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Research Article

Wave Scattering in Inhomogeneous Strings

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The scattering of waves in an inhomogeneous infinite string with one change in the density of the string is well known. In this paper, we study the case where there are two discontinuities in the density of the string. It turns out that we can write the solution for this case as well. The form of the solution will be given in finite sums of reflected and transmitted waves over finite time. The finite sums become converging infinite series for infinite time.

1. Introduction

The scattering of waves in an infinite string due to a single change in the density of the string has been studied in graduate text books such as [1, 2]. In [1] the author notes that in practice a scattering of this sort occurs in physical systems such as submarine cables or telephone lines when a join in the system creates a transmitted as well as a reflected wave. Both of these waves can be computed in terms of the incoming signal. For practical purposes it is desirable to suppress the reflected waves by attaching, at the join, a damping mechanism or a point mass so that the reflected waves do not interfere with the incoming signals. All three cases of no suppression, suppression by damping, and suppression by point mass have been studied in [1] by assigning appropriate boundary conditions at the join. In this paper we study the case where the string has more than one change in its density and no suppression occurs at the joins. We write the form of the solution when there are only two changes in the density. We also show what the reflected and transmitted waves due to an incident wave on both sides of a join at an arbitrary point in the string look like. This will allow us to theoretically write the form of the reflected and transmitted waves for higher number of changes. But because of the repetitive bouncing back and forth of the waves between each interface writing a general formula for the solution of the wave problem with more than two discontinuities will be impossible. Just imagine that all the waves inside one pair of joins get transmitted to the neighboring pair and in turn bounce against the joins and create waves of their own. Writing a formula to account for all of these waves will require a horrendous amount of work if at all possible, and we are not going to tackle it here.

To fix the ideas, consider two semi-infinite strings with densities ρ_1 and ρ_2, $\rho_1 \neq \rho_2$, joined together at the origin O. The speed of the waves traveling along the two pieces is c_1 and c_2, respectively. An incoming wave $f(x - c_1 t)$, such that $f(s) = 0,\ s > 0$, will be scattered to partly reflected and partly outgoing at $x = 0$. Suppose the resulting wave $u(x,t)$ satisfies that

$$u_{tt} - c^2(x) u_{xx} = 0, \quad (x,t) \in R \times (0,\infty), \tag{1.1}$$

where

$$c(x) = \begin{cases} c_1, & x \leq 0, \\ c_2, & x > 0. \end{cases} \tag{1.2}$$

Under the geometric continuity (the string is continuous at $x = 0$) and dynamical continuity (the transverse force is continuous at $x = 0$) conditions,

$$\begin{aligned} u(0-,t) &= u(0+,t), \quad t \geq 0, \\ u_x(0-,t) &= u_x(0+,t), \quad t \geq 0, \end{aligned} \tag{1.3}$$

and the initial conditions,

$$\begin{aligned} u(x,0) &= f(x), \quad x \in R, \\ u_t(x,0) &= -c_1 f'(x), \quad x \in R, \end{aligned} \tag{1.4}$$

the solution is well known to be (see [1, 2])

$$u(x,t) = \begin{cases} f(x - c_1 t) + \dfrac{c_2 - c_1}{c_2 + c_1} f(-x - c_1 t), & x < 0, \\ \dfrac{2c_2}{c_2 + c_1} f\left(\dfrac{c_1}{c_2}(x - c_2 t)\right), & x > 0. \end{cases} \tag{1.5}$$

Now, consider the above setup extended to a string made of three pieces in the intervals $(-\infty, 0)$, $(0, \sigma)$, and (σ, ∞) for some constant $\sigma > 0$. Denote the densities and their corresponding wave speeds by ρ_1, ρ_2, ρ_3 and c_1, c_2, c_3, respectively. We will study the behavior of an incoming wave $f(x - c_1 t)$, where $f(s) = 0,\ s > 0$, at the interfaces and show how, under similar conditions as above, similar formulas for the scattered waves can be written. We note that each outgoing wave from an interface will be incident on the next, and the reflected waves at the next interface will be incident on the previous. Each incident wave will, in turn, be scattered into reflected and outgoing waves at the interfaces. We will write the solution to the wave problem as sums of these scattered waves.

2. The Problem

Consider the problem

$$u_{tt} - c^2(x)u_{xx} = 0, \quad (x,t) \in R \times (0,\infty),$$
$$c(x) = \begin{cases} c_1, & x \le 0, \\ c_2, & 0 < x \le \sigma, \\ c_3, & x > \sigma. \end{cases} \tag{2.1}$$

Assume that a wave $f(x - c_1 t) \in C^2(R)$, where $f(s) \equiv 0,\ s > 0$, is incoming. At $x = 0$ the wave scatters as follows:

$$u_1(x,t) = \begin{cases} f(x - c_1 t) + \dfrac{c_2 - c_1}{c_1 + c_2} f(-x - c_1 t), & x < 0, \\ \dfrac{2c_2}{c_1 + c_2} f\left(\dfrac{c_1}{c_2}(x - c_2 t)\right), & 0 \le x \le \sigma. \end{cases} \tag{2.2}$$

The transmitted wave $(2c_2/(c_1 + c_2)) f((c_1/c_2)(x - c_2 t))$ becomes incident at the point $x = \sigma$. Let us denote

$$g(x - c_2 t) = \frac{2c_2}{c_1 + c_2} f\left(\frac{c_1}{c_2}(x - c_2 t)\right), \quad 0 \le x < \sigma. \tag{2.3}$$

We will show here, for the sake of completeness, that g also scatters at $x = \sigma$ and creates new waves.

