Research Article

Characteristic Number Associated to Mass Linear Pairs

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Let Δ be a Delzant polytope in $\mathbb{R}^n$ and $\mathbf{b} \in \mathbb{Z}^n$. Let E denote the symplectic fibration over S^2 determined by the pair $(\Delta, \mathbf{b})$. Under certain hypotheses, we prove the equivalence between the fact that $(\Delta, \mathbf{b})$ is a mass linear pair (McDuff and Tolman, 2010) and the vanishing of a characteristic number of E . Denoting by $\text{Ham}(M_\Delta)$, the Hamiltonian group of the symplectic manifold defined by Δ , we determine loops in $\text{Ham}(M_\Delta)$ that define infinite cyclic subgroups in $\pi_1(\text{Ham}(M_\Delta))$ when Δ satisfies any of the following conditions: (i) it is the trapezium associated with a Hirzebruch sur-face, (ii) it is a Δ_p bundle over Δ_1 , and (iii) Δ is the truncated simplex associated with the one point blowup of $\mathbb{C}P^n$.

1. Introduction

Let (N, Ω) be a closed connected symplectic $2n$ -manifold. By $\text{Ham}(N, \Omega)$, we denote the Hamiltonian group of (N, Ω) [1, 2]. Associated with a loop ψ in $\text{Ham}(N, \Omega)$, there exist characteristic numbers which are invariant under deformation of ψ . These invariants are defined in terms of characteristic classes of fibre bundles, and their explicit values are not easy to calculate, in general. Here, we will consider a particular invariant I , whose definition will be recalled below. By proving the nonvanishing of I for certain loops, we will deduce the existence of infinity cyclic subgroups of $\pi_1(\text{Ham}(N, \Omega))$, when N is a toric manifold. The vanishing of the invariant I on particular loops in $\text{Ham}(N, \Omega)$ is related with the concept of mass linear pair, which has been developed in [3]. In this introduction, we will state the main results of the paper and will give a schematic exposition of the concepts involved in these statements.

A loop ψ in $\text{Ham}(N, \Omega)$ determines a Hamiltonian fibre bundle $E \rightarrow S^2$ with standard fibre N , via the clutching construction. Various characteristic numbers for the fibre bundle E

have been defined in [4]. These numbers give rise to topological invariants of the loop ψ . In this paper, we will consider only the following characteristic number:

$$I(\psi) := \int_E c_1(\text{VTE})c^n, \quad (1.1)$$

where VTE is the vertical tangent bundle of E , and $c \in H^2(E, \mathbb{R})$ is the coupling class of the fibration $E \rightarrow S^2$ [1, 5]. $I(\psi)$ depends only on the homotopy class of the loop ψ . Moreover, the map

$$I : \psi \in \pi_1(\text{Ham}(N, \Omega)) \mapsto I(\psi) \in \mathbb{R} \quad (1.2)$$

is an $\mathbb{R}$ -valued group homomorphism [4].

Our purpose is to study this characteristic number when N is a toric manifold and ψ is a 1-parameter subgroup of $\text{Ham}(N)$ defined by the toric action. The referred 1-parameter subgroup is determined by an element $\mathbf{b}$ in the integer lattice of the Lie algebra of the corresponding torus. On the other hand, a toric symplectic manifold is determined by its moment polytope. For a general polytope, a mass linear function on it is a linear function “whose value on the center of mass of the polytope depends linearly on the positions of the supporting hyperplanes” [3]. In this paper, we will relate the vanishing of the number $I(\psi)$ with the fact that $\mathbf{b}$ defines a mass linear function on the polytope associated with the toric manifold. In the following paragraphs, we provide a more detailed exposition of this relation.

Let T be the torus $(\text{U}(1))^n$ and $\Delta = \Delta(\mathbf{n}, k)$ the polytope in $\mathfrak{t}^*$ with m facets defined by

$$\Delta(\mathbf{n}, k) = \bigcap_{j=1}^m \{x \in \mathfrak{t}^* : \langle x, \mathbf{n}_j \rangle \leq k_j\}, \quad (1.3)$$

where $k_j \in \mathbb{R}$ and the $\mathbf{n}_j \in \mathfrak{t}$ are the outward conormals to the facets. The facet defined by the equation $\langle x, \mathbf{n}_j \rangle = k_j$ will be denoted F_j , and we put $\text{Cm}(\Delta)$ for the mass center of the polytope Δ .

In [3], the chamber $\mathcal{C}_\Delta$ of Δ is defined as the set of $k' \in \mathbb{R}^m$ such that the polytope $\Delta' := \Delta(\mathbf{n}, k')$ is analogous to Δ ; that is, the intersection $\cap_{j \in J} F_j$ is nonempty if and only if $\cap_{j \in J} F'_j \neq \emptyset$ for any $J \subset \{1, \dots, m\}$. When we consider only polytopes which belong to the chamber of a fixed polytope, we delete the $\mathbf{n}$ in the notation introduced in (1.3).

Further, McDuff and Tolman [3] introduced the concept of mass linear pair. Given the polytope Δ and $\mathbf{b} \in \mathfrak{t}$, the pair $(\Delta, \mathbf{b})$ is mass linear if the map

$$k \in \mathcal{C}_\Delta \mapsto \langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle \in \mathbb{R} \quad (1.4)$$

is linear. That is,

$$\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \sum_j R_j k_j + C, \quad (1.5)$$

where R_j and C are constant.

