equation:

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}(x)u_{ij} + \sum_{i=1}^{n} b_i(x)u_i + c(x)u = 0, \tag{1}$$

in supposition that $\|a_{ij}(x)\|$ is a real symmetric matrix, moreover $\omega(x)$ is a positive measurable function satisfying the doubling condition: for concentric balls B_R^x or R and $2R$ radius, there exists such a constant γ

$$\omega(B_R^x) \geq \gamma\omega(B_{2R}^x), \tag{2}$$

where for the measurable sets E. $\omega(E)$ means $\int_E \omega(y)\,dy$

$$\gamma|\xi|^2\omega(x) \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \gamma^{-1}\omega(x)|\xi|^2; \quad \xi \in E_n,\ x \in D, \tag{3}$$

$$a_{ij}(x) \in C^1_\omega\left(\overline{D}\right); \quad i, j, 1, \ldots, n, \tag{4}$$

$$|b_i(x)| \le b_0; -b_0 \le c(x) \le 0; \quad i = 1, \ldots, n;\ x \in D. \tag{5}$$

Here $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$; $i, j = 1, \ldots, n$; $\gamma \in (0, 1]$ and $b_0 \ge 0$ are constants. Besides we will suppose that the lower coefficients of the operator $\mathcal{L}$ are measurable functions in D. Let $\lambda \in (0, 1)$ be a number. Denote by $C^{0,\lambda}(D)$ a Banach space of the functions $u(x)$ defined in D with the finite norm;

$$\|u\|_{C^\lambda_\omega(D)} = \sup_{x\in D} \omega(x)|u(x)| + \sup_{\substack{x,y\in D\\ x\ne y}} \frac{|u(x) - u(y)|\omega}{|x - y|^\lambda}. \tag{6}$$

The compact $E \subset \overline{D}$ is called removable with respect to (1) in the space $C^\lambda_\omega(D)$ if from

$$\mathcal{L}u = 0, \quad x \in D \setminus E, \quad u|_{\partial D\setminus E} = 0, \quad u(x) \in C^\lambda_\omega(D), \tag{7}$$

it follows that $u(x) \equiv 0$ in D.

The aim of the given paper is finding sufficient condition of removability of a compact with respect to (1) in the space $C^\lambda_\omega(D)$. This problem have been investigated by many researchers. For the Laplace equation the corresponding result was found by Carleson [1]. Concerning the second-order elliptic equations of divergent structure, we show in this direction the papers [2, 3]. For a class of nondivergent elliptic equations of the second order with discontinuous coefficients the removability condition for a compact in the space $C^\lambda(D)$ was found in [4]. mention also papers [5–7] in which the conditions of removability for a compact in the space of continuous functions have been obtained. The removable sets of solutions of the second-order elliptic and parabolic equations in nondivergent form were considered in [8–10]. In [11], Kilpelinen and Zhong have studied the divergent quasilinear equation without minor members and proved the removability of a compact. Removable sets for pointwise solutions of elliptic partial differential equations were found by Diederich [12]. Removable singularities of solutions of linear partial differential equations were considered in [13]. Removable sets at the boundary for subharmonic functions have been investigated by Dahlberg [14]. Denote by $B_R(z)$ and $S_R(z)$ the ball $\{x : |x - z| < R\}$ and the sphere $\{x : |x - z| = R\}$ of radius R with the center at the point $z \in E_n$ respectively. We will need the following generalization of mean value theorem belonging to Gerver and Landis [15] in weight case.

Lemma 1. *Let the domain G be situated between the spheres $S_R(0)$ and $S_{2R}(0)$, moreover let the intersection $\partial G \cap \{x : R < |x| < 2R\}$ be a smooth surface. Further, let in $\overline{G}$ the uniformly positive*

definite matrix $\|a_{ij}(x)\|$; $i, j = 1, \ldots, n$ *and the function* $u(x) \in C^2(G) \cap C^1_\omega(\overline{G})$ *be given. Then there exists the piecewise smooth surface* Σ *dividing in* G *the spheres* $S_R(0)$ *and* $S_{2R}(0)$ *such that*

$$\int_\Sigma \omega \left| \frac{\partial u}{\partial \nu} \right| ds \leq K \operatorname*{osc}_G u \cdot \frac{\omega(G)}{R^2}. \tag{8}$$

Here $K > 0$ *is a constant depending only on the matrix* $\|a_{ij}(x)\|$ *and* n, *and* $\partial u / \partial \nu$ *is a derivative by a conormal determined by the equality*

$$\frac{\partial u(x)}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \cos\left(\overline{n}, x_j\right)^{1/2}, \tag{9}$$

where $\cos(\overline{n}, x_j)$; $j = 1, \ldots, n$ *are direction cosines of a unit external normal vector to* Σ.

