Research Article

Existence of Almost Periodic Solutions for a Class of Abstract Impulsive Differential Equations

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We study the existence of piecewise almost periodic solutions for a class of abstract impulsive semilinear differential equations.

1. Introduction

Due to their numerous applications to model the dynamics of evolving processes arising in mechanics, electrical engineering, medicine, population dynamics, among many others important areas of technology and science, the theory of impulsive differential equations has attracted the attention of researchers. It is well known that there are many evolutionary processes in which the system is subject to abrupt changes of state at certain moments of time between intervals of continuous evolution. Such changes can be approximated as being instantaneous changes at states, or in the form of impulses. We refer the reader to [1–3] for numerous examples and applications.

Several of the systems mentioned in the preceding paragraph can be modeled by impulsive differential equations. The literature related to ordinary differential equations with impulses is very extensive and we refer the reader to [2, 4] and the references therein. Impulsive ordinary differential equations appear frequently in applications involving nonlinear systems. It is worth to point out here that the analysis of such systems with discontinuities is different from that for continuous systems (see for instance [5]). This has been a special motivation to attract the attention of authors to the theory of systems with impulses. For this reason the theory of impulsive differential equations has drawn the attention of many authors in recent years, see for instance [6–11].

On the other hand, the study of the existence of periodic solutions, as well as its numerous generalizations to almost periodic solutions, asymptotically almost periodic solutions, almost automorphic solutions, asymptotically almost automorphic solutions, pseudo almost periodic solutions, and so forth, is one of the most attracting topics in the qualitative theory of differential equations due both to its mathematical interest as to their applications in various areas of applied science. Likewise, the existence of periodic and almost periodic solutions of impulsive ordinary differential equations has been considered by many authors. The reader can see [4, 12–19]. Recently Stamov [16] have investigated the existence of almost periodic solutions for the Lasota-Ważewska model and the exponential stability of these solutions, while Ahmad and Stamov [14] have investigated the existence of almost periodic processes corresponding to competitive nonautonomous Lotka-Volterra systems described by integrodifferential equations with infinite delay. However, the study of almost periodic solutions of impulsive partial differential equations are not sufficiently considered in the literature. This fact is the main motivation of the present work.

In this work we are concerned with nonautonomous abstract differential equations. Let $\mathbb{X}$ be a Banach space endowed with the norm $\|\cdot\|_{\mathbb{X}}$. Specifically, we study the existence of almost periodic mild solutions for a class of impulsive differential equation of the form

$$\begin{aligned} u'(t) &= A(t)u(t) + F(u(t)) + f(t), \quad t \in \mathbb{R}, \quad t \neq t_i, \quad i \in \mathbb{Z}, \\ \Delta u(t_i) &= I_i(u(t_i)), \end{aligned} \tag{1.1}$$

where $t_i \in \mathbb{R}$ for $i \in \mathbb{Z}$ and $A(t) : D(A(t)) \subseteq \mathbb{X} \rightarrow \mathbb{X}$, $t \in \mathbb{R}$, are closed linear operators that satisfy some properties that will be established later. Moreover, $f : \mathbb{R} \rightarrow \mathbb{X}$ and $F, I_i : \mathbb{X} \rightarrow \mathbb{X}$ for $i \in \mathbb{Z}$ are appropriate functions. In addition, the symbol $\Delta \xi(t)$ represents the jump of the function ξ at t , which is defined by $\Delta \xi(t) = \xi(t^+) - \xi(t^-)$, where the notations $\xi(t^+)$ and $\xi(t^-)$ represent, respectively, the right hand side and the left hand side limits of the function ξ at t .

Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be Banach spaces, the symbol $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ stands for the Banach space formed by all bounded linear operators from $\mathbb{Y}$ into $\mathbb{X}$ endowed with the uniform operator topology. In particular, we abbreviate this notation by $\mathcal{L}(\mathbb{X})$ when $\mathbb{Y} = \mathbb{X}$. Moreover, for $r > 0$, we denote by $B_r(\mathbb{X})$ the closed ball consisting of all $x \in \mathbb{X}$ such that $\|x\| \leq r$. For a linear operator A , we denote by $\rho(A)$ its resolvent set and, for $\lambda \in \rho(A)$, $R(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent operator. In addition, we denote by $\text{AP}(\mathbb{X})$ the space of almost periodic functions from $\mathbb{R}$ into $\mathbb{X}$ in the Bohr sense. It is well known that $\text{AP}(\mathbb{X})$ endowed with the norm of uniform convergence is a Banach space. For details about the almost periodic functions we refer to [20].

We now give a brief summary of this work. In Section 2 we recall some definitions and properties that are needed to establish our results. In Section 3, we establish our results about the existence of almost periodic mild solutions to the impulsive Cauchy problem (1.1). Finally, in Section 4 we apply our abstract results to a concrete system with impulses.

2. Preliminaries

We begin by defining the countable subsets of $\mathbb{R}$ where the impulsive effects are concentrated.

Let $\mathcal{B}$ be the set consisting all real sequences $T = \{t_i\}_{i \in \mathbb{Z}}$ such that $\alpha = \inf_{i \in \mathbb{Z}} (t_{i+1} - t_i) > 0$. It is immediate that this condition implies that $\lim_{i \rightarrow \infty} t_i = \infty$ and $\lim_{i \rightarrow -\infty} t_i = -\infty$.

Remark 2.1. If $T = \{t_i\}_{i \in \mathbb{Z}} \in \mathcal{B}$ and $t_i < t \leq t_{i+1}$, then

$$t - t_k = t - t_i + t_i - t_k \geq t - t_i + (i - k)\alpha, \quad (2.1)$$

for $k \leq i$.

Definition 2.2. The sequence $T = \{t_k\}_{k \in \mathbb{Z}} \in \mathcal{B}$ is said to be almost periodic if for each $\epsilon > 0$ there is a relatively dense set Q_ϵ in $\mathbb{R}$ having the following property: for each $\tau \in Q_\epsilon$ there is an integer $q \in \mathbb{Z}$ such that $|t_{i+q} - t_i - \tau| < \epsilon$ for all $i \in \mathbb{Z}$.

Given a sequence $T = \{t_i\}_{i \in \mathbb{Z}} \in \mathcal{B}$, we consider $t_i^j = t_{i+j} - t_i$, for $i, j \in \mathbb{Z}$.

Definition 2.3 (see [4]). The sequence $T = \{t_i\}_{i \in \mathbb{Z}} \in \mathcal{B}$ is said to be uniformly almost periodic if for each $\epsilon > 0$ there exists a relatively dense set Q_ϵ in $\mathbb{Z}$ such that

$$|t_{i+q}^j - t_i^j| < \epsilon, \quad (2.2)$$

for all $i, j \in \mathbb{Z}$ and $q \in Q_\epsilon$.

Remark 2.4. It has been established in [4, Lemma 27] that if T is uniformly almost periodic, then it is almost periodic.

For $T = \{t_i\}_{i \in \mathbb{Z}} \in \mathcal{B}$ fixed, let $\mathcal{PC}_T(\mathbb{R}, \mathbb{X})$ be the space consisting of all piecewise continuous functions $x : \mathbb{R} \rightarrow \mathbb{X}$ such that $x(\cdot)$ is continuous at t for every $t \notin T$ and $x(t_i) = x(t_i^-)$ for all $i \in \mathbb{Z}$, where, as it is usual, we have denoted by $x(t^-)$ the left hand limit of $x(\cdot)$ at t . Hence a function $x \in \mathcal{PC}_T(\mathbb{R}, \mathbb{X})$ if x is left continuous on $\mathbb{R}$, right continuous on $\mathbb{R} \setminus T$ and it can have jump discontinuities on the right hand side at the points of T .