Theorem 1. *The incoming wave g given by* (2.3) *scatters at $x = \sigma$, for $t > \sigma/c_2$, in the form*

$$u_2(x,t) = \begin{cases} g(x - c_2 t) + \dfrac{c_3 - c_2}{c_2 + c_3} g(2\sigma - (x + c_2 t)), & 0 < x \le \sigma, \\ \dfrac{2c_3}{c_2 + c_3} g\left(\dfrac{c_3 - c_2}{c_3}\sigma + \dfrac{c_2}{c_3}(x - c_3 t)\right), & x > \sigma. \end{cases} \tag{2.4}$$

Proof. First note that $g(x - c_2 t) = g(x - \sigma)$ when $t = \sigma/c_2$, and $g(x - \sigma) = 0,\ x > \sigma$, by the definition of g in (2.3). Consider the wave function $u(x,t)$, which we assume as $u(x,t) = u_1(x,t)$ on $(-\infty, 0) \times (0, \sigma/c_2]$. The solution to the wave equation

$$u_{tt} - c^2(x)u_{xx} = 0, \quad (x,t) \in (0,\infty) \times \left(\frac{\sigma}{c_2}, \infty\right), \tag{2.5}$$

where

$$c(x) = \begin{cases} c_2, & 0 < x \le \sigma, \\ c_3, & x > \sigma, \end{cases} \tag{2.6}$$

is of the form

$$u(x,t) = \begin{cases} j(x - c_2 t) + q(x + c_2 t), & 0 < x < \sigma, \\ l(x - c_3 t) + m(x + c_3 t), & 0 > \sigma. \end{cases} \tag{2.7}$$

We use the geometric and dynamical continuity conditions,

$$\begin{aligned} u(\sigma-,t) &= u(\sigma+,t), \quad t \geq \frac{\sigma}{c_2}, \\ u_x(\sigma-,t) &= u_x(\sigma+,t), \quad t \geq \frac{\sigma}{c_2}, \end{aligned} \tag{2.8}$$

and the initial conditions,

$$\begin{aligned} u\left(x, \frac{\sigma}{c_2}\right) &= g(x-\sigma), \quad x \in (0,\infty), \\ u_t\left(x, \frac{\sigma}{c_2}\right) &= -c_2 g'(x-\sigma), \quad x \in (0,\infty), \end{aligned} \tag{2.9}$$

to find j, q, l, and m. From continuity of the wave function $u(x,t)$ at $x = \sigma$, we have

$$j(\sigma - c_2 t) + q(\sigma + c_2 t) = l(\sigma - c_3 t) + m(\sigma + c_3 t). \tag{2.10}$$

Using dynamical condition (2.8) at $x = \sigma$ and differentiating equation (2.10) with respect to t, we have the system

$$\begin{aligned} j'(\sigma - c_2 t) + q'(\sigma + c_2 t) &= l'(\sigma - c_3 t) + m'(\sigma + c_3 t) - c_2 j'(\sigma - c_2 t) + c_2 q'(\sigma + c_2 t) \\ &= -c_3 l'(\sigma - c_3 t) + c_3 m'(\sigma + c_3 t). \end{aligned} \tag{2.11}$$

Solving the system (2.11) for q' and j' and integrating, we have

$$q(\sigma + c_2 t) = \frac{c_3 - c_2}{2c_3} l(\sigma - c_3 t) + \frac{c_3 + c_2}{2c_3} m(\sigma + c_3 t), \tag{2.12}$$

$$j(\sigma - c_2 t) = \frac{c_3 + c_2}{2c_3} l(\sigma - c_3 t) + \frac{c_3 - c_2}{2c_3} m(\sigma + c_3 t). \tag{2.13}$$

From initial conditions (2.9) at $t = \sigma/c_2$,

$$\begin{aligned} u\left(x, \frac{\sigma}{c_2}\right) &= g(x-\sigma) \\ &= j(x-\sigma) + q(x+\sigma), \quad x < \sigma, \\ u_t\left(x, \frac{\sigma}{c_2}\right) &= -c_2 g'(x-\sigma) \\ &= -c_2 j'(x-\sigma) + c_2 q'(x+\sigma), \quad x < \sigma. \end{aligned} \tag{2.14}$$

The right-hand sides of (2.14) provide a system that can be solved for j and q:

$$j(x-\sigma) = g(x-\sigma), \quad x < \sigma, \tag{2.15}$$

$$q(x+\sigma) = 0, \quad x < \sigma. \tag{2.16}$$

At the time $t = \sigma/c_2$ and $x > \sigma$, we have

$$g(x-\sigma) = u\left(x, \frac{\sigma}{c_2}\right) = l\left(x - \frac{c_3}{c_2}\sigma\right) + m\left(x + \frac{c_3}{c_2}\sigma\right) = 0, \quad x > \sigma, \tag{2.17}$$

$$-c_2 g'(x-\sigma) = u_t\left(x, \frac{\sigma}{c_2}\right) = -c_3 l'\left(x - \frac{c_3}{c_2}\sigma\right) + c_3 m'\left(x + \frac{c_3}{c_2}\sigma\right) = 0, \quad x > \sigma, \tag{2.18}$$

