International Scholarly Research Network
ISRN Mathematical Analysis
Volume 2011, Article ID 621901, 14 pages
doi:10.5402/2011/621901

Research Article

An Iterative Approximation Method for a Common Fixed Point of Two Pseudocontractive Mappings

Habtu Zegeye

Department of Mathematics, University of Botswana, Private Bag 00704, Gaborone, Botswana

Correspondence should be addressed to Habtu Zegeye, habtuzh@yahoo.com

Received 24 January 2011; Accepted 22 March 2011

Academic Editors: G. L. Karakostas, G. Mantica, O. Miyagaki, N. Shioji, and C. Zhu

We introduce an iterative process for finding an element in the common fixed point sets of two continuous pseudocontractive mappings. As a consequence, we provide an approximation method for a common fixed point of a finite family of pseudocontractive mappings. Furthermore, our convergence theorem is applied to a convex minimization problem. Our theorems extend and unify most of the results that have been proved for this class of nonlinear mappings.

1. Introduction

Let H be a real Hilbert space. A mapping T with domain $D(T) \subset H$ and range $R(T)$ in H is called *pseudocontractive* if for each $x, y \in D(T)$ we have

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2. \tag{1.1}$$

T is called *strongly pseudocontractive* if there exists $k \in (0,1)$ such that

$$\langle x - y, Tx - Ty \rangle \leq k\|x - y\|^2, \quad \forall x, y \in D(T), \tag{1.2}$$

and T is said to be *k-strict pseudocontractive* if there exists a constant $0 \leq k < 1$ such that

$$\langle x - y, Tx - Ty \rangle \leq \|x - y\|^2 - k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in D(T). \tag{1.3}$$

The operator T is called *Lipschitzian* if there exists $L \geq 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in D(T)$. If $L = 1$, then T is called *nonexpansive*, and if $L \in [0,1)$, then T is called a *contraction*. As a result of Kato [1], it follows from inequality (1.1) that T is pseudocontractive if and only if the inequality

$$\|x-y\| \leq \|(1+t)(x-y)-t(Tx-Ty)\| \tag{1.4}$$

holds for each $x, y \in D(T)$ and for all $t > 0$.

Apart from being an important generalization of nonexpansive, strongly pseudocontractive and k-strict pseudocontractive mappings, interest in pseudocontractive mappings stems mainly from their firm connection with the important class of nonlinear *accretive* operators, where a mapping A with domain $D(A)$ and range $R(A)$ in H is called *accretive* if the inequality

$$\|x-y\| \leq \|x-y+s(Ax-Ay)\| \tag{1.5}$$

holds for every $x, y \in D(A)$ and for all $s > 0$. We observe that A is accretive if and only if $T := I - A$ is pseudocontractive, and thus a zero of A, $N(A) := \{x \in D(A) : Ax = 0\}$, is a fixed point of T, $F(T) := \{x \in D(T) : Tx = x\}$. It is now well known that if A is accretive then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems. Consequently, considerable research efforts have been devoted to iterative methods for approximating fixed points of T when T is pseudocontractive (see, e.g., [2–4] and the references contained therein).

Construction of fixed points of nonexpansive mappings via Mann's algorithm [5] has extensively been investigated recently in the literature (see, e.g., [6, 7] and references therein). Related works can also be found in [7–18]. Mann's algorithm is defined by $x_0 \in K$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.6}$$

where $\{\alpha_n\}$ is a real control sequence in the interval $(0,1)$. If T is a nonexpansive mapping with a fixed point and if the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm (1.6) converges weakly to a fixed point of T (this is indeed true in a uniformly convex Banach space with a Fréchet differentiable norm [7]). However, this convergence is in general not strong (see the counterexample in [19]; see also [20]).

For a sequence $\{\alpha_n\}$ of real numbers in $(0,1)$ and an arbitrary $u \in C$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$ and

$$x_{n+1} := \alpha_{n+1}u + (1-\alpha_{n+1})T(x_n), \quad n \geq 0, \tag{1.7}$$

where T is a nonexpansive mapping of C into itself. Halpern [11] was the first to study the convergence of Algorithm (1.7) in the framework of Hilbert spaces. Lions [14] and Wittmann [21] improved the result of Halpern by proving strong convergence of $\{x_n\}$ to a fixed point of T if the real sequence $\{\alpha_n\}$ satisfies certain conditions. Reich [22], Shioji and Takahashi [16], and Zegeye and Shahzad [23] extend the result of Wittmann [21] to the case of Banach space.