Definition 2.5 (see [4]). Let $T = \{t_i\}_{i \in \mathbb{Z}} \in \mathcal{B}$. The function $\varphi \in \mathcal{PC}_T(\mathbb{R}, \mathbb{X})$ is said to be T -piecewise almost periodic if the following conditions are fulfilled.

- (i) The set T is almost periodic.
- (ii) For every $\epsilon > 0$ there exists $\delta > 0$ such that $\|x(t) - x(t')\| < \epsilon$ for all $t, t' \in (t_i, t_{i+1})$ such that $|t - t'| < \delta$ and some $i \in \mathbb{Z}$.
- (iii) For each $\epsilon > 0$ there exists a relatively dense set Ω_ϵ in $\mathbb{R}$ such that if $\tau \in \Omega_\epsilon$, then

$$\|\varphi(t + \tau) - \varphi(t)\|_{\mathbb{X}} < \epsilon, \quad (2.3)$$

for all $t \in \mathbb{R}$ satisfying the condition $|t - t_i| > \epsilon$ for all $i \in \mathbb{Z}$.

For a fixed sequence $T = \{t_i\}_{i \in \mathbb{Z}}$ as in the Definition 2.2, we denote by $\text{AP}_T(\mathbb{X})$ the space formed for all T -piecewise almost periodic function. It is well known that the space $\text{AP}_T(\mathbb{X})$ endowed with the norm of the uniform convergence $\|\cdot\|_\infty$ is a Banach space.

For more details about the properties of piecewise almost periodic functions we refer the reader to [4]. We only mention here a pair of properties which are essentials for our developments.

Lemma 2.6. *Let $g \in \text{AP}_T(\mathbb{X})$. Then $\mathcal{R}(g)$ is a relatively compact subset of $\mathbb{X}$.*

Proof. Let $\epsilon > 0$ be fixed. It follows from the Definition 2.5(ii) that there exists $\delta > 0$ such that

$$\|g(t) - g(t')\| < \epsilon, \quad (2.4)$$

for all $t, t' \in (t_i, t_{i+1})$ such that $|t - t'| < \delta$ and $i \in \mathbb{Z}$. We can assume that $\delta < \epsilon$. Applying now the Definition 2.5(iii) we infer the existence of $L > 0$ having the following property. In any interval of length greater or equal that L there exists τ such that $\|g(t + \tau) - g(t)\|_{\mathbb{X}} < \delta$ for all $t \in \mathbb{R}$ satisfying the condition $|t - t_i| \geq \delta$ for all $i \in \mathbb{Z}$. Hence, if $t \in (t_i, t_{i+1})$, $|t - t_i| > \delta$ and $|t - t_{i+1}| > \delta$, we can select $\tau \in [-t, L - t]$ and

$$g(t) \in g(t + \tau) + B_\delta \subseteq g([0, L]) + B_\delta. \quad (2.5)$$

Since g is piecewise continuous, the set $K_L = g([0, L])$ is compact. Moreover, if $t \in (t_i, t_i + \delta)$, then $\|g(t) - g(t_i + \delta)\| < \epsilon$ which implies that $g(t) \in K_L + B_\epsilon$. A similar property holds when $t \in (t_{i+1} - \delta, t_{i+1})$. Consequently, for every $t \in \mathbb{R}$ we have that $g(t) \in K_L + B_\epsilon$. Since ϵ was arbitrarily chosen, this completes the proof. $\square$

Definition 2.7 (see [21, Definition 2]). The sequence of maps $I_i : \mathbb{X} \rightarrow \mathbb{X}$, $i \in \mathbb{Z}$, is said to be almost periodic if for each $x \in \mathbb{X}$ and $\epsilon > 0$ there is a relatively dense set $Q_\epsilon(x)$ in $\mathbb{Z}$ such that $\|I_{i+q}(x) - I_i(x)\| < \epsilon$ for all $q \in Q_\epsilon(x)$ and $i \in \mathbb{Z}$.

The following property follows easily from the Definition 2.7.

Lemma 2.8. *Let $I_i : \mathbb{X} \rightarrow \mathbb{X}$, $i \in \mathbb{Z}$, be an almost periodic sequence of maps and let $K \subset \mathbb{X}$ a compact set. Assume that there is a positive constant L such that*

$$\|I_i(x) - I_i(y)\| \leq L\|x - y\|, \quad (2.6)$$

for all $x \in \mathbb{X}$ and $i \in \mathbb{Z}$. Then the set of sequences $\{(I_i(x))_{i \in \mathbb{Z}} : x \in K\}$ is uniformly almost periodic.

Combining Lemma 2.8 with Definition 2.5 we obtain the following property which will be essential for our developments.

Lemma 2.9. *Let $T = \{t_i\}_{i \in \mathbb{Z}} \in \mathcal{B}$ and let $\varphi \in \text{AP}_T(\mathbb{X})$. Assume that the sequence of vector-valued functions $\{I_i\}_{i \in \mathbb{Z}}$ is almost periodic. If there is a positive constant L such that*

$$\|I_i(x) - I_i(y)\| \leq L\|x - y\|, \quad (2.7)$$

for all $x \in \mathbb{X}$ and $i \in \mathbb{Z}$, then $\{I_i(\varphi(t_i))\}_{i \in \mathbb{Z}}$ is an almost periodic sequence.

Proof. To prove this lemma we are going to use the characterization of almost periodic sequences established in [4, Theorem 70]. Let $(q''_n)_n$ be a sequence of integer numbers. Since $\mathcal{R}(\varphi)$ is relatively compact, it follows from the Lemma 2.8 that $(I_i(x))_{i \in \mathbb{Z}}$ is a uniformly almost periodic sequence for $x \in \mathcal{R}(\varphi)$. Therefore, we can extract a subsequence $(q'_n)_n$ of $(q''_n)_n$ such that $I_{i+q'_n}(x)$ is convergent as $n \rightarrow \infty$ uniformly for $x \in \mathcal{R}(\varphi)$ and $i \in \mathbb{Z}$. On the other hand, as

φ is a T -piecewise almost periodic function and $\inf_{i \in \mathbb{Z}} (t_{i+1} - t_i) > 0$, it follows from [4, Lemma 37] that the sequence $\varphi(t_i)$ is almost periodic. Consequently, we can extract a subsequence $(q_n)_n$ of $(q'_n)_n$ such that $\varphi(t_{i+q_n})_n$ is convergent as $n \rightarrow \infty$ uniformly for $i \in \mathbb{Z}$. This allows us to conclude that

$$\begin{aligned}
& \|I_{i+q_m}(\varphi(t_{i+q_m})) - I_{i+q_n}(\varphi(t_{i+q_n}))\| \\
& \leq \|I_{i+q_m}(\varphi(t_{i+q_m})) - I_{i+q_m}(\varphi(t_{i+q_n}))\| + \|I_{i+q_m}(\varphi(t_{i+q_n})) - I_{i+q_n}(\varphi(t_{i+q_n}))\| \\
& \leq L\|\varphi(t_{i+q_m}) - \varphi(t_{i+q_n})\| + \|I_{i+q_m}(\varphi(t_{i+q_n})) - I_{i+q_n}(\varphi(t_{i+q_n}))\| \\
& \leq L\|\varphi(t_{i+q_m}) - \varphi(t_{i+q_n})\| + \sup_{x \in \mathcal{R}(\varphi)} \|I_{i+q_m}(x) - I_{i+q_n}(x)\|,
\end{aligned} \tag{2.8}$$

which implies that $(I_{i+q_n}(\varphi(t_{i+q_n})))_n$ is a Cauchy sequence uniformly for $i \in \mathbb{Z}$. This complete the proof. $\square$

To study the impulsive system (1.1), we recall briefly some important properties of the classical nonautonomous abstract Cauchy problem. Let $A(t) : D(A(t)) \rightarrow \mathbb{X}$ be closed linear operators. We begin by studying the homogeneous problem

$$\begin{aligned}
u'(t) &= A(t)u(t), \quad t \geq s, \quad t, s \in \mathbb{R}, \\
u(s) &= x \in \mathbb{X}.
\end{aligned} \tag{2.9}$$

In what follows we introduce the concept of evolution family associated with the problem (2.9).