Differentiating the right-hand side of (2.17) and solving the system obtained from (2.17) and (2.18), namely,

$$\begin{aligned} l'\left(x - \frac{c_3}{c_2}\sigma\right) + m'\left(x + \frac{c_3}{c_2}\sigma\right) &= 0, \quad x > \sigma, \\ -c_3 l'\left(x - \frac{c_3}{c_2}\sigma\right) + c_3 m'\left(x + \frac{c_3}{c_2}\sigma\right) &= 0, \quad x > \sigma, \end{aligned} \tag{2.19}$$

we will have

$$l\left(x - \frac{c_3}{c_2}\sigma\right) = 0, \quad x > \sigma, \tag{2.20}$$

$$m\left(x + \frac{c_3}{c_2}\sigma\right) = 0, \quad x > \sigma. \tag{2.21}$$

By (2.15), when the arguments of g and j are negative the functions are equal. This means that

$$j(x - c_2 t) = g(x - c_2 t), \quad t > \frac{\sigma}{c_2}. \tag{2.22}$$

Notice that by (2.21) if the argument of m satisfies $x + (c_3/c_2)\sigma > \sigma + (c_3/c_2)\sigma$, then $m = 0$. For $t > \sigma/c_2$ the argument of m in (2.12) and (2.13) is larger than $\sigma + (c_3/c_2)\sigma$. Therefore,

$$m(x + c_3 t) = 0, \quad x \geq \sigma, t > \frac{\sigma}{c_2}, \tag{2.23}$$

and (2.12) and (2.13) become,

$$q(\sigma + c_2 t) = \frac{c_3 - c_2}{2c_3} l(\sigma - c_3 t), \quad t > \frac{\sigma}{c_2}, \tag{2.24}$$

$$j(\sigma - c_2 t) = \frac{c_3 + c_2}{2c_3} l(\sigma - c_3 t), \quad t > \frac{\sigma}{c_2}. \tag{2.25}$$

From (2.22) and (2.25) we obtain

$$g(\sigma - c_2 t) = \frac{c_3 + c_2}{2c_3} l(\sigma - c_3 t), \quad t > \frac{\sigma}{c_2}. \tag{2.26}$$

Equation (2.26) determines l as follows:

$$l(\sigma - c_3 t) = \frac{2c_3}{c_3 + c_2} g(\sigma - c_2 t), \quad t > \frac{\sigma}{c_2}. \tag{2.27}$$

Denote the argument of l in (2.27) by τ, then

$$l(\tau) = \frac{2c_3}{c_3 + c_2} g\left(\sigma - \frac{c_2}{c_3}(\sigma - \tau)\right), \quad \tau > \frac{c_3 - c_2}{c_2}\sigma. \tag{2.28}$$

Due to (2.28),

$$l(x - c_3 t) = \frac{2c_3}{c_3 + c_2} g\left(\frac{c_3 - c_2}{c_3}\sigma + \frac{c_2}{c_3}(x - c_3 t)\right), \quad t > \frac{\sigma}{c_2}. \tag{2.29}$$

Finally, putting (2.23) and (2.27) in (2.12) determines q:

$$q(\sigma + c_2 t) = \frac{c_3 - c_2}{c_3 + c_2} g(\sigma - c_2 t), \quad t > \frac{\sigma}{c_2}. \tag{2.30}$$

A similar change of independent variable in (2.30) yields

$$q(x + c_2 t) = \frac{c_3 - c_2}{c_3 + c_2} g(2\sigma - (x + c_2 t)), \quad 0 < x < \sigma, t > \frac{\sigma}{c_2}. \tag{2.31}$$

Plugging j, m, l, and q determined by (2.22), (2.23), (2.29), and (2.31), respectively, in (2.7) yields u_2 in (2.4). □

By an argument similar to the one in Theorem 1, we can show the following. The reflected wave at $x = \sigma$, namely $((c_3 - c_2)/(c_2 + c_3))g(2\sigma - (x + c_2 t))$ in (2.4), becomes incident at the point $x = 0$. Let us denote

$$h(x + c_2 t) = \frac{c_3 - c_2}{c_2 + c_3} g(2\sigma - (x + c_2 t)), \quad 0 < x < \sigma. \tag{2.32}$$

Then, the wave h scatters at $x = 0$, for $t > 2\sigma/c_2$, as follows:

$$u_3(x,t) = \begin{cases} \dfrac{2c_1}{c_1 + c_2} h\left(\dfrac{c_2}{c_1}(x + c_1 t)\right), & x < 0, \\ h(\mathrm{x} + c_2 t) + \dfrac{c_1 - c_2}{c_1 + c_2} h(-x + c_2 t), & 0 < x < \sigma. \end{cases} \tag{2.33}$$

We note here that $h(x + c_2 t)$ is the wave that moves to the left with $h(s) = 0,\ s < 0$. It is not difficult to check that u_3 satisfies $u_3(x,0) = h(x), u_{3,t}(x,0) = c_2 h'(x), u_3(0-,t) = u_3(0+,t)$ and $u_{3,x}(0-,t) = u_{3,x}(0 + t)$.