Proof. Let $G \subset R^n$ be a bounded domain $f(x) \in C^2(G)$. Then there exists a finite number of balls $\{B^{x_\nu}_{r_\nu}\}$, $\nu = 1, 2, \ldots, N$ which cover Q_f and such that if we denote by S_ν, the surface of νth ball, then

$$\sum_{\nu=1}^{N} \int_{S_\nu} \omega(x) \left| \nabla f \right| dx < \varepsilon. \tag{10}$$

Decompose O_f into two parts: $O_f = O'_f \cup O''_f$, where O'_f is a set of points O_f for which $\nabla^2 f \neq 0$, O''_f is a set of points for which $\nabla^2 f = 0$.

The set O'_f has n-dimensional Lebesgue measure equal zero, as on the known implicit function theorem, the O'_f lies on a denumerable number of surfaces of dimension $n - 1$. If we use the absolute continuity of integral

$$\omega(D) = \int_D \omega(x) dx \tag{11}$$

with respect to Lebesque measure D and the above said, we get that the set O'_f may be included into the set D for which $\omega(D) < \eta$, $\eta > 0$ will be chosen later. Let for each point $x \in O' f$, there exist such r_x that $B^x_{r_x}$ and $B^x_{6r_x}$ are contained in $D \subset G$. Then

$$\int_{5r_x}^{6r_x} dr \int_{S^x_r} \omega(\sigma)\, d\sigma \leq \omega\left(B^x_{6r_x}\right), \tag{12}$$

therefore there exists such $5r_x \leq t \leq 6r_x$ that

$$r_x \int_{S^x_r} \omega(\sigma)\, d\sigma \leq \omega\left(B^x_{6r_x}\right). \tag{13}$$

Then

$$\begin{aligned}\int_{S_r^x} \omega(\sigma)\left|\nabla f\right| d\sigma &\le Ct \int_{S_r^x} \omega(\sigma) d\sigma \le (6C) r \left(r_x \int_{S_r^x} \omega(\sigma) d\sigma \right) \\ &\le (6C)\omega\left(B_{6r_x}^x\right) \le (6C)\gamma^{-3}\omega\left(B_{r_x}^x\right) \le c_0 \omega\left(B_{t/5}^x\right),\end{aligned} \tag{14}$$

where $C = \sup D|\nabla^2 f|$, $\alpha = \operatorname{diam} G$, $c_0 = (6C)\gamma^{-3}$.

Now by a Banach process [4, page 126] from the ball system $\{B_{t/5}^x\}$ we choose such a denumerable number of notintersecting balls $\{B_{t_\nu/5}^{x_\nu}\}$, $\nu = 1, 2, \ldots, N$ that the ball of five-times greater radius $\{B_{t_\nu}^{x_\nu}\}$ cover the whole O_f' set. We again denote these balls by $\{B_{t_\nu/5}^{x_\nu}\}$, $\nu = 1, 2, \ldots, N$ and their surface by S_ν'. Then by virtue of (5)

$$\sum_{\nu=1}^{\infty} \int_{S_\nu'} \omega(\sigma)\left|\nabla f\right| d\sigma \le C_0 \omega(G) < C_0 \eta. \tag{15}$$

Now let $x \in O_f''$. Then

$$\int_{5r_x}^{6r_x} dr \int_{S_r^x} \omega(\sigma) d\sigma \le \omega\left(B_{6r_x}^x\right). \tag{16}$$

Therefore there exists such $5r_x \le t \le 6r_x$ that

$$r_x \int_{S_r^x} \omega(\sigma)\, d\sigma \le \omega\left(B_{6r_x}^x\right). \tag{17}$$

Assign arbitrary $\eta > 0$. By virtue of that $|\nabla f|_{S_t^x} \le \eta \cdot t$, for sufficiently small t we have

$$\begin{aligned}\int_{S_t^x} \omega(\sigma)\left|\nabla f\right| d\sigma &\le \eta t \int_{S_t^x} \omega(\sigma) d\sigma \le (2\eta)\left(r_x \int_{S_t^x} \omega(\sigma) d\sigma \right) \\ &\le (2\eta)\omega\left(B_{2r_x}^x\right) \le (6C) \le (2\eta)^{\gamma-1}\omega\left(B_{t/5}^x\right) \le \eta C_1 \omega\left(B_{t/5}^x\right).\end{aligned} \tag{18}$$

Again by means of Banach process and by virtue of (43) we get

$$\sum_{\nu=1}^{N} \int_{S_\nu^n} \omega(\sigma)\left|\nabla f\right| d\sigma \le \eta \cdot C_1 \omega(D), \tag{19}$$

where S_ν^n is the surface of balls in the second covering.

Combining the spherical surfaces S'_ν and S''_ν we get that the open balls system covers the closed set O_f. Then a finite subcovering may be choosing from it. Let them be the balls $B_1, B_2, \ldots, B_x$ and their surfaces are $S_1, S_2, \ldots, S_N$. We get from inequalities (4) and (7)

$$\sum_{\nu=1}^{N} \int_{S_\nu} |\nabla f| \omega(\sigma) d\sigma \leq \left[C_1 \omega\left(\overline{D}\right) + C_0\right] \eta. \tag{20}$$

Put now $\varepsilon = [C_1 \omega(\overline{D}) + C_0]\eta$.