Definition 2.10. A family of bounded linear operators $\{U(t, s), t \geq s\} \subset \mathcal{L}(\mathbb{X})$ is called a strongly continuous evolution family if the following conditions are fulfilled.

- (a) $U(t, t) = I$.
- (b) $U(t, s)U(s, r) = U(t, r)$ for every $r \leq s \leq t$.
- (c) For each $x \in \mathbb{X}$ the function $(t, s) \rightarrow U(t, s)x$ is continuous.

If for each $s \leq t$ and each $x \in D(A(s))$ we have that $U(t, s)(D(A(s))) \subseteq D(A(t))$, and the function $t \rightarrow U(t, s)x$ is continuously differentiable with respect to t and a solution of the problem (2.9), then the evolution family $U(t, s)$ is said to be generated by $\{A(t) : t \in \mathbb{R}\}$.

The existence of an evolution family allows us to establish a variation of constants formula to solve the inhomogeneous problem

$$\begin{aligned}
u'(t) &= A(t)u(t) + f(t), \quad t > s, \quad t, s \in \mathbb{R}, \\
u(s) &= x \in \mathbb{X},
\end{aligned} \tag{2.10}$$

where f is a locally integrable function. We refer the reader to [22] for a discussion about this matter. We only recall here that the function

$$u(t) = U(t, s)u_0 + \int_s^t U(t, \xi)f(\xi)d\xi, \quad t \geq s, \quad (2.11)$$

is said to be the mild solution of the problem (2.10).

The problem of establishing the conditions under which $\{A(t) : t \in \mathbb{R}\}$ generates an evolution family that has been studied by several authors. We refer to [23] for a discussion about this subject. In the rest of this work we use the terminology used in [23]. Specifically we consider the following conditions.

(AT1) There are constants $K \geq 0$, $\omega \in \mathbb{R}$, and $\phi \in (\pi/2, \pi)$ such that $\lambda \in \rho(A(t))$ and

$$\|R(\lambda, A(t))\| \leq \frac{K}{1 + |\lambda - \omega|}, \quad (2.12)$$

for $t \in \mathbb{R}$ and $\lambda \in \{w\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda - w)| \leq \phi\}$.

(AT2) There are constants $L \geq 0$ and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$ such that

$$|\lambda|^\nu \|(\omega I - A(t))R(\lambda + \omega, A(t))(R(\omega, A(t)) - R(\omega, A(t)))\| \leq L|t - s|^\mu, \quad (2.13)$$

for $t, s \in \mathbb{R}$ and $|\arg \lambda| \leq \phi$.

The concept of exponential dichotomy is a well-known technique used to study the asymptotic behavior of solutions of differential equations. In the next definition we precise this concept [23, 24].

Definition 2.11. An evolution family $U(t, s)$, $t, s \in \mathbb{R}$, $s \leq t$, on a Banach space $\mathbb{X}$ has an exponential dichotomy if there are a uniformly bounded and strongly continuous map $P : \mathbb{R} \rightarrow \mathcal{L}(\mathbb{X})$ such that each $P(t)$ is a projection, and constants $N, \mu > 0$ such that for $t \geq s$ the following conditions hold.

$$(ex1) \quad U(t, s)P(s) = P(t)U(t, s).$$

$$(ex2) \quad \text{Let } Q(t) = I - P(t). \text{ The restriction } U_Q(t, s) : Q(s)\mathbb{X} \rightarrow Q(t)\mathbb{X} \text{ of } U(t, s) \text{ is invertible} \\ \text{(we set } U_Q(s, t) = U_Q(t, s)^{-1}).$$

$$(ex3) \quad \|U(t, s)P(s)\| \leq Ne^{-\mu(t-s)}, \text{ and } \|U_Q(s, t)Q(t)\| \leq Ne^{-\mu(t-s)}.$$

In this case, the operator-valued map $\Gamma(t, s)$ given by $\Gamma(t, s) = U(t, s)P(s)$, for $t \geq s$ and $\Gamma(t, s) = -U_Q(t, s)Q(s)$, for $t < s$ is called Green's function corresponding to $U(\cdot, \cdot)$ and $P(\cdot)$.

The existence, properties, and examples of Green's functions can be founded in [23].

We only mention here a few properties directly related with our objectives. If f is a bounded function, it follows from (2.11) that the unique bounded mild solution of the equation

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \quad (2.14)$$

is given by

$$u(t) = \int_{-\infty}^{\infty} \Gamma(t, \xi) f(\xi) d\xi, \quad t \in \mathbb{R}. \quad (2.15)$$

Assuming that the conditions (AT1)-(AT2) hold, we introduce the following condition.

$$(R) \quad R(\omega, A(\cdot)) \in \text{AP}(\mathcal{L}(\mathbb{X})).$$

Lemma 2.12 (see [23, Proposition 5.8]). *Assume that the conditions (AT1)-(AT2) and (R) hold. If $U(\cdot, \cdot)$ has an exponential dichotomy with constants $N, \mu > 0$, then for each $\epsilon > 0$ and $h > 0$ there is a relatively dense set $\Omega_{\epsilon, h}$ such that*

$$\|\Gamma(t + \tau, s + \tau) - \Gamma(t, s)\| \leq \epsilon e^{-(\mu/2)|t-s|}, \quad |t - s| > h, \quad t, s \in \mathbb{R}, \quad \tau \in \Omega_{\epsilon, h}. \quad (2.16)$$

We abbreviate this property by writing $\Gamma \in \text{AP}(\mathcal{L}(\mathbb{X}))$.

Remark 2.13. As was pointed out in [25], under the hypotheses of the Lemma 2.12, for all sequence of real numbers $(t'_n)_{n \geq 0}$ and $h > 0$ there is a subsequence $(t_n)_{n \geq 0}$ such that $\Gamma(t + t_n, s + t_n)x$ is uniformly convergent for all $x \in X$ and $t, s \in \mathbb{R}$ with $|t - s| > h$. Taking $h = 1, 1/2, \dots$, and using the method of diagonal choice, it is possible to show that $\Gamma(t + t_n, s + t_n)x$ is uniform convergent with respect to $t, s \in \mathbb{R}$ with $t \neq s$ for each $x \in X$. Moreover, since $P(\cdot) \in \text{AP}(\mathcal{L}(\mathbb{X}))$ and $\Gamma(t, t) = P(t)$, then the preceding assertion holds for all $t, s \in \mathbb{R}$.

3. Existence of Solutions

In this section we keep the notations introduced in the Section 2. In addition, we will assume that $T = \{t_i\}_{i \in \mathbb{Z}} \in \mathcal{B}$ is an uniformly almost periodic sequence of moments, and we consider as standing hypotheses that conditions (AT1)-(AT2) hold and that $U(t, s)$ is an evolution family generated by the operators $A(t)$ with an exponential dichotomy $P(t)$. Initially, we consider the abstract cauchy problem with impulses

$$\begin{aligned} u'(t) &= A(t)u(t) + f(t), \quad t \in \mathbb{R}, \quad t \neq t_i, \quad i \in \mathbb{Z}, \\ \Delta u(t_i) &= x_i, \quad i \in \mathbb{Z}, \end{aligned} \quad (3.1)$$

where $f \in \mathcal{PC}_T(\mathbb{R}, \mathbb{X})$.