In this manner, the forms of outgoing waves through the interfaces $x = 0$ and $x = \sigma$ and the ones bouncing back and forth between the two are determined. In order to write the solution to the problem

$$u_{tt} - c^2(x) u_{xx} = 0, \quad (x,t) \in (x,t) \in R \times (0,\infty), \tag{2.34}$$

where,

$$c(x) = \begin{cases} c_1, & x \le 0, \\ c_2, & 0 < x \le \sigma, \\ c_3, & x > \sigma, \end{cases} \tag{2.35}$$

subject to

$$\begin{gathered} u(0-,t) = u(0+,t), \quad t \ge 0, \\ u(x,0) = f(x), \qquad f(s) \equiv 0, \quad s > 0, \\ u_t(x,0) = -c_1 f'(x), \quad x \in R, \\ u_x(0-,t) = u_x(0+,t), \quad t \ge 0, \\ u(\sigma-,t) = u(\sigma+,t), \quad t \ge 0, \\ u_x(\sigma-,t) = u_x(\sigma+,t), \quad t \ge 0, \end{gathered} \tag{2.36}$$

for σ, c_1, c_2, c_3 positive constants, and the incoming wave $f(x - c_1 t)$ for some $f \in C^2(R)$, we need sums of the waves in each interval $(-\infty, 0)$, $(0, \sigma)$, and (σ, ∞). One way to write such solution is through the use of composition of the arguments of the scattered waves. For this purpose we introduce the following functions:

$$\begin{gathered} L(\eta) = \frac{c_3 - c_2}{c_3}\sigma + \frac{c_2}{c_3}\eta, \\ R(\eta) = \frac{c_2}{c_1}\eta, \\ M(\eta) = 2\sigma - \eta, \\ S(\eta) = -\eta. \end{gathered} \tag{2.37}$$

Let us write the functions g and h introduced in (2.3) and (2.32), respectively, in terms of the variable η:

$$g(\eta) = \frac{2c_2}{c_1 + c_2} f\left(\frac{c_1}{c_2}\eta\right), \tag{2.38}$$

$$h(\eta) = \frac{c_3 - c_2}{c_3 + c_2} g(2\sigma - \eta). \tag{2.39}$$

From M in (2.37) and (2.38)-(2.39), h can be written in terms of f as follows:

$$h(\eta) = \frac{c_3 - c_2}{c_3 + c_2} \cdot \frac{2c_2}{c_1 + c_2} f\left(\frac{c_1}{c_2} M(\eta)\right). \tag{2.40}$$

In terms of the functions in (2.37), (2.38) and (2.40), (2.4) and (2.33) in the time intervals $[0, 2\sigma/c_2)$ and $[2\sigma/c_2, 3\sigma/c_2)$ will, respectively, become

$$u_2(x,t) = \begin{cases} \dfrac{2c_2}{c_1 + c_2} f\left(\dfrac{c_1}{c_2}(x - c_2 t)\right) + \dfrac{c_3 - c_2}{c_3 + c_2} \cdot \dfrac{2c_2}{c_1 + c_2} f\left(\dfrac{c_1}{c_2} M(x + c_2 t)\right), & 0 < x < \sigma, \\ \dfrac{2c_3}{c_2 + c_3} \cdot \dfrac{2c_2}{c_1 + c_2} f\left(\dfrac{c_1}{c_2} L(x - c_3 t)\right), & x > \sigma, \end{cases} \tag{2.41}$$

$$u_3(x,t) = \begin{cases} \dfrac{2c_1}{c_1 + c_2} \cdot \dfrac{c_3 - c_2}{c_3 + c_2} \cdot \dfrac{2c_2}{c_1 + c_2} f\left(\dfrac{c_1}{c_2} M \circ R(x + c_1 t)\right), & x < 0, \\ \dfrac{c_3 - c_2}{c_3 + c_2} \dfrac{2c_2}{c_1 + c_2} f\left(\dfrac{c_1}{c_2} M(x + c_2 t)\right) + \dfrac{c_1 - c_2}{c_1 + c_2} \cdot \dfrac{c_3 - c_2}{c_3 + c_2} \cdot \dfrac{2c_2}{c_1 + c_2} & \\ \quad \cdot f\left(\dfrac{c_1}{c_2} M \circ S(x - c_2 t)\right), & 0 < x < \sigma. \end{cases} \tag{2.42}$$

Now, we begin to write the solution of the wave problem (2.34)-(2.36) in each interval $(-\infty, 0)$, $(0, \sigma)$, and (σ, ∞) using (2.2), (2.41), and (2.42). Consider the following interpretations of (2.2), (2.41), and (2.42). First we have the following wave on the interval $(-\infty, 0)$:

$$f(x - c_1 t). \tag{2.43}$$

When this wave hits the join at 0 from the left, by (2.2) the reflected wave gets a coefficient of the form $(c_2 - c_1)/(c_2 + c_1)$ and an argument change to $S(x + c_1 t)$, that is, the reflected wave corresponding to (2.43) is

$$\frac{c_2 - c_1}{c_2 + c_1} f(S(x + c_1 t)), \quad x < 0, \tag{2.44}$$

S is as given in (2.37). The transmitted wave due to (2.43) on the other hand gets a coefficient of $2c_2/(c_1+c_2)$ and argument change to $(c_1/c_2)(x-c_2 t)$, that is, the transmitted wave traveling to the right at 0 is

$$\frac{2c_2}{c_1 + c_2} f\left(\frac{c_1}{c_2}(x - c_2 t)\right), \quad 0 < x < \sigma. \tag{2.45}$$

Now let us see what changes the signal (2.45) undergo when it hits the join at σ on the left. From (2.41) part of the signal gets reflected to the left acquiring the coefficient $(c_3 - c_2)/(c_3 + c_2)$ and argument change to $M(x + c_2 t)$. The transmitted wave traveling right acquires the coefficient $2c_3/(c_2 + c_3)$ and the argument change $L(x - c_3 t)$. Lastly, by (2.42) a signal that hits the join at 0 on the right its transmitted part acquires the coefficient $2c_1/(c_1 + c_2)$ and the change of argument $R(x + c_1 t)$. Its reflected part acquires the coefficient $(c_1 - c_2)/(c_1 + c_2)$ and the argument change of $S(x - c_2 t)$.