Let us assume that Δ is a Delzant polytope [6]. We shall denote by $(M_\Delta, \omega_\Delta, \mu_\Delta)$ the toric manifold determined by Δ ($\mu_\Delta : M \rightarrow \mathfrak{t}^*$ being the corresponding moment map). Given $\mathbf{b}$, an element in the integer lattice of $\mathfrak{t}$, we shall write $\varphi_{\mathbf{b}}$ for the loop of Hamiltonian diffeomorphisms of $(M_\Delta, \omega_\Delta)$ defined by $\mathbf{b}$ through the toric action. We will let $I(\Delta; \mathbf{b})$ for the characteristic number $I(\varphi_{\mathbf{b}})$. When we consider only polytopes in the chamber of a given polytope, we will write $I(k; \mathbf{b})$ instead of $I(\Delta(k); \mathbf{b})$ for k in this chamber.

The group G of the translations defined by the elements of $\mathfrak{t}^*$ acts freely on $\mathcal{C}_\Delta$. We put $r := m - n$ for the dimension of the quotient $\mathcal{C}_\Delta/G$. Thus, r is the number of effective parameters which characterize the polytopes in $\mathcal{C}_\Delta$ considered as “physical bodies.”

We will prove the following theorem.

Theorem 1.1. *Let $(\Delta, \mathbf{b})$ be a pair consisting of a Delzant polytope in $\mathfrak{t}^*$ and an element in the integer lattice of $\mathfrak{t}$. If $r \leq 2$, the following statements are equivalent:*

- (a) $I(k; \mathbf{b}) = 0$, for all $k \in \mathcal{C}_\Delta$.
- (b) $(\Delta, \mathbf{b})$ is a mass linear pair as in (1.5), with $\sum_j R_j = 0$.

In [7], by direct computation, we proved the equivalence between the vanishing of $I(k; \mathbf{b})$ on $\mathcal{C}_\Delta$ and the fact that $(\Delta, \mathbf{b})$ is a mass linear pair, when Δ satisfies any of the following conditions:

- (i) it is the trapezium associated with a Hirzebruch surface,
- (ii) it is a Δ_p bundle over Δ_1 [3], and
- (iii) Δ is the truncated simplex associated with the one point blowup of $\mathbb{C}P^n$.

On the other hand, when Δ is any of these polytopes (i)–(iii), the number r is equal to 2; thus, from Theorem 1.1 and the result of [7], it follows that condition $\sum_j R_j = 0$ is satisfied by all the mass linear pairs $(\Delta, \mathbf{b})$. This fact can also be proved by direct calculation (Propositions 4.1, 4.6, and 4.9). So, Theorem 1.1, together with these propositions, generalize the result proved in [7].

Although the homotopy type of the Hamiltonian groups $\text{Ham}(N, \Omega)$ is known only for some symplectic manifolds [8], the invariant I allows us to identify nontrivial elements in $\pi_1(\text{Ham}(N, \Omega))$. As I is a group homomorphism, from Theorem 1.1, we deduce that a sufficient condition for $\varphi_{\mathbf{b}}$ to generate an infinite cyclic subgroup in $\pi_1(\text{Ham}(M_\Delta, \omega_\Delta))$ is that the above condition (b) does not hold for $(\Delta, \mathbf{b})$. More precisely, we have the following consequence of Theorem 1.1.

Theorem 1.2. *Given the Delzant polytope Δ and $\mathbf{b}$ an element in the integer lattice of $\mathfrak{t}$. If $r \leq 2$ and $(\Delta, \mathbf{b})$ is not mass linear, then $\varphi_{\mathbf{b}}$ generates an infinite cyclic subgroup in $\pi_1(\text{Ham}(M_{\Delta(k)}, \omega_{\Delta(k)}))$, for all $k \in \mathcal{C}_\Delta$.*

In the proof of Theorem 1.1, a formula for the characteristic number $I(\varphi_{\mathbf{b}})$ obtained in [9] plays a crucial role. This formula gives $I(\varphi_{\mathbf{b}})$ in terms of the integrals, on the facets of the polytope, of the normalized Hamiltonian function corresponding to the loop $\varphi_{\mathbf{b}}$ (see (2.13)). From this expression for $I(\varphi_{\mathbf{b}})$, we will deduce a relation between the directional derivative of map (1.4) along the vector $(1, \dots, 1)$ of $\mathbb{R}^m$, the Euclidean volume of $\Delta(k)$ and $I(k; \mathbf{b})$ (see (3.5)). From this relation, it is easy to complete the proof of Theorem 1.1.

This paper is organized as follows. In Section 2, we study the characteristic number $I(k; \mathbf{b})$ when $(\Delta, \mathbf{b})$ is a linear pair and k varies in the chamber of Δ ; we prove that $I(k; \mathbf{b})$ is a homogeneous polynomial of the k_j (Proposition 2.4).