In 2000, Moudafi [24] introduced viscosity approximation method and proved that if H is a real Hilbert space, for given $x_0 \in C$, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n)T(x_n), \quad n \geq 0, \tag{1.8}$$

where $f : C \rightarrow C$ is a contraction mapping and $\{\alpha_n\} \subset (0,1)$ satisfies certain conditions, converges strongly to a common fixed point of T. Moudafi [24] generalizes Halpern's theorems in the direction of viscosity approximations. In [25], Zegeye et al. extended Moudafi's result to the class of Lipschitz pseudocontractive mappings in Banach spaces more general than Hilbert spaces. Viscosity approximations are very important because they are applied to convex optimization, linear programming, monotone inclusions, and elliptic differential equations.

Our concern now is the following. Is it possible to construct a viscosity approximation sequence that converges strongly to a fixed point of pseudocontractive mappings more general than nonexpansive mappings?

In this paper, motivated and inspired by the work of Halpern [11], Moudafi [24], and the methods of Takahashi and Zembayashi [26], we introduce a viscosity approximation method for finding a common fixed point of two continuous pseudocontractive mappings. As a consequence, we provide an approximation method for a common fixed point of finite family of pseudocontractive mappings. This provides affirmative answer to the above concern. Furthermore, we apply our convergence theorem to the convex minimization problem. Our theorems extend and unify most of the results that have been proved for this important class of nonlinear operators.

2. Preliminaries

Let C be closed and convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C. In connection with metric projection, we have the following lemma.

Lemma 2.1. *Let C be a nonempty convex subset of a Hilbert space H. Let $x \in H$ and $x_0 \in C$. Then, $x_0 = P_C x$ if and only if*

$$\langle z - x_0, x_0 - x\rangle \geq 0, \quad \forall z \in C. \tag{2.2}$$

Lemma 2.2 (see [27]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \sigma_n, \quad n \geq 0, \tag{2.3}$$

where (i) $\{\gamma_n\} \subset [0,1]$, $\sum \gamma_n = \infty$ and (ii) $\limsup_{n\to\infty} \sigma_n/\gamma_n \leq 0$ or $\sum |\sigma_n| < \infty$. Then, $a_n \rightarrow 0$ as $n \rightarrow \infty$.

By a similar argument in [28], we have the following lemma.

Lemma 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous accretive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Az \rangle + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \tag{2.4}$$

Moreover, by a similar argument of the proof of Lemmas 2.8 and 2.9 of [26], we get the following lemma.

Lemma 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous accretive mapping. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x := \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\} \tag{2.5}$$

for all $x \in H$. Then, the following hold:

(1) *T_r is single valued;*

(2) *T_r is firmly nonexpansive type mapping, that is, for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle; \tag{2.6}$$

(3) *$F(T_r) = \mathrm{VI}(C, A)$;*

(4) *$\mathrm{VI}(C, A)$ is closed and convex.*

3. Main Results

In the sequel, we will make use of the following lemmas.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Tz \rangle - \frac{1}{r}\langle y - z, (1 + r)z - x \rangle \leq 0, \quad \forall y \in C. \tag{3.1}$$

Proof. Let $x \in H$ and $r > 0$. Let $A := I - T$, where I is the identity mapping on C. Then, clearly A is continuous accretive mapping. Thus, by Lemma 2.3, there exists $z \in C$ such that $\langle y - z, Az \rangle + (1/r)\langle y - z, z - x \rangle \geq 0$, for all $y \in C$. But this is equivalent to $\langle y - z, Tz \rangle - (1/r)\langle y - z, (1 + r)z - x \rangle \leq 0$, for all $y \in C$. Hence, the lemma holds. □

Lemma 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be continuous pseudocontractive mapping. For $r > 0$ and $x \in H$, define a mapping $F_r : H \to C$ as follows:*

$$F_r x := \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r}\langle y - z, (1 + r)z - x \rangle \leq 0, \ \forall y \in C \right\} \tag{3.2}$$

for all $x \in H$. Then, the following hold:

(1) *F_r is single valued;*

(2) *F_r is firmly nonexpansive type mapping, that is, for all $x, y \in H$,*

$$\|F_r x - F_r y\|^2 \leq \langle F_r x - F_r y, x - y\rangle; \tag{3.3}$$

(3) *$F(F_r) = F(T)$;*

(4) *$F(T)$ is closed and convex.*

Proof. We note that $\langle y - z, Tz\rangle - (1/r)\langle y - z, (1+r)z - x\rangle \leq 0$, for all $y \in C$, is equivalent to $\langle y - z, Az\rangle + (1/r)\langle y - z, z - x\rangle \geq 0$, for all $y \in C$, where $A := I - T$ is continuous accretive mapping and I the identity mapping on C. Moreover, as T is self-map, we have that $\mathrm{VI}(C, A) = F(T)$. Thus, by Lemma 2.4, the conclusions of (1)–(4) hold. □