Definition 3.1. A function $u \in \mathcal{PC}_T(\mathbb{R}, \mathbb{X})$ is said to be a piecewise continuous mild solution of the problem (3.1) if for all $t \in \mathbb{R}$, $t_i < t \leq t_{i+1}$, we have

$$u(t) = U(t, t_i)u(t_i) + U(t, t_i)x_i + \int_{t_i}^t U(t, s)f(s)ds. \quad (3.2)$$

If we assume that f is bounded and that the set $\{\|x_i\| : i \in \mathbb{Z}\}$ is bounded, it follows from (2.15) that the unique bounded piecewise continuous mild solution of the problem (3.1) is given by

$$u(t) = \int_{-\infty}^{\infty} \Gamma(t, \xi)f(\xi)d\xi + \sum_{t_i < t} \Gamma(t, t_i)x_i, \quad t \in \mathbb{R}. \quad (3.3)$$

Our results of existence of solutions for the problem (1.1) are based in several properties of this construction, which we will collect in the following lemma.

Lemma 3.2. Let $T = \{t_i\}_{i \in \mathbb{Z}} \in \mathcal{B}$ uniformly almost periodic. Assume that $f \in \text{AP}_T(\mathbb{X})$, $\Gamma \in \text{AP}(\mathcal{L}(\mathbb{X}))$ and the sequence $\{x_i : i \in \mathbb{Z}\}$ is almost periodic. Then for each $\epsilon > 0$ there are relatively dense sets Ω_ϵ of $\mathbb{R}$ and Q_ϵ of $\mathbb{Z}$ such that the following conditions hold.

- (i) $\|f(t + \tau) - f(t)\| < \epsilon$ for all $t \in \mathbb{R}$, $|t - t_i| > \epsilon$, $\tau \in \Omega_\epsilon$ and $i \in \mathbb{Z}$.
- (ii) $\|\Gamma(t + \tau, s + \tau) - \Gamma(t, s)\| < \epsilon$ for all $t, s \in \mathbb{R}$, $|t - s| > 0$, $|s - t_i| > \epsilon$, $|t - t_i| > \epsilon$, $\tau \in \Omega_\epsilon$ and $i \in \mathbb{Z}$.
- (iii) $\|x_{i+q} - x_i\|_{\mathbb{X}} < \epsilon$ for all $q \in Q_\epsilon$ and $i \in \mathbb{Z}$.
- (iv) $|t_{i+q} - t_i - \tau| < \epsilon$ for all $q \in Q_\epsilon$ and $\tau \in \Omega_\epsilon$ and $i \in \mathbb{Z}$.

Remark 3.3. The proof of Lemma 3.2 is based on the technique of finding common almost periods. We refer the reader to [4] for a discussion about this method.

Lemma 3.4. Assume that the conditions of Lemma 2.12 hold. Let $x \in \text{AP}_T(\mathbb{X})$ and let $v(t)$ be given by

$$v(t) = \int_{-\infty}^{\infty} \Gamma(t, \xi)x(\xi)d\xi, \quad t \in \mathbb{R}. \quad (3.4)$$

Then the function $v \in \text{AP}(\mathbb{X})$.

Proof. For $\epsilon > 0$, let Ω_ϵ be a relatively dense set of $\mathbb{R}$ formed by ϵ -periods of Γ and x such as in the Lemma 3.2. For $\tau \in \Omega_\epsilon$ we have

$$\begin{aligned} v(t + \tau) - v(t) &= \int_{-\infty}^{\infty} (\Gamma(t + \tau, s) - \Gamma(t, s))x(s)ds \\ &= \int_{-\infty}^t (\Gamma(t + \tau, s) - \Gamma(t, s))x(s)ds + \int_t^{\infty} (\Gamma(t + \tau, s) - \Gamma(t, s))x(s)ds. \end{aligned} \quad (3.5)$$

The first term on the right hand side of (3.5) can be decomposed as

$$\begin{aligned}
 \int_{-\infty}^t (\Gamma(t + \tau, s) - \Gamma(t, s))x(s)ds &= \int_{-\infty}^t (\Gamma(t + \tau, s + \tau) - \Gamma(t, s))x(s)ds \\
 &\quad + \int_{-\infty}^t \Gamma(t + \tau, s + \tau)(x(s + \tau) - x(s))ds \\
 &= J_1 + J_2.
 \end{aligned} \tag{3.6}$$

Let $h > 0$. Applying the Lemma 2.12, we can estimate the first term on the right hand side of the above expression as

$$\begin{aligned}
 \|J_1\| &= \left\| \int_{-\infty}^t (\Gamma(t + \tau, s + \tau) - \Gamma(t, s))x(s)ds \right\| \\
 &\leq \|x\|_{\infty} \int_{-\infty}^{t-h} \|\Gamma(t + \tau, s + \tau) - \Gamma(t, s)\|ds + \int_{t-h}^t \|(\Gamma(t + \tau, s + \tau) - \Gamma(t, s))f(s)\|ds \\
 &\leq \epsilon \|x\|_{\infty} \int_{-\infty}^{t-h} e^{-(\mu/2)(t-s)} ds + \int_{t-h}^t \|(\Gamma(t + \tau, s + \tau) - \Gamma(t, s))x(s)\|ds \\
 &\leq \frac{2}{\mu} \|x\|_{\infty} \epsilon + 2N \|x\|_{\infty} h.
 \end{aligned} \tag{3.7}$$

In order to estimate the second term on the right hand side of (3.6), we assume that $t_i < t \leq t_{i+1}$. For $0 < h < \alpha/2$, then we can split the integral $\int_{-\infty}^t \|\Gamma(t + \tau, s + \tau)(x(s + \tau) - x(s))\|ds$ appropriately to obtain

$$\begin{aligned}
 \|J_2\| &\leq \int_{-\infty}^t \|\Gamma(t + \tau, s + \tau)(x(s + \tau) - x(s))\|ds \\
 &\leq \sum_{j=-\infty}^{i-1} \int_{t_j+h}^{t_{j+1}-h} \|\Gamma(t + \tau, s + \tau)(x(s + \tau) - x(s))\|ds \\
 &\quad + \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+h} \|\Gamma(t + \tau, s + \tau)(x(s + \tau) - x(s))\|ds \\
 &\quad + \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-h}^{t_{j+1}} \|\Gamma(t + \tau, s + \tau)(x(s + \tau) - x(s))\|ds \\
 &\quad + \int_{t_i}^t \|\Gamma(t + \tau, s + \tau)(x(s + \tau) - x(s))\|ds.
 \end{aligned} \tag{3.8}$$

Since $x \in \text{AP}_T(\mathbb{X})$ we have that $\|x(s + \tau) - x(s)\| < \epsilon$ for all $s \in [t_j + h, t_{j+1} - h]$ and $j \in \mathbb{Z}, j \leq i$.