In Section 3, we prove Theorem 1.1. In Proposition 3.5, a sufficient geometric condition for the Delzant polytope Δ to admit a mass linear pair $(\Delta, \mathbf{b})$ is given. For a Delzant polytope Δ , Proposition 3.6 gives a necessary condition for the vanishing of $I(k; \mathbf{b})$ on $\mathcal{C}_\Delta$. We also express $\sum_j R_j$ in terms of the displacement of the center of mass $\text{Cm}(\Delta(k))$ produced by the change $k_j \rightarrow k_j + 1$ (Proposition 3.7).

Section 4 concerns the form which Theorem 1.2 adopts, when Δ is a Delzant polytope of the particular types (i)–(iii) mentioned above (see Corollary 4.2, Theorems 4.5, and 4.8). We also prove that, in these particular cases, if $(\Delta, \mathbf{b})$ is a mass linear pair, then $\sum_j R_j = 0$.

2. A Characteristic Number

Let us suppose that the polytope Δ defined in (1.3) is a Delzant polytope in $\mathfrak{t}^*$. Following [10], we recall some points of the construction of $(M_\Delta, \omega_\Delta, \mu_\Delta)$ from the polytope Δ . We put $\tilde{T} := (S^1)^{m-n}$. The $\mathbf{n}_i$ determine weights $w_j \in \tilde{\mathfrak{t}}^*$, $j = 1, \dots, m$ for a $\tilde{T}$ -action on $\mathbb{C}^m$. Then moment map for this action is

$$J : z \in \mathbb{C}^m \mapsto J(z) = \pi \sum_{j=1}^m |z_j|^2 w_j \in \tilde{\mathfrak{t}}^*. \quad (2.1)$$

The k_i define a regular value σ for J , and the manifold M_Δ is the following orbit space

$$M_\Delta = \frac{\left\{ z \in \mathbb{C}^m : \pi \sum_{j=1}^m |z_j|^2 w_j = \sigma \right\}}{\tilde{T}}, \quad (2.2)$$

where the relation defined by $\tilde{T}$ is

$$(z_j) \simeq (z'_j) \text{ iff there is } \mathbf{y} \in \tilde{\mathfrak{t}} \text{ such that } z'_j = z_j e^{2\pi i \langle w_j, \mathbf{y} \rangle} \text{ for } j = 1, \dots, m. \quad (2.3)$$

Identifying $\tilde{\mathfrak{t}}^*$ with $\mathbb{R}^r$, $\sigma = (\sigma_1, \dots, \sigma_r)$ and each σ_a is a linear combination of the k_j .

Given a facet F of Δ , we choose a vertex p of F . After a possible change in numeration of the facets, we can assume that $F_1, \dots, F_n$ intersect at p . In this numeration, $F = F_j$, for some $j \in \{1, \dots, n\}$.

If we write $z_a = \rho_a e^{i\theta_a}$, then the symplectic form can be written on $\{[z] \in M : z_a \neq 0, \text{ for all } a\}$

$$\omega_\Delta = \frac{1}{2} \sum_{i=1}^n d\rho_i^2 \wedge d\varphi_i, \quad (2.4)$$

with φ_i an angular variable, linear combination of the θ_a .

The action of $T = (S^1)^n$ on M_Δ

$$(\alpha_1, \dots, \alpha_n)[z_1, \dots, z_m] := [\alpha_1 z_1, \dots, \alpha_n z_n, z_{n+1}, \dots, z_m] \quad (2.5)$$

endows M_Δ with a structure of toric manifold. Identifying $\mathfrak{t}^*$ with $\mathbb{R}^n$, the moment map $\mu_\Delta : M_\Delta \rightarrow \mathfrak{t} = \mathbb{R}^n$ is defined by

$$\mu_\Delta([z]) = \pi(\rho_1^2, \dots, \rho_n^2) + (d_1, \dots, d_n), \quad (2.6)$$

where the constants d_i are linear combinations of the k_j and

$$\text{im } \mu_\Delta = \Delta. \quad (2.7)$$

The facet $F = F_j$ of Δ is the image by μ_Δ of the submanifold

$$D_j = \{[z_1, \dots, z_m] \in M_\Delta \mid z_j = 0\}. \quad (2.8)$$

We write $x_i := \pi\rho_i^2$, then

$$\int_{M_\Delta} (\omega_\Delta)^n = n! \int_\Delta dx_1 \cdots dx_n. \quad (2.9)$$

Let $\mathbf{b}$ be an element in the integer lattice of $\mathfrak{t}$. The normalized Hamiltonian of the circle action generated by $\mathbf{b}$ is the function f determined by

$$f = \langle \mu_\Delta, \mathbf{b} \rangle + \text{constant}, \quad \int_{M_\Delta} f (\omega_\Delta)^n = 0. \quad (2.10)$$

That is, $f = \langle \mu_\Delta, \mathbf{b} \rangle - \langle \text{Cm}(\Delta), \mathbf{b} \rangle$, where

$$\langle \text{Cm}(\Delta), \mathbf{b} \rangle = \frac{\int_M \langle \mu_\Delta, \mathbf{b} \rangle (\omega_\Delta)^n}{\int_M (\omega_\Delta)^n}. \quad (2.11)$$

Moreover,

$$\int_{M_\Delta} \langle \mu_\Delta, \mathbf{b} \rangle (\omega_\Delta)^n = n! \int_\Delta \sum_{i=1}^n b_i x_i dx_1 \cdots dx_n. \quad (2.12)$$