Following [2], assume

$$\varepsilon = \frac{\omega(D)(\operatorname{oscu}_G)}{R^2}, \tag{21}$$

and according to Lemma 1 for a given ε we will find the balls $B_1, B_2, \ldots, B_x$ and exclude them from the domain G. Put $D^* = D \setminus \bigcup_{\nu=1}^{N} B_\nu$ intersect with G^* a closed spherical layer

$$R\left(1 + \frac{1}{4}\right) \leq |x| \leq R\left(1 + \frac{1}{4}\right). \tag{22}$$

We denote the intersection by G'. We can assume that the function $u(x)$ is defined in some δ vicinity G'_δ of set G'. Take $\delta < R/4$ so that

$$\operatorname*{oscu}_{G'_\delta} \leq 2 \operatorname*{oscu}_{G}. \tag{23}$$

On a closed set G' we have $\nabla f \neq 0$. Consider on G'_δ the equation system

$$\frac{dx}{dt} = u_x. \tag{24}$$

Let some surface S touches the direction of the field at each its point, then

$$\int_S \left|\frac{\partial u}{\partial n}\right| d\sigma = 0, \tag{25}$$

since $\partial u / \partial n$ is identically equal to zero at S.

We will use it in constructing the needed surface of Σ. Tubular surfaces whose generators will be the trajectories of the system (50) constitute the basis of Σ.

They will add nothing to the integral we are interested in. These surfaces will have the form of thin tubes that cover G'. Then we shall put partitions to some of these tubes. Lets construct tubes. Denote by E the intersection of G' with sphere $|x| = R(1 + 3/4)$.

Let N be a set of points E. Where field direction of system (50) touches the sphere $|x| = R(1+3/4)$. Cover N with such an open on the sphere $|x| = R(1+3/4)$ set F that

$$\int_F \omega(x)\left|\frac{\partial u}{\partial n}\right| d\sigma \leq \frac{\omega(G)(\operatorname{osc} u_D)}{R^2}. \tag{26}$$

It will be possible if on $N(\partial u/\partial R) \equiv 0$.

Put $E' = E \setminus F$. Cover E' on the sphere by a finite number of open domains with piece-wise smooth boundaries. We shall call them cells. We shall control their diameters in estimation of integrals that we need. The surface remarked by the trajectories lying in the ball $|x| \leq (7/4)R$ and passing through the bounds of cells we shall call tube.

So, we obtained a finite number of tubes. The tube is called open if not interesting, this tube one can join by a broken line the point of its corresponding cell with a spherical layer $(5/4)R - \delta < |x| < (7/4)R$. Choose the diameters of cells so small that the trajectory beams passing through each cell could differ no more than $\delta/2n$.

By choose of cells diameters the tubes will be contained in

$$\frac{5}{4}R - \delta < |x| < \frac{5}{4}R. \tag{27}$$

Let also the cell diameter be chosen so small that the surface that is orthogonal to one trajectory of the tube intersects the other trajectories of the tube at an angle more than $\pi/4$.

Cut off the open tube by the hypersurface in the place where it has been imbedded into the layer

$$\frac{5}{4}R - \frac{\delta}{2} < |x| < \frac{5}{4}R \tag{28}$$

at first so that the edges of this tube be embedded into this layer.

Denote these cutoff tubes by $T_1, T_2, \ldots, T_S$. If each open tube is divided with a partition, then a set-theoretical sum of closed tubes, tubes $T_1, T_2, \ldots, T_S$ their partitions spheres $S_1, S_2, \ldots, S_{N,}$ and the set F on the sphere $|x| = (7/4)R$ divides the spheres $|x| = R$ and $|x| = 2R$. Note that $\int_S \omega|\partial u/\partial n| d\sigma$ along the surface of each tube equals to zero, since $\partial u/\partial n$ identically equals to zero.

Now we have to choose partitions so that the integral $\int_S \omega|\partial u/\partial n| d\sigma$ was of the desired value. Denote by U_i the domain bounded by T_i with corresponding cell and hypersurface cutting off this tube. We have $U_i \cap U_j = \emptyset$ and therefore

$$\sum_{i=1}^{m} \omega(U_i) < 2\omega(D). \tag{29}$$

Consider a tube T_i and corresponding domain U_i. Choose any trajectory on this tube. Denote it by L_i. The length $\mu_i L_i$ of the curve L_i satisfies the inequality

$$\mu_i L_i \geq \frac{R}{2}. \tag{30}$$

Let introduce on L_i a parameter l (length of the arc), counted from the cell. By $\sigma_i(l)$ denote the cross-section by U_i hypersurface passing thought the point, corresponding to l and orthogonal to the trajectory L_i at this point. Let the diameter of cells be so small

$$\int_{L_i} dl \int_{\sigma_i(l)} \omega(x) d\sigma < 2\omega(U_i). \tag{31}$$

Then by Chebyshev inequality a set H points $l \in L_i$ where

$$\int_{\sigma_i(l)} \omega(x) d\sigma > \frac{8}{R}\omega(U_i) \tag{32}$$

satisfies the inequality $\mu_i H < R/4$ and hence by virtue of (55) for $E = L_i \setminus H$ it is valid and

$$\mu_1 E > R/4. \tag{33}$$

At the points of the curve L_i the derivative $\partial u/\partial l$ preserves its sign, and therefore

$$\int_E \left|\frac{\partial u}{\partial l}\right| dl \le \int_{L_i} \left|\frac{\partial u}{\partial l}\right| dl \le \operatorname*{osc}_{D'_\delta} u. \tag{34}$$