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_i : C \to C$, for $i = 1, 2$, be continuous pseudocontractive mappings. Then, in what follows, $T_{r_n}, F_{r_n} : H \to C$ are defined as follows. For $x \in H$ and $\{r_n\} \subset (0, \infty)$, define

$$\begin{aligned} T_{r_n} x &:= \left\{ z \in C : \langle y - z, T_1 z\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x\rangle \leq 0,\ \forall y \in C \right\}, \\ F_{r_n} x &:= \left\{ z \in C : \langle y - z, T_2 z\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x\rangle \leq 0,\ \forall y \in C \right\}. \end{aligned} \tag{3.4}$$

Now, we prove our main convergence theorem.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_i : C \to C$, for $i = 1, 2$, be continuous pseudocontractive mappings such that $F := \bigcap_{i=1}^{2} F(T_i) \neq \emptyset$. Let f be a contraction of C into itself, and let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_{r_n} F_{r_n} x_n, \tag{3.5}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\liminf_{n\to\infty} r_n > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then, the sequence $\{x_n\}_{n\geq 1}$ converges strongly to $z \in F$, where $z = P_F f(z)$.

Proof. Let $Q = P_F$. Then, Qf is a contraction of C into C. In fact, we have that

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq \beta\|x - y\|, \tag{3.6}$$

for all $x, y \in C$, where β is contraction constant of f. So Qf is a contraction of C into itself. Since C is closed subset of H, there exists a unique element z of C such that $z = Qf(z)$.

Let $v \in F$, and let $u_n := T_{r_n} w_n$, where $w_n := F_{r_n} x_n$. Then, we have from Lemma 3.2 that

$$\|u_n - v\| = \|T_{r_n} w_n - T_{r_n} v\| \leq \|w_n - v\| = \|F_{r_n} x_n - F_{r_n} v\| \leq \|x_n - v\|. \tag{3.7}$$

Moreover, from (3.5) and (3.7), we get that

$$\begin{aligned}
\|x_{n+1}-v\| &= \left\|\alpha_n\left(f(x_n)-v\right)+(1-\alpha_n)(T_{r_n}F_{r_n}x_n-v)\right\| \\
&\le \alpha_n\left\|f(x_n)-v\right\|+(1-\alpha_n)\|u_n-v\| \\
&\le \alpha_n\left\|f(x_n)-v\right\|+(1-\alpha_n)\|x_n-v\| \\
&\le \alpha_n\left\|f(x_n)-f(v)\right\|+\alpha_n\left\|f(v)-v\right\|+(1-\alpha_n)\|x_n-v\| \\
&\le \alpha_n\beta\|x_n-v\|+\alpha_n\left\|f(v)-v\right\|+(1-\alpha_n)\|x_n-v\| \\
&= (1-(1-\beta)\alpha_n)\|x_n-v\|+(1-\beta)\alpha_n\left(\frac{1}{1-\beta}\left\|f(v)-v\right\|\right) \\
&\le \max\left\{\|x_n-v\|,\frac{1}{1-\beta}\left\|f(v)-v\right\|\right\}.
\end{aligned} \tag{3.8}$$

By induction, we get that

$$\|x_n-v\|\le \max\left\{\|x_1-v\|,\frac{1}{1-\beta}\left\|f(v)-v\right\|\right\},\quad n\ge 1. \tag{3.9}$$

Therefore, $\{x_n\}$ is bounded. Consequently, we get that $\{w_n\}$, $\{T_{r_n}w_n\}$, $\{F_{r_n}x_n\}$, and $\{f(x_n)\}$ are bounded. Next, we show that $\|x_{n+1}-x_n\|\to 0$. But from (3.5) we have that

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \left\|\alpha_n f(x_n)+(1-\alpha_n)u_n-\alpha_{n-1}f(x_{n-1})-(1-\alpha_{n-1})u_{n-1}\right\| \\
&\le \left\|\alpha_n f(x_n)-\alpha_n f(x_{n-1})+\alpha_n f(x_{n-1})-\alpha_{n-1}f(x_{n-1})\right. \\
&\qquad \left.+\ (1-\alpha_n)u_n-(1-\alpha_n)u_{n-1}+(1-\alpha_n)u_{n-1}-(1-\alpha_{n-1})u_{n-1}\right\| \\
&\le \alpha_n\beta\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|K+(1-\alpha_n)\cdot\|u_n-u_{n-1}\| \\
&\le \alpha_n\beta\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|K+(1-\alpha_n)\cdot\|w_n-w_{n-1}\|,
\end{aligned} \tag{3.10}$$