Note that the signals inside the interval $(0, \sigma)$ keep scattering, but the ones that are transmitted outside this interval continue traveling to the right or left forever. So, now let us look at a few more wave signals that are produced inside $(0, \sigma)$. Look at the wave $((c_1 - c_2)/(c_1 + c_2)) \cdot ((c_3 - c_2)/(c_3 + c_2)) \cdot ((2c_2)/(c_1 + c_2)) \cdot f((c_1/c_2) M \circ S(x - c_2 t))$ in (2.42). As it travels to the right it reaches the join σ as an incident wave on the left. By the argument above its reflected component picks up a coefficient of $(c_3 - c_2)/(c_3 + c_2)$ and an argument change of $Mx + c_2 t$. The transmitted component picks up a coefficient of $2c_3/(c_2 + c_3)$ and an argument change $L(x - c_3 t)$. These two wave are, respectively,

$$\frac{c_3 - c_2}{c_3 + c_2} \cdot \frac{c_1 - c_2}{c_1 + c_2} \cdot \frac{c_3 - c_2}{c_3 + c_2} \cdot \frac{2c_2}{c_1 + c_2} \cdot f\left(\frac{c_1}{c_2} M \circ S \circ M(x + c_2 t)\right), \tag{2.46}$$

$$\frac{2c_3}{c_2 + c_3} \cdot \frac{c_1 - c_2}{c_1 + c_2} \cdot \frac{c_3 - c_2}{c_3 + c_2} \cdot \frac{2c_2}{c_1 + c_2} \cdot f\left(\frac{c_1}{c_2} M \circ S \circ L(x - c_3 t)\right). \tag{2.47}$$

Now the wave (2.46) is incident on the join 0 from the right. The argument above shows that the transmitted part must pick up the coefficient $2c_1/(c_1 + c_2)$ and the change of argument $R(x + c_1 t)$, and the reflected part picks up the coefficient $(c_1 - c_2)/(c_1 + c_2)$ and the argument change of $S(x - c_2 t)$. These two new waves are given below, respectively,

$$\frac{2c_1}{c_1 + c_2} \cdot \frac{c_3 - c_2}{c_3 + c_2} \cdot \frac{c_1 - c_2}{c_1 + c_2} \cdot \frac{c_3 - c_2}{c_3 + c_2} \cdot \frac{2c_2}{c_1 + c_2} \cdot f\left(\frac{c_1}{c_2} M \circ S \circ M \circ R(x + c_1 t)\right), \tag{2.48}$$

$$\frac{c_1 - c_2}{c_1 + c_2} \cdot \frac{c_3 - c_2}{c_3 + c_2} \cdot \frac{c_1 - c_2}{c_1 + c_2} \cdot \frac{c_3 - c_2}{c_3 + c_2} \cdot \frac{2c_2}{c_1 + c_2} \cdot f\left(\frac{c_1}{c_2} M \circ S \circ M \circ S(x - c_2 t)\right). \tag{2.49}$$

The waves (2.47) and (2.48) will be moving to the right and left away from the interval $(0,\sigma)$ indefinitely. But the wave (2.49) will be incident on the join σ on the left, and the process of reflection and transmission repeats as before. It is not too difficult now to decipher the general pattern of the reflected and transmitted waves in $(0,\sigma)$ and the outgoing waves outside of this interval. Here they are. The waves outgoing in $x<0$ are of the form

$$\begin{gathered}\frac{c_2-c_1}{c_1+c_2}f(-x-c_1t),\\ \frac{4c_1c_2}{(c_1+c_2)^2}\cdot\left(\frac{c_3-c_2}{c_3+c_2}\right)^{(k-1)/2}\cdot\left(\frac{c_1-c_2}{c_1+c_2}\right)^{(k-3)/2}\cdot f\left(\frac{c_1}{c_2}(M\circ S)^{(k-3)/2}\circ M\circ R(x+c_1t)\right),\\ x<0,\quad 0<t<\frac{(2i+1)\sigma}{c_2},\quad k=3,5,7,\ldots.\end{gathered}\tag{2.50}$$

The waves in $(0,\sigma)$ moving left are

$$\begin{gathered}\left(\frac{c_1-c_2}{c_1+c_2}\right)^{(j-2)/2}\cdot\left(\frac{c_3-c_2}{c_3+c_2}\right)^{j/2}\cdot\frac{2c_2}{c_1+c_2}\cdot f\left(\frac{c_1}{c_2}(M\circ S)^{(j-2)/2}\circ M(x+c_2t)\right),\\ 0<x<\sigma,\quad 0<t<\frac{2i\sigma}{c_2},\quad j=2,4,6,\ldots.\end{gathered}\tag{2.51}$$