An expression for the value of the invariant $I(\psi_{\mathbf{b}})$ in terms of integrals of the Hamiltonian function has been obtained in Section 4 of [9] (see also [11, 12])

$$I(\Delta; \mathbf{b}) := I(\psi_{\mathbf{b}}) = -n \sum_{F \text{ facet}} N(F), \quad (2.13)$$

where the contribution $N(F)$ of the above facet $F = F_j$ (with $j = 1, \dots, n$) is

$$\begin{aligned} N_j &:= N(F) = (n-1)! \int_{F_j} f dx_1 \cdots \widehat{dx_j} \cdots dx_n \\ &= (n-1)! \left(\int_{F_j} \langle \mu_\Delta, \mathbf{b} \rangle dx_1 \cdots \widehat{dx_j} \cdots dx_n - \langle \text{Cm}(\Delta), \mathbf{b} \rangle \int_{F_j} dx_1 \cdots \widehat{dx_j} \cdots dx_n \right), \end{aligned} \quad (2.14)$$

with $dx_1 \cdots \widehat{dx_j} \cdots dx_n := dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n$.

Given $\Delta = \Delta(\mathbf{n}, k)$, we consider the polytope $\Delta' = \Delta(\mathbf{n}, k')$ obtained from Δ by the translation defined by a vector a of $\mathfrak{t}^*$. As we said, we write $I(k; \mathbf{b})$ and $I(k'; \mathbf{b})$ for the corresponding characteristic numbers. According to the construction of the respective toric manifolds,

$$M_{\Delta'} = M_\Delta, \quad \omega_{\Delta'} = \omega_\Delta, \quad \mu_{\Delta'} = \mu_\Delta + a. \quad (2.15)$$

But the *normalized* Hamiltonians f and f' corresponding to the action of $\mathbf{b}$ on M_Δ and $M_{\Delta'}$ are equal. Thus, it follows from (2.13) that $I(k; \mathbf{b}) = I(k'; \mathbf{b})$. More precisely, we have the evident proposition.

Proposition 2.1. *If a is an arbitrary vector of $\mathfrak{t}^*$, then $I(k; \mathbf{b}) = I(k'; \mathbf{b})$, for $k'_j = k_j + \langle a, \mathbf{n}_j \rangle$, $j = 1, \dots, m$.*

By Proposition 2.1, we can assume that all d_j in (2.6) are zero for the determination of $I(k; \mathbf{b})$.

The following lemma is elementary.

Lemma 2.2. *If*

$$S_n(\tau) := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \leq \tau, \ 0 \leq x_j, \ \forall j \right\}, \quad (2.16)$$

then

$$\int_{S_n(\tau)} f(x_1, \dots, x_n) dx_1 \cdots dx_n = \begin{cases} \frac{\tau^n}{n!} & \text{if } f = 1 \\ c \frac{\tau^{n+c}}{(n+c)!} & \text{if } f = x_i^c, \ c = 1, 2 \\ \frac{\tau^{n+2}}{(n+2)!} & \text{if } f = x_i x_j, \ i \neq j. \end{cases} \quad (2.17)$$

More general, if $c_1, \dots, c_n \in \mathbb{R}_{>0}$, we put

$$S_n(c, \tau) := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n c_i x_i \leq \tau, \ 0 \leq x_j, \ \forall j \right\}, \quad (2.18)$$

then

$$\int_{S_n(c,\tau)} dx_1 \cdots dx_n = \frac{1}{n!} \prod_{i=1}^n \frac{\tau}{c_i}, \quad \int_{S_n(c,\tau)} x_j dx_1 \cdots dx_n = \frac{1}{(n+1)!} \frac{\tau}{c_j} \prod_{i=1}^n \frac{\tau}{c_i}. \quad (2.19)$$

Thus, in the particular case that $\Delta = S_n(c, \tau)$, the integral $\int_{M_\Delta} (\omega_\Delta)^n$ is a monomial of degree n in τ , and $\int_{M_\Delta} \langle \mu_\Delta, \mathbf{b} \rangle (\omega_\Delta)^n$ is a monomial of degree $n+1$.

We return to the general case in which Δ is the polytope defined in (1.3). Its vertices are the solutions to

$$\langle x, \mathbf{n}_{j_a} \rangle = k_{j_a}, \quad a = 1, \dots, n; \quad (2.20)$$

hence, the coordinates of any vertex of Δ are linear combinations of the k_j .

A hyperplane in $\mathbb{R}^n$ through a vertex $(x_1^0, \dots, x_n^0)$ of Δ is given by an equation of the form

$$\langle x, \mathbf{n} \rangle = \langle x^0, \mathbf{n} \rangle =: \kappa. \quad (2.21)$$

Thus, the independent term κ is a linear combination (l.c.) of the k_j . Moreover, the coordinates of the common point of n hyperplanes

$$\langle x, \tilde{\mathbf{n}}_i \rangle = \kappa_i, \quad (2.22)$$

with κ_i l.c. of, the k_j are also l.c. of the k_j .

By drawing hyperplanes through vertices of Δ (or more generally, through points which are the intersection of n hyperplanes as (2.22)), we can obtain a family $\{\beta S\}$ of subsets of Δ such that

- (a) each βS is the transformed of a simplex $S_n(b, \tau)$ by an element of the group of Euclidean motions in $\mathbb{R}^n$.
- (b) For $\alpha \neq \beta$, $\alpha S \cap \beta S$ is a subset of the border of αS .
- (c) $\bigcup_{\beta} \beta S = \Delta$.