where $K=2\sup\{\|f(x_n)\|+\|u_n\|:n\in\mathbf{N}\}$. Moreover, since $w_n=F_{r_n}x_n$ and $w_{n+1}=F_{r_{n+1}}x_{n+1}$, we get that

$$\left\langle y-w_n,T_2w_n\right\rangle-\frac{1}{r_n}\left\langle y-w_n,(1+r_n)w_n-x_n\right\rangle\le 0,\quad \forall y\in C, \tag{3.11}$$

$$\left\langle y-w_{n+1},T_2w_{n+1}\right\rangle-\frac{1}{r_{n+1}}\left\langle y-w_{n+1},(1+r_{n+1})w_{n+1}-x_{n+1}\right\rangle\le 0,\quad \forall y\in C. \tag{3.12}$$

Putting $y := w_{n+1}$ in (3.11) and $y := w_n$ in (3.12), we get that

$$\langle w_{n+1} - w_n, T_2 w_n \rangle - \frac{1}{r_n} \langle w_{n+1} - w_n, (1 + r_n) w_n - x_n \rangle \le 0, \tag{3.13}$$

$$\langle w_n - w_{n+1}, T_2 w_{n+1} \rangle - \frac{1}{r_{n+1}} \langle w_n - w_{n+1}, (1 + r_{n+1}) w_{n+1} - x_{n+1} \rangle \le 0. \tag{3.14}$$

Adding (3.13) and (3.14), we have

$$\langle w_{n+1} - w_n, T_2 w_n - T_2 w_{n+1} \rangle - \left\langle w_{n+1} - w_n, \frac{(1 + r_n) w_n - x_n}{r_n} - \frac{(1 + r_{n+1}) w_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \le 0, \tag{3.15}$$

which implies that

$$\langle w_{n+1} - w_n, (w_{n+1} - T_2 w_{n+1}) - (w_n - T_2 w_n) \rangle - \left\langle w_{n+1} - w_n, \frac{w_n - x_n}{r_n} - \frac{w_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \le 0. \tag{3.16}$$

Now, using the fact that T_2 is pseudocontractive, we get that

$$\left\langle w_{n+1} - w_n, \frac{w_n - x_n}{r_n} - \frac{w_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \ge 0, \tag{3.17}$$

and hence

$$\left\langle w_{n+1} - w_n, w_n - w_{n+1} + w_{n+1} - x_n - \frac{r_n}{r_{n+1}} (w_{n+1} - x_{n+1}) \right\rangle \ge 0. \tag{3.18}$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in \mathbf{N}$. Then, we have

$$\begin{aligned} \|w_{n+1} - w_n\|^2 &\le \left\langle w_{n+1} - w_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(w_{n+1} - x_{n+1}) \right\rangle \\ &\le \|w_{n+1} - w_n\| \left\{ \|x_{n+1} - x_n\| + \left| \left(1 - \frac{r_n}{r_{n+1}}\right) \right| \cdot \|w_{n+1} - x_{n+1}\| \right\}, \end{aligned} \tag{3.19}$$

and hence from (3.19) we obtain that

$$\begin{aligned} \|w_{n+1} - w_n\| &\le \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \cdot \|w_{n+1} - x_{n+1}\| \\ &\le \|x_{n+1} - x_n\| + \frac{1}{b} |r_{n+1} - r_n| L, \end{aligned} \tag{3.20}$$

where $L = \sup\{\|w_n - x_n\| : n \in \mathbf{N}\}$. Furthermore, from (3.10) and (3.20), we have that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \alpha_n\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K \\ &\quad + (1-\alpha_n)\left(\|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|L\right) \\ &= \left(1 - \alpha_n(1-\beta)\right)\|x_n - x_{n-1}\| + K|\alpha_n - \alpha_{n-1}| \\ &\quad + (1-\alpha_n)\frac{L}{b}|r_n - r_{n-1}|.\end{aligned} \tag{3.21}$$

Now, using conditions of $\{\alpha_n\}$, $\{r_n\}$ and Lemma 2.2, we have that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.22}$$

Consequently, from (3.20) and (3.22), we obtain that

$$\lim_{n\to\infty}\|w_{n+1} - w_n\| = 0. \tag{3.23}$$

Similarly, taking $u_n = T_{r_n}w_n$ and $u_{n+1} = T_{r_{n+1}}w_{n+1}$ and following the method used for w_n, we get that $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$. Furthermore, since $x_n = \alpha_{n-1}f(x_{n-1}) + (1-\alpha_{n-1})u_{n-1}$, we have that

$$\begin{aligned}\|x_n - u_n\| &\leq \|x_n - u_{n-1}\| + \|u_{n-1} - u_n\| \\ &\leq \alpha_{n-1}\left\|f(x_{n-1}) - u_{n-1}\right\| + \|u_{n-1} - u_n\|.\end{aligned} \tag{3.24}$$