The waves in $(0,\sigma)$ moving right are

$$\begin{gathered}\frac{2c_2}{c_1+c_2}f\left(\frac{c_1}{c_2}(x-c_2t)\right),\\ \left(\frac{c_3-c_2}{c_3+c_2}\right)^{(k-1)/2}\cdot\left(\frac{c_1-c_2}{c_1+c_2}\right)^{(k-1)/2}\cdot\frac{2c_2}{c_1+c_2}\cdot f\left(\frac{c_1}{c_2}(M\circ S)^{(k-1)/2}(x-c_2t)\right),\\ 0<x<\sigma,\quad k=3,5,7,\ldots.\end{gathered}\tag{2.52}$$

Finally the ones moving right for $x>\sigma$ are

$$\begin{gathered}\frac{4c_2c_3}{(c_1+c_2)(c_2+c_3)}\left(\frac{c_1-c_2}{c_1+c_2}\cdot\frac{c_3-c_2}{c_3+c_2}\right)^{(j-2)/2}\cdot f\left(\frac{c_1}{c_2}(M\circ S)^{(j-2)/2}\circ L(x-c_3t)\right),\\ x>\sigma,\quad 0<t<\frac{2i\sigma}{c_2},\quad j=2,4,6,\ldots.\end{gathered}\tag{2.53}$$

Now, we recall that u_1 given in (2.2) is only good for the time interval $(0, \sigma/c_2)$, and after $t = \sigma/c_2$, the transmitted part hits the join at σ and splits. Therefore, in this time interval we have the wave v_1 defined as

$$v_1(x,t) = \begin{cases} f(x - c_1 t) + \dfrac{c_2 - c_1}{c_2 + c_1} f(-x - c_1 t), & x < 0, \\ \dfrac{2c_2}{c_1 + c_2} f\left(\dfrac{c_1}{c_2}(x - c_2 t)\right), & 0 < x < \sigma, \\ 0, & x > \sigma. \end{cases} \tag{2.54}$$

We note that v_1 is the original incoming signal $f(x - c_1 t)$ incident on 0 plus the reflected and transmitted waves. The transmitted wave has not reached the join σ yet, and so there is no wave beyond the point $x = \sigma$. Therefore v_1 in (2.54) satisfies the wave problem (2.34)–(2.36) in this time interval. We now bring in another wave v_2 in the interval $(\sigma/c_2, 2\sigma/c_2)$. We define it by

$$v_2(x,t) = \begin{cases} 0, & x < 0, \\ \dfrac{c_3 - c_2}{c_3 + c_2} \cdot \dfrac{2c_2}{c_1 + c_2} f\left(\dfrac{c_1}{c_2} M(x + c_2 t)\right), & 0 < x < \sigma, \\ \dfrac{2c_3}{c_2 + c_3} \cdot \dfrac{2c_2}{c_1 + c_2} f\left(\dfrac{c_1}{c_2} L(x - c_3 t)\right), & x > \sigma. \end{cases} \tag{2.55}$$

The function v_2 in (2.55) represents the reflected and transmitted waves at σ, caused by the incident signal $(2c_2/(c_1+c_2)) f((c_1/c_2)(x - c_2 t))$. We see this incident wave in the definition of v_1 in (2.54). We argue that the sum of the wave in (2.54) and (2.55), $v_1 + v_2$, over the interval $(0, 2\sigma/c_2)$ satisfies the wave problem (2.34)–(2.36). The fact that they satisfy the wave (2.34) over each space interval is clear. The boundary conditions at 0 follow from the fact that v_1 is the solution to the wave problem (1.1)–(1.4) and the function $((c_3 - c_2)/(c_3 + c_2)) \cdot (2c_2/(c_1 + c_2)) f((c_1/c_2) M(x + c_2 t)) = ((c_3 - c_2)/(c_3 + c_2)) \cdot (2c_2/(c_1 + c_2)) f((c_1/c_2)(2\sigma - (x + c_2 t))) = 0$ for $x = 0,\ t < 2\sigma/c_2$. The boundary conditions at $x = \sigma$ are satisfied, because the sum of the parts for $0 < x < \sigma$ and $x > \sigma$ is the the solution of the wave problem (2.5), (2.8)–(2.9), with g given by (2.3) there. The initial conditions are satisfied because v_1 satisfies them, and $v_2(x, 0) \equiv 0$ due to the argument of f staying positive when $t = 0$.

Now we consider wave v_3 over the interval $(2\sigma/c_2, 3\sigma/c_2)$. It is defined by

$$v_3(x,t) = \begin{cases} \dfrac{4c_1 c_2}{(c_1 + c_2)^2} \cdot \left(\dfrac{c_3 - c_2}{c_3 + c_2}\right) \cdot f\left(\dfrac{c_1}{c_2} M \circ R(x + c_1 t)\right), & x < 0, \\ \dfrac{2c_2}{c_1 + c_2} \dfrac{c_3 - c_2}{c_3 + c_2} \dfrac{c_1 - c_2}{c_1 + c_2} f\left(\dfrac{c_1}{c_2} (M \circ S)(x - c_2 t)\right), & 0 < x < \sigma, \\ 0, & x > \sigma. \end{cases} \tag{2.56}$$