Hence, by using (65) and a mean value theorem for one variable function we find that there exists $l_0 \in E$

$$\left\|\frac{\partial u}{\partial l}\right\|_{l=l_0} \le \frac{4}{R} \operatorname*{osc}_{D'_\delta} u. \tag{35}$$

But on the other hand

$$\left\|\frac{\partial u}{\partial l}\right\|_{l=l_0} = |\nabla u|_{l=l_0}. \tag{36}$$

Together with (67) it gives

$$|\nabla u|_{l=l_0} \int_{\sigma_i(l_0)} \omega(x)\, d\sigma \le \frac{8}{R}\frac{4}{R}\omega(U_i)(\operatorname{osc} u_D). \tag{37}$$

Now, let the diameter of cells be still so small that

$$\int_{\sigma_i(l_0)} \omega(x)|\nabla u| d\sigma \le \frac{16 \cdot 4}{R}\omega(U_i)(\operatorname{osc} u_D), \tag{38}$$

(we can do it, since the derivatives $\partial u/\partial x_i$ are uniformly continuous). Therefore according to (53)

$$\sum_{i=1}^{S}\int_{\sigma_i(l_0)}\omega(x)|\nabla u|d\sigma \le \frac{16\cdot 4}{R}\omega(U_i)(\mathrm{osc}\,u_D). \tag{39}$$

Define by Σ a set-theoretical sum of all closed tubes, all open tubes T_i, all $\sigma_i(l_0)$, all spheres S_i and sets F on the sphere $|x| = (7/4)R$. Then, we get by (4), (49), (51), and (73)

$$\int_{\Sigma}\omega(x)\left|\frac{\partial u}{\partial n}\right|d\sigma \le K\frac{\omega(D)(\mathrm{osc}\,u_D)}{R^p}. \tag{40}$$

Then, we get by (4), (49), (51), (73)

$$\int_{\Sigma}\omega(x)\left|\frac{\partial u}{\partial n}\right|d\sigma \le K\frac{\omega(D)(\mathrm{osc}\,u_D)}{R^p}. \tag{41}$$

The lemma is proved. □

Denote by $W^1_{2,\omega}(D)$ the Banach space of the functions $u(x)$ defined in D with the finite norm

$$\|u\|_{W^1_{2,\omega}(D)} = \left(\int_D \omega\left(u^2 + \sum_{i=1}^{n} u_i^2\right)dx\right)^{1/2}, \tag{42}$$

and let $\overset{o}{W}{}^1_{2,\omega}(D)$ be a completion of $C_0^\infty(D)$ by the norm of the space $W^1_{2,\omega}(D)$.

By $m_H^s(A)$ we will denote the Hausdorff measure of the set A of order $s > 0$. Further, everywhere the notation $C(\cdots)$ means that the positive constant C depends only on the content of brackets.

Theorem 2. *Let D be a bounded domain in E_n and let $E \subset \overline{D}$ be a compact. If with respect to the coefficients of the operator $\mathcal{L}$ the conditions* (3)–(5) *are fulfilled, then for removability of the compact E with respect to the* (1) *in the space $C^\lambda_\omega(D)$ it sufficies that*

$$m_H^{n-2+\lambda}(E) = 0. \tag{43}$$

Proof. At first we show that without loss of generality we can suppose the condition $\partial D \in C^1$ is fulfilled. Suppose that the condition (43) provides the removability of the compact E for the domains, whose boundary is the surface of the class C^1, but $\partial D \in C^1$, and by fulfilling (43) the compact E is not removable. Then the problem (7) has a nontrivial solution $u(x)$, moreover $u|_E = f(x)$ and $f(x) \neq 0$. We always can suppose the lowest coefficients of the operator $\mathcal{L}$ is infinitely differentiable in D. Moreover, without loss of generality, we'll suppose that the coefficients of the operator $\mathcal{L}$ are extended to a ball $B \supset \overline{D}$ with saving the conditions (3)–(5).