Thus, since $\alpha_n \to 0$, we obtain that

$$\|x_n - u_n\| \longrightarrow 0. \tag{3.25}$$

Moreover, for $v \in F$, using Lemma 3.2, we get that

$$\begin{aligned}\|w_n - v\|^2 &= \|F_{r_n}x_n - F_{r_n}v\|^2 \\ &\leq \langle F_{r_n}x_n - F_{r_n}v, x_n - v\rangle \\ &= \langle w_n - v, x_n - v\rangle \\ &= \frac{1}{2}\left(\|w_n - v\|^2 + \|x_n - v\|^2 - \|x_n - w_n\|^2\right),\end{aligned} \tag{3.26}$$

and hence

$$\|w_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - w_n\|^2. \tag{3.27}$$

Therefore, from (3.5), the convexity of $\|\cdot\|^2$, (3.7) and (3.27) we get that

$$\begin{aligned}\|x_{n+1}-v\|^2 &= \left\|\alpha_n f(x_n)+(1-\alpha_n)u_n-v\right\|^2\\ &\le \alpha_n\left\|f(x_n)-v\right\|^2+(1-\alpha_n)\|u_n-v\|^2\\ &\le \alpha_n\left\|f(x_n)-v\right\|^2+(1-\alpha_n)\|w_n-v\|^2\\ &\le \alpha_n\left\|f(x_n)-v\right\|^2+(1-\alpha_n)\left(\|x_n-v\|^2-\|x_n-w_n\|^2\right)\\ &\le \alpha_n\left\|f(x_n)-v\right\|^2+\|x_n-v\|^2-(1-\alpha_n)\|x_n-w_n\|^2,\end{aligned} \tag{3.28}$$

and hence

$$\begin{aligned}(1-\alpha_n)\|x_n-w_n\|^2 &\le \alpha_n\left\|f(x_n)-v\right\|^2+\|x_n-v\|^2-\|x_{n+1}-v\|^2\\ &\le \alpha_n\left\|f(x_n)-v\right\|^2+\|x_n-x_{n+1}\|(\|x_n-v\|+\|x_{n+1}-\mathrm{v}\|).\end{aligned} \tag{3.29}$$

So we have $\|x_n-w_n\| \to 0$ as $n \to \infty$. This implies with (3.25) that $\|u_n-w_n\| \le \|u_n-x_n\| + \|x_n-w_n\| \to 0$ as $n \to \infty$.

Next, we show that

$$\limsup_{n\to\infty}\langle f(z)-z, x_n-z\rangle \le 0, \tag{3.30}$$

where $z = P_F f(z)$. To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z)-z, x_n-z\rangle = \lim_{i\to\infty}\langle f(z)-z, x_{n_i}-z\rangle. \tag{3.31}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and $w \in H$ such that $x_{n_{i_j}} \rightharpoonup w$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup w$. Since $\{x_{n_i}\} \subset C$ and C is convex and closed, we get that $w \in C$. Moreover, since $x_n - w_n \to 0$ as $n \to \infty$, we have that $w_{n_i} \rightharpoonup w$. Now, we show that $w \in F$. Note that, from the definition of w_{n_i}, we have

$$\langle y-w_{n_i}, T_2 w_{n_i}\rangle - \frac{1}{r_{n_i}}\langle y-w_{n_i}, (r_{n_i}+1)w_{n_i}-x_{n_i}\rangle \le 0, \quad \forall y \in C. \tag{3.32}$$

Put $z_t = tv + (1-t)w$ for all $t \in (0,1]$ and $v \in C$. Consequently, we get that $z_t \in C$. From (3.32) and pseudocontractivity of T_2, it follows that

$$\begin{aligned}
\langle w_{n_i} - z_t, T_2 z_t \rangle &\geq \langle w_{n_i} - z_t, T_2 z_t \rangle + \langle z_t - w_{n_i}, T_2 w_{n_i} \rangle - \frac{1}{r_{n_i}} \langle z_t - w_{n_i}, (1 + r_{n_i}) w_{n_i} - x_{n_i} \rangle \\
&= -\langle z_t - w_{n_i}, T_2 z_t - T_2 w_{n_i} \rangle - \frac{1}{r_{n_i}} \langle z_t - w_{n_i}, w_{n_i} - x_{n_i} \rangle - \langle z_t - w_{n_i}, w_{n_i} \rangle \\
&\geq -\|z_t - w_{n_i}\|^2 - \frac{1}{r_{n_i}} \langle z_t - w_{n_i}, w_{n_i} - x_{n_i} \rangle - \langle z_t - w_{n_i}, w_n \rangle \\
&= \langle w_{n_i} - z_t, z_t \rangle - \left\langle z_t - w_{n_i}, \frac{w_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle.
\end{aligned} \tag{3.33}$$