Moreover, in view of that

$$t - s \geq t - t_i + t_i - s \geq t - t_i + t_i - (t_{j+1} - h) \geq t - t_i + \alpha(i - 1 - j) + h, \quad (3.9)$$

we can estimate

$$\begin{aligned} & \sum_{j=-\infty}^{i-1} \int_{t_j+h}^{t_{j+1}-h} \|\Gamma(t+\tau, s+\tau)(x(s+\tau) - x(s))\| ds \\ & \leq \epsilon \sum_{j=-\infty}^{i-1} \int_{t_j+h}^{t_{j+1}-h} \|\Gamma(t+\tau, s+\tau)\| ds \\ & \leq \epsilon N \sum_{j=-\infty}^{i-1} \int_{t_j+h}^{t_{j+1}-h} e^{-\mu(t-s)} ds \\ & \leq \frac{\epsilon N}{\mu} \sum_{j=-\infty}^{i-1} e^{-\mu(t-t_{j+1}+h)} \\ & \leq \frac{\epsilon N}{\mu} \sum_{j=-\infty}^{i-1} e^{-\mu\alpha(i-j-1)} \\ & \leq \frac{\epsilon N}{\mu(1 - e^{-\mu\alpha})}. \end{aligned} \quad (3.10)$$

To estimate the second term on the right hand side of (3.8) we observe

$$\begin{aligned} & \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+1}-h} \|\Gamma(t+\tau, s+\tau)(x(s+\tau) - x(s))\| ds \\ & \leq 2N\|x\|_{\infty} \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+1}-h} e^{-\mu(t-s)} ds \\ & \leq 2N\epsilon\|x\|_{\infty} e^{-\mu h} \sum_{j=-\infty}^{i-1} e^{-\mu(t-t_j)} \\ & \leq 2N\epsilon e^{-\mu(h+t-t_i)}\|x\|_{\infty} \sum_{j=-\infty}^{i-1} e^{-\mu\alpha(i-j)} \\ & \leq \frac{2N\epsilon\|x\|_{\infty}}{1 - e^{-\mu\alpha}}. \end{aligned} \quad (3.11)$$

We obtain similar estimates for the third and fourth term on the right hand side of (3.8). Combining these estimate, we get that $\|J_1\| \leq O(\epsilon)$ and $\|J_2\| \leq O(\epsilon)$ independent of $t \in \mathbb{R}$.

Furthermore, proceeding in a similar way we can show that

$$\left\| \int_t^\infty (\Gamma(t + \tau, s) - \Gamma(t, s))x(s)ds \right\| = O(\epsilon), \quad (3.12)$$

which completes the proof that $v(\cdot)$ is an almost periodic function in the Bohr's sense. $\square$

We introduce a new condition on the evolution family $U(\cdot, \cdot)$.

(H) For each $x \in \mathbb{X}$, $U(t + h, t)x \rightarrow x$ as $h \rightarrow 0^+$ uniformly for $t \in \mathbb{R}$.

Lemma 3.5. *Assume that the assumptions (R) and (H) hold. Let K be a compact subset of $\mathbb{X}$. Then $\|\Gamma(t + h, s + h')x - \Gamma(t, s)x\| \rightarrow 0$, as $h, h' \rightarrow 0^+$ uniformly for $s, t \in \mathbb{R}$ and $x \in K$.*

Proof. Since $P \in \text{AP}(\mathcal{L}(\mathbb{X}))$ [23, Corollary 5.1] the set $W = \{P(s)x : x \in K, s \in \mathbb{R}\}$ is relatively compact in $\mathbb{X}$. Moreover, using (H), we have

$$\Gamma(s + h, s)x - P(s)x = [U(s + h, s) - I]P(s)x = [U(s + h, s) - I]y \rightarrow 0, \quad h \rightarrow 0, \quad (3.13)$$

uniformly for $y \in W$.

We will prove separately that $\Gamma(t + h, s)x - \Gamma(t, s)x \rightarrow 0$ and $\Gamma(t, s + h)x - \Gamma(t, s)x \rightarrow 0$ as $h \rightarrow 0$ uniformly for $s, t \in \mathbb{R}$ and $x \in \mathbb{X}$. Assume first that $h > 0$. We have

$$\Gamma(t + h, s)x - \Gamma(t, s)x = [U(t + h, t) - I]\Gamma(t, s)x \quad (3.14)$$

$$= [U(t + h, t) - I, \Gamma(t, s) - P(s)]x + [U(t + h, t) - I]P(s)x. \quad (3.15)$$

Using (3.13), (3.15), and assumption (H) we deduce that there exists $\beta > 0$ such that

$$\Gamma(t + h, s)x - \Gamma(t, s)x \rightarrow 0, \quad h \rightarrow 0, \quad (3.16)$$

uniformly for $s, t \in \mathbb{R}$ with $t - s \leq \beta$ and $x \in \mathbb{X}$. Similarly, the assertion (3.16) is immediate when $t - s > 0$ is large enough, because $\|\Gamma(t, s)\| \rightarrow 0$, as $t - s \rightarrow \infty$. We consider now $0 < \beta < a$ and $\beta \leq t - s \leq a$. It follows from [23, Theorem 5.9] that the function $r \mapsto \Gamma(t + r, s + r)$ is almost periodic. This implies that the set $W' = \{\Gamma(t, s)x : x \in K, \beta \leq t - s \leq a\}$ is relatively compact in $\mathbb{X}$. Using again the assumption (H) we get that $[U(t + h, t) - I]y \rightarrow 0$, $h \rightarrow 0$, uniformly for $t \in \mathbb{R}$ and $y \in W'$. Applying (3.14) we also obtain (3.16). A similar argument shows the assertion for $h < 0$.

Now we estimate $\Gamma(t, s + h)x - \Gamma(t, s)x$ as $h \rightarrow 0$. Assume now that $h > 0$. Using again the assumption (H) and the fact that $P(\cdot)$ is almost periodic, from the decomposition

$$\Gamma(t, s + h)x - \Gamma(t, s)x = U(t, s + h)[P(s + h)x - U(s + h, s)P(s)x], \quad (3.17)$$

it follows easily that $\Gamma(t, s + h)x - \Gamma(t, s)x$ as $h \rightarrow 0$ uniformly for $s, t \in \mathbb{R}$ and $x \in \mathbb{X}$. Arguing in a similar way we obtain the property for $h < 0$. $\square$

Lemma 3.6. *Assume that the assumptions (R) and (H) hold. Let $(x_i)_{i \in \mathbb{Z}}$ be a discrete almost periodic sequence and let $g(t)$ be given by*

$$g(t) = \Gamma(t, t_i)x_i, \quad t_i < t \leq t_{i+1}, \quad i \in \mathbb{Z}. \quad (3.18)$$

Then the function $g \in \text{AP}_T(\mathbb{X})$.

Proof. It is clear that g is T -piecewise continuous. Let $\epsilon, \beta > 0$ be fixed. Let $M > 0$ be a constant such that $\|x_j\| \leq M$ for all $j \in \mathbb{Z}$. Moreover, applying the Lemma 3.5 we can choose $\delta > 0$ such that

$$\|\Gamma(t, s)x_j - \Gamma(t', s')x_j\| \leq \frac{\epsilon}{3}, \quad (3.19)$$

for $|t - t'| \leq \delta$, $|s - s'| \leq \delta$, and for all $j \in \mathbb{Z}$.

We consider $0 < \epsilon' = \min\{\epsilon/3M, \epsilon/3N, \beta, \delta\}$.

Applying the Lemma 3.2 we infer that there are relatively dense sets Ω of $\mathbb{R}$ and Q of $\mathbb{Z}$ such that the following conditions hold.

- (i) $\|\Gamma(t + \tau, s + \tau) - \Gamma(t, s)\| < \epsilon'$ for all $t, s \in \mathbb{R}$, $|t - s| > 0$, $|s - t_j| > \epsilon'$, $|t - t_j| > \epsilon'$, $\tau \in \Omega$ and $j \in \mathbb{Z}$.
- (ii) $\|x_{j+q} - x_j\|_{\mathbb{X}} < \epsilon'$ for all $q \in Q$ and $j \in \mathbb{Z}$.
- (iii) $|t_{j+q} - t_j - \tau| < \epsilon'$ for all $q \in Q$, $\tau \in \Omega$ and $i \in \mathbb{Z}$.