Let $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \min\{f(x), 0\}$, and $u^\pm(x)$ be generalized by Wiener (see [15]) solutions of the boundary value problems

$$\mathcal{L}u^\pm = 0, \quad x \in D \setminus E, \quad u^\pm|_{\partial D \setminus E} = 0, \quad u^\pm|_E = f^\pm. \tag{44}$$

Evidently, $u(x) = u^+(x) + u^-(x)$. Further, let D' be such a domain that $\partial D' \in C^1, \overline{D} \subset D'$, $\overline{D'} \subset B$, and $\vartheta^\pm(x)$ be solutions of the problems

$$\mathcal{L}\vartheta^\pm = 0, \quad x \in D' \setminus E, \quad \vartheta^\pm|_{\partial D'} = 0, \quad \vartheta^\pm|_E = f^\pm, \quad \vartheta^\pm(x) \in C^\lambda_\omega(D'). \tag{45}$$

By the maximum principle for $x \in D$,

$$0 \le u^+(x) \le \vartheta^+(x), \qquad \vartheta^-(x) \le u^-(x) \le 0. \tag{46}$$

But according to our supposition, $\vartheta^+(x) \equiv \vartheta^-(x) \equiv 0$. Hence, it follows that $u(x) \equiv 0$. So, we'll suppose that $\partial D \in C^1$. Now, let $u(x)$ be a solution of the problem (7), and the condition (43) be fulfilled. Give an arbitrary $\varepsilon > 0$. Then there exists a sufficiently small positive number δ and a system of the balls $\{B_{r_k}(x^k)\}$, $k = 1, 2, \ldots$, such that $r_k < \delta$, $E \subset \bigcup_{k=1}^\infty B_{r_k}(x^k)$ and

$$\sum_{k=1}^\infty r_k^{n-2+\lambda} < \varepsilon. \tag{47}$$

Consider a system of the spheres $\{B_{2r_k}(x^k)\}$, and let $D_k = D \cap B_{2r_k}(x^k), k = 1, 2, \ldots$. Without loss of generality we can suppose that the cover $\{B_{2r_k}(x^k)\}$ has a finite multiplicity $a_0(n)$. By the Landis-Gerver theorem, for every k, there exists a piece-wise smooth surface Σ_k dividing in D_k the spheres $S_{r_k}(x^k)$ and $S_{2r_k}(x^k)$, such that

$$\int_{\Sigma_k} \omega \left| \frac{\partial u}{\partial \nu} \right| ds \le K \operatorname*{oscu}_{D_k} \frac{\omega(D_k)}{r_k^2}. \tag{48}$$

Since $u(x) \in C^\lambda_\omega(D)$, there exists a constant $H_1 > 0$ depending only on the function $u(x)$ such that

$$\operatorname*{oscu}_{D_k} \omega \le H_1 (2r_k)^\lambda. \tag{49}$$

Besides,

$$\omega(D_k) \le \operatorname{mes}_n B_{2r_k}\left(x^k\right) = \Omega_n 2^n r_k^n; \quad k = 1, 2, \ldots, \tag{50}$$

where $\Omega_n = \mathrm{mes}_n B_1(0)$. Using (49) and (50) in (48), we get

$$\int_{\Sigma_k} \omega \left| \frac{\partial u}{\partial \nu} \right| ds \leq C_1 r_k^{n-2+\lambda}; \ k = 1, 2, \ldots, \tag{51}$$

where $C_1 = KH_1 2^{n+\lambda}$.

Let D_Σ be an open set situated in $D \setminus E$ whose boundary consists of unification of Σ and Γ, where $\Sigma = \bigcup_{k=1}^{\infty} \Sigma_k$, $\Gamma = \partial D \setminus \bigcup_{k=1}^{\infty} D_k^+$, D_k^+ is a part of D_k remaining after the removing of points situated between Σ and $S_{2r_k}(x^k)$; $k = 1, 2, \ldots$. Denote by D'_Σ the arbitrary connected component D_Σ, and by $\mathcal{M}$ we denote the elliptic operator of divergent structure

$$\mathcal{M} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right). \tag{52}$$

According to Green formula for any functions $z(x)$ and $W(x)$ belonging to the intersection $C^2(D'_\Sigma) \cap C^1(\overline{D}'_\Sigma)$, we have

$$\int_{D'_\Sigma} (z\mathcal{M}\beta - \beta\mathcal{M}z)\, dx = \int_{\partial D'_\Sigma} \left(z \frac{\partial \beta}{\partial \nu} - \beta \frac{\partial z}{\partial \nu} \right) ds. \tag{53}$$

Since $\partial D \in C^1$, then $u(x) \in C^1(D'_\Sigma) \cap C^1(\overline{D'_\Sigma})(x) \in C^1(\overline{D}'_\Sigma)$ (see [16]). From (53) choosing the functions $z = 1$, $\beta = \omega u^2$, we have

$$\int_{D'_\Sigma} \mathcal{M}\left(\omega u^2\right) dx = 2 \int_{\partial D'_\Sigma} \omega u \frac{\partial u}{\partial \nu} ds + \int_{\partial D_\Sigma} \omega_{x_i} u^2 ds. \tag{54}$$

But $|u(x)| \leq M < \infty$ for $x \in \overline{D}$. Let us put the condition

$$\omega_{x_i} < c\omega. \tag{$*$}$$

By virtue of condition (52) and $\int_{\partial D_\Sigma} \omega u^2 ds < C_3 M\varepsilon$, subject to (51) and (47), we conclude

$$\begin{aligned} \int_{D'_\Sigma} \mathcal{M}\left(\omega u^2\right) dx &\leq 2Ma_0 \sum_{k=1}^{\infty} \int_{\Sigma_k} \omega \left| \frac{\partial u}{\partial \nu} \right| ds + \int_{D'_\Sigma} \omega u^2 dx \\ &\leq 2Ma_0 C_1 \sum_{k=1}^{\infty} r_k^{n-2+\alpha} + \varepsilon M c_2 < C_3 \varepsilon, \end{aligned} \tag{55}$$

where $C_3 = 2Ma_0 C_1$.