Then, since $w_n - x_n \to 0$, as $n \to \infty$, we obtain that $(w_{n_i} - x_{n_i})/r_{n_i} \to 0$ as $i \to \infty$. Thus, as $i \to \infty$, it follows that

$$\langle w - z_t, T_2 z_t \rangle \geq \langle w - z_t, z_t \rangle, \tag{3.34}$$

and hence

$$-\langle v - w, T_2 z_t \rangle \geq -\langle v - w, z_t \rangle \quad \forall v \in C. \tag{3.35}$$

Letting $t \to 0$ and using the fact that T_2 is continuous, we obtain that

$$-\langle v - w, T_2 w \rangle \geq -\langle v - w, w \rangle \quad \forall v \in C. \tag{3.36}$$

Now, let $v = T_2 w$. Then, we obtain that $w = T_2 w$, and hence $w \in F(T_2)$. Furthermore, the fact that $u_n - w_n \to 0$ and $w_{n_i} \rightharpoonup w$ imply that $u_{n_i} \rightharpoonup w$, following the method used for w_n, we obtain that $w \in F(T_1)$, and hence $w \in \bigcap_{i=1}^{2} F(T_i)$. Therefore, since $z = P_F f(z)$, by Lemma 2.1, we have

$$\begin{aligned}
\limsup_{n \to \infty} \langle f(z) - z, x_n - z \rangle &= \lim_{i \to \infty} \langle f(z) - z, x_{n_i} - z \rangle \\
&= \langle f(z) - z, w - z \rangle \leq 0.
\end{aligned} \tag{3.37}$$

Now, we show that $x_n \to z$ as $n \to \infty$. From $x_{n+1} - z = \alpha_n (f(x_n) - z) + (1 - \alpha_n)(u_n - z)$, we have that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1 - \alpha_n)^2 \|u_n - z\|^2 + 2\alpha_n \langle f(x_n) - z, x_{n+1} - z \rangle \\
&= (1 - \alpha_n)^2 \|u_n - z\|^2 + 2\alpha_n \langle f(x_n) - f(z), x_{n+1} - z \rangle + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\
&\leq (1 - \alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n \beta \|x_n - z\| \cdot \|x_{n+1} - z\| + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\
&\leq (1 - \alpha_n)^2 \|x_n - z\|^2 + \alpha_n \beta \left\{ \|x_n - z\|^2 + \|x_{n+1} - z\|^2 \right\} + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle.
\end{aligned} \tag{3.38}$$

This implies that,

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \frac{(1-\alpha_n)^2+\alpha_n\beta}{1-\alpha_n\beta}\|x_n-z\|^2+\frac{2\alpha_n}{1-\alpha_n\beta}\langle f(z)-z, x_{n+1}-z\rangle \\
&= \frac{1-2\alpha_n+\alpha_n\beta}{1-\alpha_n\beta}\|x_n-z\|^2+\frac{\alpha_n^2}{1-\alpha_n\beta}\|x_n-z\|^2+\frac{2\alpha_n}{1-\alpha_n\beta}\langle f(z)-z, x_{n+1}-z\rangle \\
&\leq (1-\gamma_n)\|x_n-z\|^2+\sigma_n,
\end{aligned} \tag{3.39}$$

where $\gamma_n := 2(1-\beta)\alpha_n/(1-\alpha_n\beta)$, $\sigma_n := (2(1-\beta)\alpha_n/(1-\alpha_n\beta))\{\alpha_n M/2(1-\beta)+(1/(1-\beta))\langle f(z)-z, x_{n+1}-z\rangle\}$, for $M=\sup\{\|x_n-z\|^2 : n\in\mathbf{N}\}$. But note that $\sum_{n=1}^{\infty}\gamma_n=\infty$, $\lim_{n\to\infty}\gamma_n=0$, and $\limsup_{n\to\infty}\sigma_n/\gamma_n\leq 0$. Therefore, by Lemma 2.2, we conclude that $\{x_n\}$ converges to $z\in F$, where $z=P_F f(z)$. This completes the proof. □

If, in Theorem 3.3, $f=u\in C$ is a constant mapping, then we get $z=P_F(u)$. In fact, we have the following corollary.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_i : C\to C$, for $i=1,2$, be continuous pseudocontractive mappings such that $F:=\bigcap_{i=1}^{2}F(T_i)\neq\emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1, u\in C$ and*

$$x_{n+1}=\alpha_n u+(1-\alpha_n)T_{r_n}F_{r_n}x_n, \tag{3.40}$$

where $\{\alpha_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$ such that $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\liminf_{n\to\infty}r_n>0$, and $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$. Then, the sequence $\{x_n\}_{n\geq1}$ converges strongly to $z\in F$, where $z=P_F(u)$.