We consider $t \in \mathbb{R}$ and $i \in \mathbb{Z}$ such that $t_i < t < t_{i+1}$, $|t - t_i| > \beta$ and $|t - t_{i+1}| > \beta$. By using (iii) we select $q \in Q$ such that

$$|t_{j+q} - t_j - \tau| < \epsilon', \quad (3.20)$$

for all $j \in \mathbb{Z}$. This implies that

$$t_{i+q} - t_i - \tau < \epsilon' < \beta, \quad (3.21)$$

which in turn implies that

$$t + \tau > t_i + \beta + \tau > t_{i+q}. \quad (3.22)$$

Arguing in a similar way, we have

$$t_{i+q+1} - t_{i+1} - \tau > -\epsilon' > -\beta, \quad (3.23)$$

from which follows that

$$t_{i+q+1} > t_{i+1} - \beta + \tau > t + \tau. \quad (3.24)$$

Next we abbreviate $k = i + q$. Since $t_k < t + \tau < t_{k+1}$ we can write

$$\begin{aligned} g(t + \tau) - g(t) &= \Gamma(t + \tau, t_k)x_k - \Gamma(t, t_i)x_i \\ &= [\Gamma(t + \tau, t_i + \tau) - \Gamma(t, t_i)]x_k + \Gamma(t, t_i)(x_k - x_i) \\ &\quad + \Gamma(t + \tau, t_k)x_k - \Gamma(t + \tau, t_i + \tau)x_k. \end{aligned} \quad (3.25)$$

Using (i) we deduce that

$$\|[\Gamma(t + \tau, t_i + \tau) - \Gamma(t, t_i)]x_k\| \leq \frac{\epsilon}{3}. \quad (3.26)$$

Similarly, using (ii) we infer that

$$\|\Gamma(t, t_i)(x_k - x_i)\| \leq \frac{\epsilon}{3}. \quad (3.27)$$

Using again that $|t_{i+q} - t_i - \tau| < \epsilon' < \delta$ and (3.19), we have that

$$\|\Gamma(t + \tau, t_k)x_k - \Gamma(t + \tau, t_i + \tau)x_k\| \leq \frac{\epsilon}{3}. \quad (3.28)$$

Collecting these estimate, it follows from (3.25) that $\|g(t + \tau) - g(t)\| \leq \epsilon$ for all $\tau \in \Omega$.

Next we will prove that g is uniformly continuous on the $\cup_{i \in \mathbb{Z}}(t_i, t_{i+1})$. Let $t, h \in \mathbb{R}$, $h > 0$ such that $t_i < t, t + h < t_{i+1}$. We have

$$\begin{aligned} g(t + h) - g(t) &= \Gamma(t + h, t_i)x_i - \Gamma(t, t_i)x_i \\ &= [\Gamma(t + h, t) - \Gamma(t, t)]\Gamma(t, t_i)x_i. \end{aligned} \quad (3.29)$$

We choose appropriately $0 < \beta < a$ and we analyze $\Gamma(t, t_i)x_i$ in the cases $t - t_i < \beta$, $\beta \leq t - t_i \leq a$ and $a < t - t_i$. Arguing as in the proof of the Lemma 3.5 we conclude that $g(t + h) - g(t)$ converges to zero as $h \rightarrow 0$ independent of t and i . $\square$

The next theorem is the main result of this section. In this statement we denote

$$C = \frac{2N}{\mu} (\|f\|_\infty + \|F(0)\|) + \frac{N}{1 - e^{-\mu a}} \sup_{j \in \mathbb{Z}} \|I_j(0)\|. \quad (3.30)$$

Theorem 3.7. Assume that the conditions (R) and (H) hold. Let $f \in \text{AP}_T(\mathbb{X})$ and let $F, I_i : \mathbb{X} \rightarrow \mathbb{X}$, $i \in \mathbb{Z}$ be maps that satisfy the following conditions.

(a) The family $\{I_i : i \in \mathbb{Z}\}$ is almost periodic and there exists a constant $L_1 \geq 0$ such that

$$\|I_i(x) - I_i(y)\| \leq L_1 \|x - y\|, \quad (3.31)$$

for all $x, y \in \mathbb{X}$ and $i \in \mathbb{Z}$.

(b) There exists a nondecreasing function $L_2 : [0, \infty) \rightarrow [0, \infty)$ such that

$$\|F(x) - F(y)\| \leq L_2(r) \|x - y\|, \quad (3.32)$$

for all $x, y \in \mathbb{X}$, $\|x - y\| \leq r$.

If there exists $r > 0$ such that

$$\frac{C}{r} + N \left(\frac{L_1}{1 - e^{-\mu\alpha}} + \frac{2L_2(r)}{\mu} \right) < 1, \quad (3.33)$$

then the problem (1.1) has a unique T -piecewise continuous almost periodic mild solution $u(\cdot)$ which satisfies

$$u(t) = \int_{-\infty}^{\infty} \Gamma(t, s) f(s) ds + \int_{-\infty}^{\infty} \Gamma(t, s) F(u(s)) ds + \sum_{t_i < t} \Gamma(t, t_i) I_i(u(t_i)). \quad (3.34)$$

Proof. Let $\Lambda : \text{AP}_T(\mathbb{X}) \rightarrow \mathcal{PC}_T(\mathbb{R}, \mathbb{X})$ be the operator defined by

$$\Lambda u(t) = \int_{-\infty}^{\infty} \Gamma(t, s) f(s) ds + \int_{-\infty}^{\infty} \Gamma(t, s) F(u(s)) ds + \sum_{t_i < t} \Gamma(t, t_i) I_i(u(t_i)). \quad (3.35)$$

Since the function $t \mapsto F(u(t))$ belongs to the space $\text{AP}_T(\mathbb{X})$, it follows from Lemma 3.4 that the function $t \mapsto \int_{-\infty}^{\infty} \Gamma(t, s) f(s) ds + \int_{-\infty}^{\infty} \Gamma(t, s) F(u(s)) ds$ also belongs to the space $\text{AP}_T(\mathbb{X})$.

Let $x_i = I_i(u(t_i))$ for $i \in \mathbb{Z}$. We define $g_k : \mathbb{R} \rightarrow \mathbb{X}$ by $g_k(t) = \Gamma(t, t_{i-k}) x_{i-k}$ for $t_i < t \leq t_{i+1}$. It follows from Lemma 3.6 that $g_k \in \text{AP}_T(\mathbb{X})$. Moreover,

$$\|g_k(t)\| = \|\Gamma(t, t_{i-k}) x_{i-k}\| \leq N \sup_{i \in \mathbb{Z}} \|x_i\| e^{-\mu(t-t_{i-k})} \leq N \sup_{i \in \mathbb{Z}} \|x_i\| e^{-\mu(t-t_i)} e^{-\mu\alpha k}. \quad (3.36)$$

Therefore, the series $\sum_{k=0}^{\infty} g_k$ is uniformly convergent on $\mathbb{R}$. Consequently, $\sum_{k=0}^{\infty} g_k \in \text{AP}_T(\mathbb{X})$. Since $\sum_{t_i < t} \Gamma(t, t_i) I_i(u(t_i)) = \sum_{k=0}^{\infty} g_k$, and combining this property with the previous assertion, it follows that $\Lambda u \in \text{AP}_T(\mathbb{X})$.