On the other hand

$$\mathcal{M}\left(\omega u^2\right) = 6u\omega\mathcal{M}(u) + 2\sum_{i,j=1}^{n}\omega a_{ij}u_iu_j + (2u+1)\sum_{i,j=1}^{n}a_{ij}u_{x_j}\omega_{x_i} + \sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial x_i}u\omega_{x_j} + \sum_{i,j=1}^{n}a_{ij}u\omega_{x_ix_j} \tag{56}$$

and besides,

$$\mathcal{M}u = \mathcal{L}u + \sum_{i=1}^{n}d_i(x)u_i - c(x)u, \tag{57}$$

where

$$d_i(x) = \sum_{j=1}^{n}\frac{\partial a_{ij}(x)}{\partial x_j} - b_i(x), \qquad i = 1,\ldots,n. \tag{58}$$

It is evident that by virtue of conditions (4) and (5) $|d_i(x)| \le d_0 < \infty;\ i = 1,\ldots,n$. Thus, from (55) we obtain

$$\begin{aligned} &6\int_{D'_\Sigma} u\omega\sum_{i=1}^{n}d_i(x)u_i dx - 6\int_{D'_\Sigma} u^2c(x)dx + 2\int_{D'_\Sigma}\sum_{i,j=1}^{n}\omega(x)a_{ij}u_iu_j dx \\ &+(2u+1)\int_{D'_\Sigma}\sum_{i,j=1}^{n}a_{ij}u_j\omega_{x_i}dx + \int_{D'_\Sigma}\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial x_j}u\omega_{x_i}dx + |\nabla u|^2dx \\ &+\int_{D'_\Sigma}\sum_{i,j=1}^{n}a_{ij}u\omega_{x_ix_j}dx < C_3\varepsilon. \end{aligned} \tag{59}$$

Hence, for any $\alpha > 0$ it follows that

$$\begin{aligned} 2\gamma\int_{D'_\Sigma}\omega|\nabla u|^2dx &< 6d_0\int_{D'_\Sigma}\omega|u||u_i|dx + 6\int_{D'_\Sigma}u^2\omega(x) + (2u+1)\int_{D'_\Sigma}a_{ij}u_j\omega_{x_i}dx \\ &+ d_0\int_{D'_\Sigma}u\omega_{x_i}^2dx + \int_{D'_\Sigma}a_{ij}u\omega_{x_ix_j} + C_3\varepsilon \le 6\frac{d_0}{\varepsilon}\int_{D'_\Sigma}|u|^2dx \\ &+ 6\frac{d_0\varepsilon}{2}\int_{D'_\Sigma}\omega^2|\nabla u|^2dx + (2n+1)\int_{D'_\Sigma}u_j\omega dx \\ &+ d_0\int_{D'_\Sigma}u\omega dx + \gamma C_4\ \varepsilon \le 6\frac{d_0}{\varepsilon}M\mathrm{mes}_nD\ + \frac{(2M+1)\gamma}{\varepsilon}\mathrm{mes}_nD \\ &+ d_0M\omega(D) + \gamma C_4M\omega(D) + C_3\varepsilon. \end{aligned} \tag{60}$$

If we take into account that

$$\left|\omega_{x_i x_j}\right| < C_4 \omega(x), \tag{61}$$

then from here we have that

$$\int_{D'_\Sigma} \omega^2 |\nabla u|^2 dx \le C_5, \tag{62}$$

where $C_5 = (6d_0 + (2M+1))M\mathrm{mes}_n D + (d_0 M + \gamma C_4 M)\omega(D) + C_3/\gamma$. Without loss of generality we assume that $\varepsilon \le 1$. Hence we have $\int_D \omega^2 |\nabla u|^2 dx \le C_6$.

Thus $u(x) \in W^1_{2,\omega}(D)$. From the boundary condition and $\mathrm{mes}_{n-1}(\partial D \cap E) = 0$ we get $u(x) \in W^1_{2,\omega}(D)$. Now, let $\sigma \ge 2$ be a number which will be chosen later, $D^+_\Sigma = \{x : x \in D'_\Sigma, u(x) > 0\}$. Without loss of generality, we suppose that the set D^+_Σ is not empty. Supposing in (53) $z = 1$, $\beta = \omega u^\sigma$, we get

$$\begin{aligned}\int_{D^+_\Sigma} \mathcal{M}(\omega u^\sigma)\, dx &= \sigma \int_{\partial D^+_\Sigma} \left(\omega_\nu u^\sigma + \sigma u^{\sigma-1} \frac{\partial u}{\partial \nu}\right) ds \\ &\le M^\sigma \int_{\partial D^+_\Sigma} \omega ds + \sigma M^{\sigma-1} \int_{\partial D^+_\Sigma} \left|\frac{\partial u}{\partial \nu}\right| ds \le C_5(a_0, M, \sigma, C_1)\varepsilon.\end{aligned} \tag{63}$$