Thus, by construction, each facet of βS is contained in a hyperplane π of the form $\langle x, \mathbf{n} \rangle = \kappa$, with κ l.c. of the k_j .

On the other hand, the hyperplane π is transformed by an element of $\text{SO}(n)$ in an hyperplane $\langle x, \mathbf{n}' \rangle = \kappa$. If $\mathcal{T}$ is a translation in $\mathbb{R}^n$ which applies $S_n(b, \tau)$ onto βS , then this transformation maps $(0, \dots, 0)$ in a vertex $a = (a_1, \dots, a_n)$ of βS . So, the translation $\mathcal{T}$ transforms π in $\langle x, \mathbf{n} \rangle = \kappa + \langle a, \mathbf{n} \rangle =: \kappa'$. As each a_j is a l.c. of the k_j , so is κ' . Hence, any element of the group of Euclidean motions in $\mathbb{R}^n$ which maps $S_n(b, \tau)$ onto βS transforms the hyperplane π

$$\langle x, \mathbf{n}' \rangle = \kappa', \quad (2.23)$$

with κ' a l.c. of the k_j .

Let assume that $(R\mathcal{T}_a)(S(b, \tau)) = {}_\beta S$, with $R \in \text{SO}(n)$ and $\mathcal{T}_a$ the translation defined by a . Then the oblique facet of $S(b, \tau)$, contained in the hyperplane $\sum b_i x_i = \tau$, is the image by $T_{-a}R^{-1}$ of a facet of ${}_\beta S$, which in turn is contained in a hyperplane of (2.23) (κ' being a l. c. of the k_j). The argument of the preceding paragraph applied to R^{-1} and $\mathcal{T}_{-a}$ proves that τ is a l. c. of the k_j . Hence, by (2.19), the integral

$$\int_{{}_\beta S} dx_1 \cdots dx_n = \int_{S_n(b, \tau)} dx_1 \cdots dx_n \quad (2.24)$$

is a monomial of degree n of a l. c. of the k_j . Thus,

$$\int_M (\omega_\Delta)^n = \sum_{\beta} \int_{{}_\beta S} dx_1 \cdots dx_n, \quad (2.25)$$

is a homogeneous polynomial of degree n of the k_j .

Similarly,

$$\int_{M_\Delta} \langle \mu_\Delta, \mathbf{b} \rangle (\omega_\Delta)^n \quad (2.26)$$

is a homogeneous polynomial of degree $n + 1$ of the k_j . Analogous results hold for

$$\int_{F_j} dx_1 \cdots \widehat{dx_j} \cdots dx_n, \quad \int_{F_j} \langle \mu_\Delta, \mathbf{b} \rangle dx_1 \cdots \widehat{dx_j} \cdots dx_n. \quad (2.27)$$

From formulas (2.9)–(2.14) together with the preceding argument, it follows the following proposition.

Proposition 2.3. *Given a Delzant polytope Δ , if $\mathbf{b}$ belongs to the integer lattice of $\mathfrak{t}$, then $I(k; \mathbf{b})$ is a rational function of the k_j , for $k \in \mathcal{C}_\Delta$.*

Analogously, we have the following proposition.

Proposition 2.4. *If $(\Delta, \mathbf{b})$ is mass linear pair, then $I(k; \mathbf{b})$ is a homogeneous polynomial in the k_j of degree n , when $k \in \mathcal{C}_\Delta$.*

We will use the following simple lemma in the proof of Theorem 1.1.

Lemma 2.5. *If $\widehat{k}_j = sk_j$ for $j = 1, \dots, m$, with $s \in \mathbb{R}$, then $\text{Cm}(\Delta(\mathbf{n}, \widehat{k})) = s \text{Cm}(\Delta(\mathbf{n}, k))$.*

Proof. The vertices of $\Delta(\mathbf{n}, k)$ are the solutions of (2.20), and the vertices of $\Delta(\mathbf{n}, \widehat{k})$ are the solutions of $\langle x, \mathbf{n}_{j_a} \rangle = sk_{j_a}$, with $a = 1, \dots, n$. Thus, the vertices of $\Delta(\mathbf{n}, \widehat{k})$ are those of $\Delta(\mathbf{n}, k)$ multiplied by s . $\square$

The lemma also follows from the fact that (2.25) and (2.26) are homogeneous polynomials of degree n and $n + 1$, respectively.

3. Proof of Theorem 1.1

Let us assume that the polytope Δ defined by (1.3) is Delzant and let k be an element of $\mathcal{C}_\Delta$. We denote by $M_{(k)}$, $\omega_{(k)}$, and $\mu_{(k)}$, the manifold, the symplectic structure and the moment map (resp.) determined by $\Delta(k)$. The facets of $\Delta(k)$ will be denoted by $F_{(k)j}$.