This is the wave due to the scattering of $((c_3 - c_2)/(c_3 + c_2)) \cdot (2c_2/(c_1 + c_2)) f((c_1/c_2) M(x + c_2 t))$, at the join 0. This incident wave can be seen in the definition of v_2 in (2.55). We claim that the sum $v_1 + v_2 + v_3$, where v_1, v_2, v_3 are given by (2.54)–(2.56), satisfies the wave problem (2.34)–(2.36) over the interval $(0, 3\sigma/c_2)$. We note that v_3 when added to the incident wave

$((c_3-c_2)/(c_3+c_2))\cdot(2c_2/(c_1+c_2))f((c_1/c_2)M(x+c_2t))$ is the wave u_3 that was given in (2.33) except for the notation. This can be seen by noting that h in (2.33) is defined in terms of g in (2.32) and g is given in terms of f in (2.3). We constructed u_3 so that it satisfies the boundary conditions $u(0-,t)=u(0+,t), u_x(0-,t), u_x(0+,t)$ in (2.36), as well as the wave equation (2.34) for $x<\sigma$. The sum $v_1+v_2+v_3$ contains u_3 for $x<\sigma$. In light of the previous argument about v_1+v_2 and the way v_3 contributes to the sum, it is not difficult to see why $v_1+v_2+v_3$ satisfies the boundary conditions in (2.36) and the wave equation in (2.34) in the interval $(0,3\sigma/c_2)$. The fact that at time $0<t<\sigma/c_2$ all terms involving f except $f(x-c_1t)$ are zero in $v_1+v_2+v_3$ shows that the initial conditions of (2.30) are also satisfied.

If one waits other σ/c_2 units of time, another scattering happens at σ from the wave $(2c_2/(c_1+c_2))((c_3-c_2)/(c_3+c_2))((c_1-c_2)/(c_1+c_2))f((c_1/c_2)(M\circ S)(x-c_2t))$ and then again at 0 from the reflected wave of $(2c_2/(c_1+c_2))((c_3-c_2)/(c_3+c_2))((c_1-c_2)/(c_1+c_2))f((c_1/c_2)(M\circ S)(x-c_2t))$. This process continues forever, and new waves appear every σ/c_2 units of time. The functions $v_4, v_5, \ldots$ can be defined as before such that their sum v_i, $i=1,2,3\ldots$ will satisfy the wave problem (2.34)–(2.36) over longer and longer time intervals. In the statement below we write the solution in the form of a finite series whose upper limit depends on the number of σ/c_2 elapsed. In doing so we use the general forms of the waves in (2.50)–(2.53).

We summarize these results in the following theorem.

Theorem 2. *Let $f(x-c_1t), f(s)=0,\ s>0,\ f\in C^2(R)$ be an incoming wave. Then the solution to the problem* (2.34)–(2.36) *in the time interval $0\le t<i\sigma/c_2,\ i=1,2,3,\ldots$ is given by*

$$u(x,t)=\begin{cases} f(x-c_1t)+\dfrac{c_2-c_1}{c_2+c_1}f(-x-c_1t), & x<0,\\ \dfrac{2c_2}{c_1+c_2}f\left(\dfrac{c_1}{c_2}(x-c_2t)\right), & 0<x<\sigma,\\ 0, & x>\sigma\end{cases}$$
$$+\Sigma_{j_{\text{even}}=2}^{i}\begin{cases} 0, & x<0,\\ \left(\dfrac{c_1-c_2}{c_1+c_2}\right)^{(j-2)/2}\cdot\left(\dfrac{c_3-c_2}{c_3+c_2}\right)^{j/2}\cdot\dfrac{2c_2}{c_1+c_2} & \\ \quad\cdot f\left(\dfrac{c_1}{c_2}(M\circ S)^{(j-2)/2}\circ M(x+c_2t)\right), & 0<x<\sigma,\\ \dfrac{4c_2c_3}{(c_1+c_2)(c_2+c_3)}\left(\dfrac{c_1-c_2}{c_1+c_2}\cdot\dfrac{c_3-c_2}{c_3+c_2}\right)^{(j-2)/2} & \\ \quad\cdot f\left(\dfrac{c_1}{c_2}(M\circ S)^{(j-2)/2}\circ L(x-c_3t)\right), & x>\sigma\end{cases}$$
$$+\Sigma_{k_{\text{odd}}=3}^{i}\begin{cases} \dfrac{4c_1c_2}{(c_1+c_2)^2}\cdot\left(\dfrac{c_3-c_2}{c_3+c_2}\right)^{(k-1)/2}\cdot\left(\dfrac{c_1-c_2}{c_1+c_2}\right)^{(k-3)/2} & \\ \quad\cdot f\left(\dfrac{c_1}{c_2}(M\circ S)^{(k-3)/2}\circ M\circ R(x+c_1t)\right), & x<0,\\ \dfrac{2c_2}{c_1+c_2}\cdot\left(\dfrac{c_3-c_2}{c_3+c_2}\cdot\dfrac{c_1-c_2}{c_1+c_2}\right)^{(k-1)/2}\cdot f\left(\dfrac{c_1}{c_2}(M\circ S)^{(k-1)/2}(x-c_2t)\right), & 0<x<\sigma,\\ 0, & x>\sigma.\end{cases}\tag{2.57}$$