This shows that $\Lambda : \text{AP}_T(\mathbb{X}) \rightarrow \text{AP}_T(\mathbb{X})$. Let $u \in \text{AP}_T(\mathbb{X})$ such that $\|u\|_\infty \leq r$. It follows from (3.35) that

$$\begin{aligned}
\|\Lambda u(t)\| &\leq \int_{-\infty}^t \|\Gamma(t,s)f(s)\|ds + \int_t^\infty \|\Gamma(t,s)f(s)\|ds \\
&\quad + \int_{-\infty}^t \|\Gamma(t,s)(F(u(s)) - F(0))\|ds + \int_{-\infty}^t \|\Gamma(t,s)F(0)\|ds \\
&\quad + \int_t^\infty \|\Gamma(t,s)(F(u(s)) - F(0))\|ds + \int_t^\infty \|\Gamma(t,s)F(0)\|ds \\
&\quad + \sum_{t_i < t} \|\Gamma(t,t_i)(I_i(u(t_i)) - I_i(0))\| + \sum_{t_i < t} \|\Gamma(t,t_i)I_i(0)\| \\
&\leq C + \frac{2N}{\mu} L_2(r)r + \frac{N}{1 - e^{-\mu\alpha}} L_1 r \\
&\leq r,
\end{aligned} \tag{3.37}$$

which implies that $\Lambda(B_r(\text{AP}_T(\mathbb{X}))) \subseteq B_r(\text{AP}_T(\mathbb{X}))$.

Finally, we verify that Λ is a contraction, let u_1, u_2 be arbitrary elements of $B_r(\text{AP}_T(\mathbb{X}))$ and $t_i < t \leq t_{i+1}$. Since $t - t_j \geq t - t_i + (i - j)\alpha$ for $j \leq i$, we get

$$\begin{aligned}
\|\Lambda u_1(t) - \Lambda u_2(t)\| &\leq \sum_{-\infty < t_j < t} \|\Gamma(t,t_j)\|_{\mathcal{L}(\mathbb{X})} \|I_j(u_1(t_j)) - I_j(u_2(t_j))\| \\
&\quad + \int_{-\infty}^\infty \|\Gamma(t,s)\|_{\mathcal{L}(\mathbb{X})} \|F(u_1(s)) - F(u_2(s))\|ds \\
&\leq \sum_{j=-\infty}^i N e^{-\mu(t-t_j)} L_1 \|u_1 - u_2\|_\infty + \int_{-\infty}^t N e^{-\mu(t-s)} L_2(r) \|u_1 - u_2\|_\infty ds \\
&\quad + \int_t^\infty N e^{-\mu(s-t)} L_2(r) \|u_1 - u_2\|_\infty ds \\
&\leq N L_1 e^{-\mu(t-t_i)} \sum_{j=-\infty}^i e^{-\mu\alpha(i-j)} \|u_1 - u_2\|_\infty + \frac{2N L_2(r)}{\mu} \|u_1 - u_2\|_\infty \\
&\leq N \left(\frac{L_1}{1 - e^{-\mu\alpha}} + \frac{2L_2(r)}{\mu} \right) \|u_1 - u_2\|_\infty.
\end{aligned} \tag{3.38}$$

It follows from (3.3) that the fixed point u of Λ is the T -piecewise continuous almost periodic mild solution of the problem (1.1). $\square$

In the case F satisfies a uniform Lipschitz condition we obtain the following immediate consequence.

Corollary 3.8. *Assume that the conditions (R) and (H) hold. Let $f \in \text{AP}_T(X)$ and let $F, I_i : \mathbb{X} \rightarrow \mathbb{X}$, $i \in \mathbb{Z}$ be maps that satisfy the following conditions.*

(a) *The family $\{I_i : i \in \mathbb{Z}\}$ is almost periodic and there exists a constant $L_1 \geq 0$ such that*

$$\|I_i(x) - I_i(y)\| \leq L_1 \|x - y\|, \quad (3.39)$$

for all $x, y \in \mathbb{X}$ and $i \in \mathbb{Z}$.

(b) *There exists a constant $L_2 \geq 0$ such that*

$$\|F(x) - F(y)\| \leq L_2 \|x - y\|, \quad (3.40)$$

for all $x, y \in \mathbb{X}$.

If

$$N\left(\frac{L_1}{1 - e^{-\mu\alpha}} + \frac{2L_2}{\mu}\right) < 1, \quad (3.41)$$

then the problem (1.1) has a unique T -piecewise continuous almost periodic mild solution.

4. Applications

As an application we consider a system described by the partial differential equation with impulses

$$\begin{aligned} \frac{\partial}{\partial t} z(t, \xi) &= \frac{\partial^2}{\partial \xi^2} z(t, \xi) + b(t)z(t, \xi) + w(t, \xi), \quad t \in \mathbb{R}, \quad \xi \in [0, \pi], \\ z(t, 0) &= z(t, \pi) = 0, \quad t \in \mathbb{R} \\ \Delta z(t_i, \xi) &= \int_0^\pi p(t_i, \eta, \xi) z(t_i, \eta) d\eta, \quad \xi \in [0, \pi], \quad i \in \mathbb{Z}. \end{aligned} \quad (4.1)$$

In this equation, we assume that $b : \mathbb{R} \rightarrow \mathbb{R}$, $w : \mathbb{R} \times [0, \pi] \rightarrow \mathbb{R}$ and $p : \mathbb{R} \times [0, \pi] \times [0, \pi] \rightarrow \mathbb{R}$ are functions that satisfy some properties to be specified later. Moreover, $T = \{t_i\}_{i \in \mathbb{Z}}$ is a uniformly almost periodic sequence of moments. Such systems arises, among other diffusion systems, in the temperature control of a heated metal bar with insulated ends. To model this system we consider $\mathbb{X} = L^2([0, \pi])$. Let A_0 be the operator defined by

$$(A_0 \varphi)(\xi) = \frac{d^2 \varphi(\xi)}{d\xi^2} \quad (4.2)$$

with domain

$$D(A_0) = \left\{ \varphi \in \mathbb{X} : \varphi \in H^2([0, \pi]), \quad \varphi(0) = \varphi(\pi) = 0 \right\}. \quad (4.3)$$

The spectrum of A_0 consists of the eigenvalues $-n^2$ for $n \in \mathbb{IN}$, with associated eigenvectors

$$\varphi_n(\xi) = \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi). \quad (4.4)$$

Furthermore, the set $\{\varphi_n : n \in \mathbb{IN}\}$ is an orthonormal basis of $\mathbb{X}$. In particular,

$$A_0\varphi = \sum_{n=1}^{\infty} -n^2 \langle \varphi, \varphi_n \rangle \varphi_n, \quad (4.5)$$

for $\varphi \in D(A_0)$. Using the above expression, one easily verifies that A_0 is the infinitesimal generator of a strongly continuous semigroup $S(t)$ given by

$$S(t)\varphi = \sum_{n=1}^{\infty} e^{-n^2 t} \langle \varphi, \varphi_n \rangle \varphi_n, \quad t \geq 0. \quad (4.6)$$

Let $A(t) = A_0 + b(t)I$ for $t \in \mathbb{R}$ defined on $D(A(t)) = D(A_0)$. We consider the following conditions.

- (a) The function $b(\cdot)$ is almost periodic.
- (b) The function $p(\cdot)$ is continuous. We denote

$$L_1 = \left(\iint_0^\pi p^2(t_i, \eta, \xi) d\eta d\xi \right)^{1/2}. \quad (4.7)$$

- (c) The function w verifies the Carathéodory conditions:

- (i) $w(t, \cdot) : [0, \pi] \rightarrow \mathbb{R}$ is measurable;
- (ii) $w(\cdot, \xi) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous a.e.;
- (iii) there exists a function $g \in L^2([0, \pi])$ such that $|w(t, \xi)| \leq g(\xi)$ for all $t \in \mathbb{R}$ and $\xi \in [0, \pi]$.