Let $\mathbf{b}$ be an element in the integer lattice of $\mathfrak{t}$. We put

$$A_{(k)} := \int_{M_{(k)}} \langle \mu_{(k)}, \mathbf{b} \rangle (\omega_{(k)})^n, \quad B_{(k)} := \int_{M_{(k)}} (\omega_{(k)})^n. \quad (3.1)$$

By (2.9), $(1/n!)B_{(k)}$ is the Euclidean volume of the polytope $\Delta(k)$. Given a facet $F_{(k)j}$, we can assume that $j \in \{1, \dots, n\}$ (see third paragraph of Section 2). So, $F_{(k)j}$ is defined by the equation $x_j = 0$. If we make an infinitesimal variation of the facet $F_{(k)j}$, by means of the translation defined by $k_j \rightarrow k_j + \epsilon$ (keeping unchanged the other k_i), then the volume of $\Delta(k)$ changes according to

$$\frac{1}{n!}B_{(k)} \longrightarrow \frac{1}{n!}B_{(k)} + \epsilon \int_{F_{(k)j}} dx_1 \cdots \widehat{dx_j} \cdots dx_n + O(\epsilon^2). \quad (3.2)$$

We write dX^j for $dx_1 \cdots \widehat{dx_j} \cdots dx_n$. Thus,

$$\frac{\partial B_{(k)}}{\partial k_j} = n! \int_{F_{(k)j}} dX^j, \quad \frac{\partial A_{(k)}}{\partial k_j} = n! \int_{F_{(k)j}} \langle \mu_{(k)}, \mathbf{b} \rangle dX^j. \quad (3.3)$$

So, by (2.11),

$$\frac{\partial}{\partial k_j} \langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \frac{n!}{(B_{(k)})^2} \left(B_{(k)} \int_{F_{(k)j}} \langle \mu_{(k)}, \mathbf{b} \rangle dX^j - A_{(k)} \int_{F_{(k)j}} dX^j \right). \quad (3.4)$$

From (2.13) and (2.14), it follows

$$\sum_{j=1}^m \frac{\partial}{\partial k_j} \langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \frac{-1}{B_{(k)}} I(k; \mathbf{b}). \quad (3.5)$$

Thus, we have proved the following proposition.

Proposition 3.1. *$I(k; \mathbf{b}) = 0$ for all $k \in \mathcal{C}_\Delta$ if and only if $\sum_{j=1}^m (\partial/\partial k_j) \langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = 0$, for all $k \in \mathcal{C}_\Delta$.*

Next, we will parametrize the quotient $\mathcal{C}_\Delta/G$ (of classes of polytopes in $\mathcal{C}_\Delta$ module translation) defined in Section 1.

After a possible renumbering, we may assume that the intersection of facets $F_1, \dots, F_n$ is a vertex of Δ . Thus, the conormals $\mathbf{n}_1, \dots, \mathbf{n}_n$ are linearly independent in $\mathfrak{t}$. So, given $k \in C_\Delta$, there is a unique $v \in \mathfrak{t}^*$, such that

$$\langle v, \mathbf{n}_i \rangle + k_i = 0, \quad i = 1, \dots, n. \quad (3.6)$$

(Expressing the $\mathbf{n}_i$ in terms of a basis of $\mathfrak{t}$ and v in the dual basis, (3.6) is a compatible and determined system of linear equations for the coordinates of v .) Moreover $v = v(k)$ depends linearly on the k_i ; that is, $\langle v(k), \mathbf{c} \rangle$ is a linear function of $k_1, \dots, k_n$, for all $\mathbf{c} \in \mathfrak{t}$.

If $m - n = 2$, we write

$$\lambda = k_{n+1} + \langle v(k), \mathbf{n}_{n+1} \rangle, \quad \tau = k_m + \langle v(k), \mathbf{n}_m \rangle, \quad (3.7)$$

where $v(k)$ the element in $\mathfrak{t}^*$ defined by (3.6). From the linearity of v with respect to the k_i , it follows that λ and τ are *linear combinations* of $k_1, \dots, k_m$.

The polytope in C_Δ defined by $(k'_1 = 0, \dots, k'_n = 0, \lambda, \tau)$ will be denoted by $\Delta_0(\lambda, \tau)$. It is the result of the translation of $\Delta(k)$ by the vector $v(k)$; that is,

$$\Delta_0(\lambda, \tau) = \Delta(k) + v(k). \quad (3.8)$$

Let $\mathbf{b}$ an element in the integer lattice of $\mathfrak{t}$, we define the function g by

$$g(\lambda, \tau) := \langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle. \quad (3.9)$$

The function g is defined on the pairs (λ, τ) such that $(0, \dots, 0, \lambda, \tau) \in C_\Delta$. By Lemma 2.5, it follows

$$g(s\lambda, s\tau) = sg(\lambda, \tau), \quad (3.10)$$

for any real number s such that $(s\lambda, s\tau)$ belongs to the domain of g . This property implies that

$$g = \lambda \frac{\partial g}{\partial \lambda} + \tau \frac{\partial g}{\partial \tau}. \quad (3.11)$$

Theorem 3.2. *If $I(k; \mathbf{b}) = 0$, for all $k \in C_\Delta$ and $r = 2$, then $\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \sum_j R_j k_j$, with R_j constant (i.e., $(\Delta, \mathbf{b})$ is a mass linear pair) and $\sum_j R_j = 0$.*