If, in Theorem 3.3, we have that $T_2\equiv I$, identity mapping on C, then we obtain the following corollary.

Corollary 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1 : C\to C$ be continuous pseudocontractive mapping such that $F(T_1)\neq\emptyset$. Let f be a contraction of C into itself, and let $\{x_n\}$ be a sequence generated by $x_1\in C$ and*

$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)T_{r_n}x_n, \tag{3.41}$$

where $\{\alpha_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$ such that $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\liminf_{n\to\infty}r_n>0$, and $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$. Then, the sequence $\{x_n\}_{n\geq1}$ converges strongly to $z\in F$, where $z=P_{F(T_1)}f(z)$.

Let H be a real Hilbert space. Let $A_i : H\to H$, for $i=1,2$, be accretive mappings. Let $T'_{r_n}x := \{z\in H : \langle y-z, (I-A_1)z\rangle-(1/r_n)\langle y-z, (1+r_n)z-x\rangle\leq 0$, for all $y\in H\}$, $F'_{r_n}x := \{z\in H : \langle y-z, (I-A_2)z\rangle-(1/r_n)\langle y-z, (1+r_n)z-x\rangle\leq 0$, for all $y\in H\}$. Then we have the following convergence theorem for a zero of two accretive mappings.

Theorem 3.6. *Let H be a real Hilbert space. Let $A_i : H \to H$, for $i = 1, 2$, be continuous accretive mappings such that $N := \bigcap_{i=1}^{2} N(A_i) \neq \emptyset$. Let f be a contraction of H into itself, and let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T'_{r_n} F'_{r_n} x_n, \tag{3.42}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ such that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\liminf_{n\to\infty} r_n > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then, the sequence $\{x_n\}_{n\geq 1}$ converges strongly to $z \in N$, where $z = P_N(f(z))$.

Proof. Let $T_i := (I - A_i)$, for $i = 1, 2$. Then, we get that T_i, for $i = 1, 2$, are continuous pseudocontractive mappings with $\bigcap_{i=1}^{2} N(A_i) = \bigcap_{i=1}^{2} F(T_i)$. Thus, the conclusion follows from Theorem 3.3. □

The proof of the following theorem can be easily obtained from the method of proof of Theorem 3.3.

Theorem 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_i : C \to C$, for $i = 1, 2, \ldots, L$, be continuous pseudocontractive mappings such that $F := \bigcap_{i=1}^{L} F(T_i) \neq \emptyset$. Let f be a contraction of C into itself, and let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) K_{1,r_n} K_{2,r_n}, \ldots, K_{N,r_n} x_n, \tag{3.43}$$

where $K_{i,r_n} x := \{z \in C : \langle y - z, T_i z\rangle - (1/r_n)\langle y - z, (1 + r_n)z - x\rangle \leq 0$, for all $y \in C\}$, for $i = 1, 2, \ldots, L$, and $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ such that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\liminf_{n\to\infty} r_n > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then, the sequence $\{x_n\}_{n\geq 1}$ converges strongly to $z \in F$, where $z = P_F(f(z))$.

4. Application

In this section, we study the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in Hilbert spaces. Let h and g be continuously Fréchet differentiable convex functionals such that the gradient of h, (∇h) and the gradient of g, (∇g) are continuous and accretive. For $\gamma > 0$ and $x \in H$, let $T''_{r_n} x := \{z \in H : \langle y - z, (I - (\nabla h))z\rangle - (1/r_n)\langle y - z, (1 + r_n)z - x\rangle \leq 0$, for all $y \in H\}$ and $F''_{r_n} x := \{z \in H : \langle y - z, (I - (\nabla g))z\rangle - (1/r_n)\langle y - z, (1 + r_n)z - x\rangle \leq 0$, for all $y \in H\}$ for all $x \in H$. Then, the following theorem holds.