But, on the other hand,

$$\begin{aligned}\mathcal{M}(u^\sigma) &= \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial \omega u^\sigma}{\partial x_j}\right) \\ &= \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\omega\left(\sigma u^{\sigma-1}\frac{\partial u}{\partial x_j}\right) + \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\omega_{x_i}\frac{\partial u^\sigma}{\partial x_j}\right)\right) \\ &= \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\omega\sigma u^{\sigma-1}\frac{\partial u}{\partial x_j}\right) + \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\sigma u^{\sigma-1}\omega x\frac{\partial u}{\partial x_j}\right) \\ &= \sigma\omega u^{\sigma-1}\mathcal{M}(u) + \sigma\omega\frac{\partial}{\partial x_i}\left(a_{ij}u^{\sigma-1}\frac{\partial u}{\partial x_j}\right) + \sigma u^{\sigma-1}\frac{\partial}{\partial x_i}\left(a_{ij}\omega\frac{\partial u}{\partial x_j}\right) + \beta \\ &= \sigma\omega u^{\sigma-1}\mathcal{M}(u) + \sigma\omega u^{\sigma-1}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sigma\omega a_{ij}u_{x_j}(\sigma-1)u^{\sigma-2}u_{x_i} \\ &\quad + \sigma u^{\sigma-1}\omega_{x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sigma u^{\sigma-1}\omega\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \beta \\ &= 3\sigma\omega u^{\sigma-1}M(u) + \sigma(\sigma-1)a_{ij}ux_i u_{x_j}u^{\sigma-2}\omega + \sigma u^{\sigma-1}\omega_{x_i}a_{ij}u_{x_j} + \beta \\ &= \sigma\int_{D^+_\Sigma} d_i(x)u_{x_i}u\omega\, dx - \sigma(\sigma-1)\int_{D^+_\Sigma} u^\sigma\omega(x)c(x)\, dx \\ &\quad + \sigma(\sigma-1)\int_{D^+_\Sigma}\sum_{i,j=1}^n u^{\sigma-2}\omega(x)a_{ij}u_iu_j dx + (2u+1)\int_{D^+_\Sigma}\sum_{i,j=1}^n a_{ij}u_j\omega_{x_j}u^{\sigma-1}.\end{aligned} \tag{64}$$

Hence, we conclude

$$\sigma(\sigma-1)\int_{D_\Sigma^+}\omega^2u^{\sigma-2}|\nabla u|^2dx \le d_0\int_{D_\Sigma^+}u^{\sigma-1}\omega u_i\,dx \le d_0\int_{D_\Sigma^+}u^{\sigma-1}\omega u_i\,dx \le \frac{d_0\varepsilon}{2}\int_{D_\Sigma^+}u^\sigma\,dx. \tag{65}$$

Let $D^+ = \{x : x \in D, u(x) > 0\}$, D_1^+ an arbitrary connected component of D^+. Subject to the arbitrariness of ε from (65) we get

$$(\sigma-1)\gamma\int_{D_1^+}\omega u^{\sigma-2}|\nabla u|^2\,dx \le d_0\int_{D_1^+}\omega u^{\sigma-1}\sum_{i=1}^n|u_i|dx. \tag{66}$$

Thus, for any $\mu > 0$

$$\begin{aligned}(\sigma-1)\gamma\int_{D_1^+}\omega u^{\sigma-2}|\nabla u|^2dx &\le \frac{d_0\mu}{2}\int_{D_1^+}\omega u^{\sigma-2}\left(\sum_{i=1}^n|u_i|\right)^2dx + \frac{d_0}{2\mu}\int_{D_1^+}\omega u^\sigma\,dx\\ &\le \frac{d_0\mu n}{2}\int_{D_1^+}\omega u^{\sigma-2}|\nabla u|^2\,dx + \frac{d_0}{2\mu}\int_{D_1^+}\omega u^\sigma\,dx.\end{aligned} \tag{67}$$

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But, on the other hand,

$$I = -\sigma\sum_{i=1}^n\int_{D_1^+}x_i\omega u^{\sigma-1}u_i\,dx = -\sum_{i=1}^n\int_{D_1^+}x_i\omega(u^\sigma)_i\,dx = n\int_{D_1^+}\omega u^\sigma\,dx \tag{68}$$

and besides, for any $\beta > 0$

$$I = \frac{\sigma\beta}{2}\int_{D_1^+}r^2\omega u^\sigma\,dx + \frac{\sigma}{2\beta}\int_{D_1^+}u^{\sigma-2}\omega^2|\nabla u|^2\,dx. \tag{69}$$