Proof. We set $f(k_1, \dots, k_m) := \langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle$. It follows from (3.8) that

$$f(k) = g(\lambda, \tau) - \langle v(k), \mathbf{b} \rangle. \quad (3.12)$$

By the hypothesis and Proposition 3.1,

$$\sum_{j=1}^m \frac{\partial f}{\partial k_j} = 0. \quad (3.13)$$

Since

$$\sum_{j=1}^m \frac{\partial f}{\partial k_j} = \frac{\partial g}{\partial \lambda} \sum_{j=1}^m \frac{\partial \lambda}{\partial k_j} + \frac{\partial g}{\partial \tau} \sum_{j=1}^m \frac{\partial \tau}{\partial k_j} - \left\langle \frac{\partial v}{\partial k_j}, \mathbf{b} \right\rangle, \quad (3.14)$$

from (3.13), we deduce

$$p \frac{\partial g}{\partial \lambda} + q \frac{\partial g}{\partial \tau} - t = 0, \quad (3.15)$$

where p, q, t stand for the following constants

$$p = \sum_{j=1}^m \frac{\partial \lambda}{\partial k_j}, \quad q = \sum_{j=1}^m \frac{\partial \tau}{\partial k_j}, \quad t = \left\langle \frac{\partial v}{\partial k_j}, \mathbf{b} \right\rangle. \quad (3.16)$$

Since $q\lambda - p\tau$ and $t\tau - qg$ are first integrals of (3.15), the general solution of this equation is

$$g(\lambda, \tau) = \left(\frac{t}{q} \right) \tau + \Phi(q\lambda - p\tau), \quad (3.17)$$

where Φ is a derivable function of one variable.

It follows from (3.11) and (3.17) that

$$\Phi(u) = u\Phi'(u). \quad (3.18)$$

Thus, $\Phi(u) = \alpha u$, with α constant. We have for f

$$f(k) = \left(\frac{b}{q} \right) \tau + \alpha(q\lambda - p\tau) - \langle v(k), \mathbf{b} \rangle. \quad (3.19)$$

In other words, f is a linear function of the k_j ; that is, $f(k) = \sum_j R_j k_j$, with R_j constant. From (3.13), it follows $\sum_j R_j = 0$. $\square$

Remark 3.3. The proof of Theorem 3.2 can be adapted to the simpler case when $r = 1$. In this case, the function $g(\lambda) = \langle \text{Cm}(\Delta_0(\lambda)), \mathbf{b} \rangle$ satisfies $p(dg/d\lambda) - t = 0$ and $g(s\lambda) = sg(\lambda)$. So, $g(\lambda) = (t/p)\lambda$ and $f(k) = (t/p)\lambda + \langle v(k), \mathbf{b} \rangle$ is a linear map of the variables k_j .

On the other hand, the proof of this theorem does not admit an adaptation to the case $r > 2$. In fact, the corresponding function Φ would be a function of $r - 1$ variables $\Phi(u_1, \dots, u_{r-1})$. The equation which corresponds to (3.18) in this case would be

$$\Phi = \sum_{i=1}^{r-1} u_i \frac{\partial \Phi}{\partial u_i}. \quad (3.20)$$

But this condition does not imply the linearity of Φ .

When $(\Delta, \mathbf{b})$ is a mass linear pair as in (1.5), by (3.5)

$$I(k; \mathbf{b}) = -B_{(k)} \sum_j R_j, \quad (3.21)$$

for all $k \in \mathcal{C}_\Delta$. From (3.21), we deduce the following proposition.

Proposition 3.4. *Let $(\Delta, \mathbf{b})$ be a mass linear pair. $I(k; \mathbf{b}) = 0$ for all $k \in \mathcal{C}_\Delta$ if and only if $\sum_j R_j = 0$.*

Proof of Theorem 1.1. It is a direct consequence of Proposition 3.4, Theorem 3.2, and the remark above. $\square$

We will deduce a sufficient condition for a Delzant polytope Δ to admit mass linear functions. We write

$$\dot{\text{Cm}}(\Delta(k)) := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \text{Cm}(\Delta(k + \check{\epsilon})), \quad (3.22)$$

with $\check{\epsilon} = (\epsilon, \dots, \epsilon)$.

Proposition 3.5. *If all points $\dot{\text{Cm}}(\Delta(k))$, for $k \in \mathcal{C}_\Delta$, belong to a hyperplane of $(\mathbb{R}^n)^*$ with a conormal vector in $\mathbb{Z}^n$ and $r \leq 2$, then Δ admits a mass linear function.*

Proof. Let $\mathbf{b} \in \mathbb{Z}^n$ be a conormal vector to the hyperplane, then

$$0 = \langle \dot{\text{Cm}}(\Delta(k)), \mathbf{b} \rangle = \left\langle \sum_j \frac{\partial}{\partial k_j} \text{Cm}(\Delta(k)), \mathbf{b} \right\rangle. \quad (3.23)$$

By (3.5), $I(k, \mathbf{b}) = 0$; Theorem 3.2 applies and $(\Delta, \mathbf{b})$ is a mass linear pair. $\square$

Proposition 3.6. *Let Δ be a Delzant polytope, such that $k = 0$ belongs to the closure of $\mathcal{C}_\Delta$. If $r \leq 2$, a necessary condition for the vanishing of $I(k; \mathbf{b})$ on $\mathcal{C}_\Delta$ is*

$$\left\langle \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \text{Cm}(\Delta(\check{\epsilon})), \mathbf{b} \right\rangle = 0. \quad (3.24)$$