Theorem 4.1. *Let H be a real Hilbert space. Let h and g be continuously Fréchet differentiable convex functionals such that the gradient of h, (∇h) and the gradient of g, (∇g) are continuous and accretive such that $N := N(\nabla h) \cap N(\nabla g) \neq \emptyset$. Let f be a contraction of H into itself, and let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T''_{r_n} F''_{r_n} x_n, \tag{4.1}$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *such that* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\liminf_{n\to\infty} r_n > 0$, *and* $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$. *Then, the sequence* $\{x_n\}_{n\geq 1}$ *converges strongly to* $z \in F$, *where* $z = P_N(f(z))$.

Proof. The conclusion follows from Theorem 3.6. We note that from the convexity and Fréchet differentiability of h and g we have $N(\nabla h) = \arg\min_{y\in C} h(y)$ and $N(\nabla g) = \arg\min_{y\in C} g(y)$. □

Remark 4.2. Our theorems extend and unify most of the results that have been proved for this important class of nonlinear operators. In particular, Theorem 3.3 extends Theorem 2.2 of Moudafi [24] and Theorem 4.1 of Iiduka and Takahashi [12] in the sense that our convergence is for the more general class of continuous pseudocontractive mappings. Moreover, this provides affirmative answer to the concern raised.

References

[1] T. Kato, "Nonlinear semigroups and evolution equations," *Journal of the Mathematical Society of Japan*, vol. 19, pp. 508–520, 1967.

[2] C. E. Chidume and S. A. Mutangadura, "An example of the Mann iteration method for Lipschitz pseudocontractions," *Proceedings of the American Mathematical Society*, vol. 129, no. 8, pp. 2359–2363, 2001.

[3] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, pp. 147–150, 1974.

[4] L. Qihou, "The convergence theorems of the sequence of Ishikawa iterates for hemicontractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 148, no. 1, pp. 55–62, 1990.

[5] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.

[6] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.

[7] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.

[8] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[9] C. E. Chidume and N. Shahzad, "Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 62, no. 6, pp. 1149–1156, 2005.

[10] C. E. Chidume, H. Zegeye, and N. Shahzad, "Convergence theorems for a common fixed point of a finite family of nonself nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2005, no. 2, pp. 233–241, 2005.

[11] B. Halpern, "Fixed points of nonexpanding maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.

[12] H. Iiduka and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 61, no. 3, pp. 341–350, 2005.

[13] S. Ishikawa, "Fixed points and iteration of a nonexpansive mapping in a Banach space," *Proceedings of the American Mathematical Society*, vol. 59, no. 1, pp. 65–71, 1976.

[14] P.-L. Lions, "Approximation de points fixes de contractions," *Comptes Rendus de l'Académie des Sciences. Séries A et B*, vol. 284, no. 21, pp. A1357–A1359, 1977.

[15] C. Martinez-Yanes and H.-K. Xu, "Strong convergence of the CQ method for fixed point iteration processes," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 64, no. 11, pp. 2400–2411, 2006.

[16] N. Shioji and W. Takahashi, "Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 125, no. 12, pp. 3641–3645, 1997.

[17] H.-K. Xu, "Strong convergence of an iterative method for nonexpansive and accretive operators," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 631–643, 2006.

[18] H. Zegeye and N. Shahzad, "Strong convergence theorems for a finite family of nonexpansive mappings and semigroups via the hybrid method," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 1, pp. 325–329, 2010.

[19] A. Genel and J. Lindenstrauss, "An example concerning fixed points," *Israel Journal of Mathematics*, vol. 22, no. 1, pp. 81–86, 1975.

[20] O. Güler, "On the convergence of the proximal point algorithm for convex minimization," *SIAM Journal on Control and Optimization*, vol. 29, no. 2, pp. 403–419, 1991.

[21] R. Wittmann, "Approximation of fixed points of nonexpansive mappings," *Archiv der Mathematik*, vol. 58, no. 5, pp. 486–491, 1992.

[22] S. Reich, "Strong convergence theorems for resolvents of accretive operators in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 75, no. 1, pp. 287–292, 1980.

[23] H. Zegeye and N. Shahzad, "Viscosity approximation methods for a common fixed point of finite family of nonexpansive mappings," *Applied Mathematics and Computation*, vol. 191, no. 1, pp. 155–163, 2007.

[24] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.

[25] H. Zegeye, N. Shahzad, and T. Mekonen, "Viscosity approximation methods for pseudocontractive mappings in Banach spaces," *Applied Mathematics and Computation*, vol. 185, no. 1, pp. 538–546, 2007.

[26] W. Takahashi and K. Zembayashi, "Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 1, pp. 45–57, 2009.

[27] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society. Second Series*, vol. 66, no. 1, pp. 240–256, 2002.

[28] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.


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