Then

$$I \le \frac{\sigma\beta}{2}\int_{D_1^+}r^2\omega u^\sigma dx + \frac{\sigma}{2\beta}\int_{D_1^+}\omega^2|\nabla u|^2u^{\sigma-2}dx, \tag{70}$$

where $r = |x|$. Denote by $k(D)$ the quantity $\sup_{x\in D}|x|$. Without loss of generality we'll suppose that $k(D) = 1$. Then

$$I \le \frac{\sigma}{2\beta}\int_{D_1^+}\omega u^\sigma dx + \frac{\sigma}{2\beta}\int_{D_1^+}\omega^2u^{\sigma-2}|\nabla u|^2dx. \tag{71}$$

Thus,

$$\left(n-\frac{\sigma\beta}{2}\right)\int_{D_1^+}\omega u^\sigma\,dx+\frac{\sigma}{2\beta}\int_{D_1^+}\omega^2u^{\sigma-2}|\nabla u|^2dx. \tag{72}$$

Now, choosing $\beta = n/\sigma$, we finally obtain

$$\int_{D_1^+}\omega u^\sigma\,dx\le\frac{\sigma^2}{n^2}\int_{D_1^+}\omega^2u^{\sigma-2}|\nabla u|^2dx. \tag{73}$$

Subject to (73) in (67), we conclude

$$(\sigma-1)\gamma\int_{D_1^+}\omega^2u^{\sigma-2}|\nabla u|^2\,dx\le\left(\frac{d_0\varepsilon n}{2}+\frac{d_0\sigma^2}{2\varepsilon n^2}\right)\int_{D_1^+}\omega^2u^{\sigma-2}|\nabla u|^2dx. \tag{74}$$

Now choose μ such that

$$(\sigma-1)\gamma>\frac{d_0\mu n}{2}+\frac{d_0\sigma^2}{2\mu n^2}. \tag{75}$$

Then from (73)–(75) it will follow that $u(x)\equiv 0$ in D_1^+, and thus $u(x)\equiv 0$ in D. Suppose that $\mu=(\sigma-1)\gamma/d_0n$. Then (75) is equivalent to the condition

$$n>\left(\frac{\sigma}{\sigma-1}\right)^2\left(\frac{d_0}{\gamma}\right)^2. \tag{76}$$

At first, suppose that

$$n>\left(\frac{d_0}{\gamma}\right)^2. \tag{77}$$

Let's choose and fix such a big $\sigma\ge 2$ that by fulfilling (77) the inequality (76) is true. Thus, the theorem is proved, if with respect to n the condition (77) is fulfilled. Show that it is true for any $n\ge 3$. For that, at first, note that if $k(D)\ne 1$, then condition (77) will take the form

$$n>\left(\frac{d_0k(D)}{\gamma}\right)^2. \tag{78}$$

Now, let the condition (77) be not fulfilled. Denote by k the least natural number for which

$$n+k>\left(\frac{d_0}{\gamma}\right)^2. \tag{79}$$

Consider $(n+k)$-dimensional semicylinder $D' = D \times (-\delta_0, \delta_0) \times \cdots \times (-\delta_0, \delta_0)$, where the number $\delta_0 > 0$ will be chosen later. Since $\omega(D) = 1$, then $\omega(D') \leq 1 + \delta_0\sqrt{k}$. Let's choose and fix δ_0 so small that along with the condition (79), the condition

$$n + k > \left(\frac{d_0\omega(D')}{\gamma}\right)^2 \tag{80}$$

was fulfilled too.

Let

$$y = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}), \qquad E' = \underbrace{E \times [-\delta_0, \delta_0] \times \cdots \times [-\delta_0, \delta_0]}_{k \text{ times}}. \tag{81}$$

Consider on the domain D' the equation

$$\mathcal{L}'_\vartheta = \sum_{i,j=1}^{n} a_{ij}(x)\vartheta_{ij} + \sum_{i=1}^{k} \frac{\partial^2 \vartheta}{\partial x_{n+i}^2} + \sum_{i=1}^{n} b_i(x)\vartheta_i + c(x)\vartheta = 0. \tag{82}$$

It is easy to see that the function $\vartheta(y) = u(x)$ is a solution of (82) in $D' \setminus E'$. Besides, $m_H^{n+k-2+\lambda}(E') = (2\delta_0)^k m_H^{n-2+\lambda}(E) = 0$, the function $\vartheta(y)$ vanishes on $(\partial D \times \underbrace{[-\delta_0, \delta_0] \times \cdots \times [-\delta_0, \delta_0]}_{k \text{ times}}) \setminus E'$ and $\partial\vartheta/\partial\nu' = 0$ at $x_{n+i} = \pm\delta_0$, $i = 1, \ldots, k$, where $\partial/\partial\nu'$ is a derivative by the conormal generated by the operator $\mathcal{L}'$. Noting that $\gamma(\mathcal{L}') = \gamma(\mathcal{L})$, $d_0(\mathcal{L}') = d_0(\mathcal{L})$ and subject to the condition (80), from the proved above we conclude that $\vartheta(y) \equiv 0$, that is, D'. The theorem is proved. □

Remark 3. As is seen from the proof, the assertion of the theorem remains valid if instead of the condition (4) it is required that the coefficients $a_{ij}(x)$ $(i, j = 1, \ldots, n)$ have to satisfy in domain D the uniform Lipschitz condition with weight.

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