Furthermore, assume that f and its derivatives of up to the order two are bounded in R. Then, the solution to (2.34)–(2.36) *over the time interval* $[0,\infty)$ *is the limit as $i \to \infty$ of the above finite time solution.*

Proof. Note that for $i = 1, u(x,t) = v_1(x,t)$ as described in (2.54), for $i = 2, u(x,t) = v_1 + v_2$, where v_2 is given in (2.55), and for $i = 3, u(x,t) = v_1 + v_2 + v_3$, where v_3 is defined in (2.56). We showed that in all these cases these sums satisfy the wave problem (2.34)–(2.36). For $i > 3$, we will be adding reflected and transmitted waves at each interface 0 or σ which, when added to their corresponding incident waves, satisfy the boundary conditions in (2.36). We also argued that the initial conditions are satisfied because for t close to 0 all terms involving f except $f(x - c_t)$ will be zero. So u solves the wave problem in $(0, i\sigma/c_2)$.

It remains to show the limiting case when $i \to \infty$. For that, let us just focus on one piece of the function u in the interval $(-\infty, 0)$:

$$u(x,t) = f(x - c_1t) + \frac{c_2 - c_1}{c_2 + c_1} f(-x - c_1t) + \sum_{k_{\text{odd}}=3}^{\infty} \frac{4c_1c_2}{(c_1 + c_2)^2} \cdot \left(\frac{c_3 - c_2}{c_3 + c_2}\right)^{(k-1)/2} \cdot \left(\frac{c_1 - c_2}{c_1 + c_2}\right)^{(k-3)/2} \cdot f\left(\frac{c_1}{c_2}(M \circ S)^{(k-3)/2} \circ M \circ R(x + c_1t)\right). \tag{2.58}$$

First notice that by definitions in (2.37) the argument of f in the summation (2.58) can be simplified to

$$\begin{aligned} &\frac{4c_1c_2}{(c_1 + c_2)^2} \cdot \left(\frac{c_3 - c_2}{c_3 + c_2}\right)^{(k-1)/2} \cdot \left(\frac{c_1 - c_2}{c_1 + c_2}\right)^{(k-3)/2} \cdot f\left(\frac{c_1}{c_2}(M \circ S)^{(k-3)/2} \circ M \circ R(x + c_1t)\right) \\ &\quad = \frac{4c_1c_2}{(c_1 + c_2)^2} \cdot \left(\frac{c_3 - c_2}{c_3 + c_2}\right)^{(k-1)/2} \cdot \left(\frac{c_1 - c_2}{c_1 + c_2}\right)^{(k-3)/2} \cdot f\left(\frac{c_1}{c_2}\left((k-1)\sigma - \frac{c_2}{c_1}(x + c_1t)\right)\right). \end{aligned} \tag{2.59}$$

The coefficients of f satisfy $|(c_3 - c_2)/(c_3 + c_2)| < 1, |(c_1 - c_2)/(c_1 + c_2)| < 1$. Let

$$r = \max\left\{\left|\frac{c_3 - c_2}{c_3 + c_2}\right|, \left|\frac{c_1 - c_2}{c_1 + c_2}\right|\right\}, \tag{2.60}$$

then $r < 1$. Now with f being bounded, the partial sums of the series (2.58) are bounded above by the partial sums of a convergent geometric series in powers of r. Therefore it is absolutely and uniformly convergent. On the other hand, the term-by-term differentiation of (2.58) in terms of t results in an extra coefficient c_1 and no extra coefficient in terms of x, as can be seen by (2.59). Since f' and f'' are also bounded, the resulting series also converge absolutely and uniformly to the derivatives of the limit of the series (2.58). A similar argument can be applied to the series for u in the intervals $(0, \sigma)$ and (σ, ∞). □

We have shown the form of the scattering of an incident wave on the right- and left-hand side of the join 0. But we have only shown the scattering of an incident wave on the left of σ. If one wanted to study a higher number of interfaces, then it would be necessary to know what happens when an incoming wave hits an interface at an arbitrary point $\sigma \neq 0$ on the right. Here, and for the completeness of the argument, we point out that the transmitted and reflected waves at $x = \sigma$ from an incoming wave $k(x + c_3)$ from the right would be as follows:

$$u_4(x,t) = \begin{cases} \dfrac{2c_2}{c_2 + c_3} k\left(\dfrac{c_2 - c_3}{c_2}\sigma + \dfrac{c_3}{c_2}(x + c_2 t)\right), & 0 < x < \sigma, \\ k(x + c_3 t) + \dfrac{c_2 - c_3}{c_2 + c_3} k(2\sigma - (x - c_3 t)), & x > \sigma. \end{cases} \tag{2.61}$$

Theoretically, this should enable us to extend the result of Theorem 2 to a higher number of discontinuities in the density of the string. The difficulty would be the ability to keep track of all incoming and outgoing waves as well as the ones that bounce back and forth between the interfaces. Since there are so many such waves, with even three interfaces, a general solution for more than two interfaces is impossible to write down. However, for a specific, relatively short length of time the solution can be found, when the discontinuities are few, say three. In this case the form of the solution will also depend on the distance between the discontinuities. Then, one can ask up to how many joins or for what length of time will the scattered waves be tractable. Another interesting question would lie in the area of inverse scattering. Since by our experience the solution contains the location σ of a discontinuity, in the absence of such knowledge will it be possible to find the coordinate of the interface from the form of the scattered waves.

References

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