In this case, the operators $A(t)$ satisfy the conditions (AT1)-(AT2). Moreover, the evolution family $U(t, s)$ generated by $A(t)$ is given by

$$U(t, s) = e^{\int_s^t b(\tau) d\tau} S(t - s), \quad s \leq t. \quad (4.8)$$

It is immediate that this evolution family satisfies the assumption (H). It follows from (c) that the function $f : \mathbb{R} \rightarrow \mathbb{X}$, $f(t) = w(t, \cdot)$ is measurable. Moreover, it follows from (b) that the map $I_i : \mathbb{X} \rightarrow \mathbb{X}$ given by

$$I_i(\varphi)(\xi) = \int_0^\pi p(t_i, \eta, \xi) \varphi(\eta) d\eta, \quad (4.9)$$

is a bounded linear map with $\|I_i\| \leq L_1$ for all $i \in \mathbb{Z}$. Consequently, if we define $u(t) = z(t, \cdot)$, the problem (4.1) can be modeled by the abstract system (1.1). In the rest of this section we assume that conditions (a), (b), and (c) are fulfilled. We assume further that w satisfies the following condition.

(d) The function w is uniformly T -piecewise almost periodic. This means that

(i) for the set $\{w(\cdot, \xi) : 0 \leq \xi \leq \pi\} \subseteq \text{AP}_T(\mathbb{R})$ and for each $\epsilon > 0$, there exists a relatively dense set Ω_ϵ in $\mathbb{R}$ such that

$$|w(t + \tau, \xi) - w(t, \xi)| < \epsilon, \quad (4.10)$$

for all $\tau \in \Omega_\epsilon$, $\xi \in [0, \pi]$ and $t \in \mathbb{R}$ satisfying the condition $|t - t_i| > \epsilon$ for all $i \in \mathbb{Z}$.

(ii) for every $\epsilon > 0$ there exists $\delta > 0$ such that $|w(t, \xi) - w(t', \xi)| < \epsilon$ for all $i \in \mathbb{Z}$, $\xi \in [0, \pi]$, $t, t' \in (t_i, t_{i+1})$ such that $|t - t'| < \delta$,

It is not difficult to show that if w satisfies conditions (c) and (d), then the function $f \in \text{AP}_T(\mathbb{X})$.

On the other hand, we consider the following condition.

(e) There are $n \in \mathbb{IN}$ and $\mu > 0$ such that $(n - 1)^2 + \mu \leq b(t) \leq n^2 - \mu$ for all $t \in \mathbb{R}$.

Assuming that this condition holds, we define $P(t) : \mathbb{X} \rightarrow \mathbb{X}$ by

$$P(t)\varphi = \sum_{k=n}^{\infty} \langle \varphi, \varphi_k \rangle \varphi_k. \quad (4.11)$$

It is immediate to see that

$$\begin{aligned} U(t, s)P(s)\varphi &= \sum_{k=n}^{\infty} \langle \varphi, \varphi_k \rangle e^{\int_s^t b(\tau) d\tau} S(t - s)\varphi_k \\ &= \sum_{k=n}^{\infty} e^{-k^2(t-s) + \int_s^t b(\tau) d\tau} \langle \varphi, \varphi_k \rangle \varphi_k, \end{aligned} \quad (4.12)$$

which implies that

$$\|U(t, s)P(s)\varphi\| \leq e^{-n^2(t-s) + \int_s^t b(\tau) d\tau} \|\varphi\|. \quad (4.13)$$

Since

$$\int_s^t b(\tau) d\tau - n^2(t - s) \leq -\mu(t - s), \quad (4.14)$$

we infer that

$$\|U(t, s)P(s)\| \leq e^{-\mu(t-s)}, \quad s \leq t. \quad (4.15)$$

Arguing in a similar way, if $Q(t) = I - P(t)$, then

$$U(t, s)Q(s)\varphi = \sum_{k=1}^{n-1} e^{-k^2(t-s) + \int_s^t b(\tau) d\tau} \langle \varphi, \varphi_k \rangle \varphi_k, \quad (4.16)$$

which implies that

$$\begin{aligned} U(t, s)Q(s)\varphi &= \sum_{k=1}^{n-1} e^{-k^2(t-s) + \int_s^t b(\tau) d\tau} \langle \varphi, \varphi_k \rangle \varphi_k, \\ U_Q(s, t)\varphi &= (U(t, s)Q(s))^{-1}\varphi = \sum_{k=1}^{n-1} e^{k^2(t-s) - \int_s^t b(\tau) d\tau} \langle \varphi, \varphi_k \rangle \varphi_k. \end{aligned} \quad (4.17)$$

Hence

$$\|U_Q(s, t)\varphi\| \leq e^{(n-1)^2(t-s) - \int_s^t b(\tau) d\tau} \|\varphi\|. \quad (4.18)$$

In view of that

$$\int_s^t b(\tau) d\tau - (n-1)^2(t-s) \geq \mu(t-s), \quad (4.19)$$

we infer that

$$\|U_Q(s, t)\| \leq e^{-\mu(t-s)}, \quad s \leq t. \quad (4.20)$$

As a consequence we can affirm that the evolution family $U(\cdot, \cdot)$ has an exponential dichotomy and the associated Green's function Γ satisfies the estimate $\|\Gamma(t, s)\| \leq e^{-\mu|t-s|}$ for $s, t \in \mathbb{R}$.

Finally, we note that for $\text{Re}(\lambda)$ enough small, the resolvent operator

$$\begin{aligned} R(\lambda, A(t)) &= (\lambda I - A_0 - b(t)I)^{-1} \\ &= (I - b(t)R(\lambda, A_0))^{-1} R(\lambda, A_0) \\ &= \sum_{k=0}^{\infty} b(t)^k R(\lambda, A_0)^{k+1}, \end{aligned} \quad (4.21)$$

where the series on the right hand side converges in the space $\mathcal{L}(\mathbb{X})$ uniformly for $t \in \mathbb{R}$. Therefore, the operator-valued function $R(\lambda, A(\cdot)) \in \text{AP}(\mathcal{L}(\mathbb{X}))$, which in turn shows that the evolution family $U(\cdot, \cdot)$ satisfies the condition (R).

The next result is an immediate consequence of the Theorem 3.7.

Theorem 4.1. *Let us assume the conditions (a)-(e) are fulfilled. If $L_1/1 - e^{-\mu\alpha} < 1$, then the problem (4.1) has a unique T -piecewise almost periodic mild solution.*

This result can be generalized to include the system

$$\begin{aligned}\frac{\partial}{\partial t} z(t, \xi) &= a(t) \frac{\partial^2}{\partial \xi^2} z(t, \xi) + b(t) z(t, \xi) + w(t, \xi), \quad t \in \mathbb{R}, \quad \xi \in [0, \pi], \\ z(t, 0) &= z(t, \pi) = 0, \quad t \in \mathbb{R}, \\ \Delta z(t_i, \xi) &= \int_0^\pi p(t_i, \eta, \xi) z(t_i, \eta) d\eta, \quad \xi \in [0, \pi], \quad i \in \mathbb{Z},\end{aligned}\tag{4.22}$$

when $a(\cdot) \in \text{AP}(\mathbb{R})$ is a positive function. In fact, in this case the evolution family $U(\cdot, \cdot)$ is given by

$$U(t, s) = e^{\int_s^t b(\tau) d\tau} S\left(\int_s^t a(\tau) d\tau\right), \quad s \leq t,\tag{4.23}$$

and we can argue as above to study this system.

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