Proof. If $I(k; \mathbf{b})$ vanishes on $\mathcal{C}_\Delta$, then $(\Delta, \mathbf{b})$ is a linear pair, by Theorem 1.1. Thus, $\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \sum_j R_j k_j + C$, on $\mathcal{C}_\Delta$. So, given $k \in \mathcal{C}_\Delta$ and ϵ small enough

$$\langle \text{Cm}(\Delta(k + \epsilon)), \mathbf{b} \rangle = \sum_j R_j k_j + \epsilon \sum_j R_j + C. \quad (3.25)$$

By Theorem 1.1, $\sum_j R_j = 0$. Thus, for any $k \in \mathcal{C}_\Delta$,

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \langle \text{Cm}(\Delta(k + \epsilon)), \mathbf{b} \rangle = 0. \quad (3.26)$$

Taking the limit as $k \rightarrow 0$,

$$0 = \lim_{k \rightarrow 0} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \langle \text{Cm}(\Delta(k + \epsilon)), \mathbf{b} \rangle = \left\langle \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \text{Cm}(\Delta(\epsilon)), \mathbf{b} \right\rangle. \quad (3.27)$$

□

Next, we will describe a geometric interpretation of the number $\sum_j R_j$. Given an arbitrary Delzant polytope Δ . If a is a vector of $\mathfrak{t}^*$, then

$$\text{Cm}(\Delta(k')) = \text{Cm}(\Delta(k)) + a, \quad (3.28)$$

if $k'_j = k_j + \langle a, \mathbf{n}_j \rangle$.

We will denote by d the element of $\mathfrak{t}^*$ defined by the following relation

$$\text{Cm}(\Delta(\tilde{k})) = \text{Cm}(\Delta(k)) + d, \quad (3.29)$$

with $\tilde{k}_j = k_j + 1$ for all j .

From (3.28) and (3.29), we have

$$\text{Cm}(\Delta(k_j + \langle d, \mathbf{n}_j \rangle)) = \text{Cm}(\Delta(k_j)) + d = \text{Cm}(\Delta(\tilde{k}_j = k_j + 1)). \quad (3.30)$$

Now, we assume that $(\Delta, \mathbf{b})$ is a mass linear pair. From (1.5), it follows

$$\begin{aligned} \langle \text{Cm}(\Delta(k_j + \langle d, \mathbf{n}_j \rangle)), \mathbf{b} \rangle &= \sum R_j k_j + \sum R_j \langle d, \mathbf{n}_j \rangle + C, \\ \langle \text{Cm}(\Delta(k_j)) + d, \mathbf{b} \rangle &= \sum R_j k_j + \langle d, \mathbf{b} \rangle + C, \quad \langle \text{Cm}(\Delta(\tilde{k})), \mathbf{b} \rangle = \sum R_j k_j + \sum R_j + C. \end{aligned} \quad (3.31)$$

These formulas allow us to state the following proposition that gives an interpretation of the sum $\sum_j R_j$ in terms of the variation of $\text{Cm}(\Delta(k))$ with the k_j .

Proposition 3.7. *Let $(\Delta, \mathbf{b})$ be a mass linear pair as in (1.5). Then,*

$$\sum_j R_j \langle d, \mathbf{n}_j \rangle = \langle d, \mathbf{b} \rangle = \sum_j R_j, \quad (3.32)$$

d being the element of $\mathfrak{t}^*$ defined by (3.29).

4. Examples

In this section, we will deduce the particular form which adopts Theorem 1.2 when Δ is a polytope of the types (i)–(iii) mentioned in the introduction. For each case, we will determine the center of mass of the corresponding polytope $\Delta(k)$ and the condition for $(\Delta, \mathbf{b})$ to be a mass linear pair. We will dedicate a subsection to each type.

4.1. Hirzebruch Surfaces

Given $r \in \mathbb{Z}_{>0}$ and $\tau, \lambda \in \mathbb{R}_{>0}$ with $\sigma := \tau - r\lambda > 0$, in [7], we considered the Hirzebruch surface N determined by these numbers. N is the quotient

$$\frac{\left\{ z \in \mathbb{C}^4 : |z_1|^2 + r|z_2|^2 + |z_4|^2 = \tau/\pi, |z_2|^2 + |z_3|^2 = \lambda/\pi \right\}}{\mathbb{T}}, \quad (4.1)$$

where the equivalence defined by $\mathbb{T} = (S^1)^2$ is given by

$$(a, b) \cdot (z_1, z_2, z_3, z_4) = (az_1, a^r b z_2, b z_3, a z_4), \quad (4.2)$$

for $(a, b) \in (S^1)^2$.

The manifold N equipped with the following $(U(1))^2$ action

$$(e_1, e_2) [z_j] = [e_1 z_1, e_2 z_2, z_3, z_4], \quad (4.3)$$

is a toric manifold. The corresponding moment polytope Δ is the trapezium in $\mathbb{R}^2$ with vertices

$$P_1 = (0, 0), \quad P_2 = (0, \lambda), \quad P_3 = (\tau, 0), \quad P_4 = (\sigma, \lambda). \quad (4.4)$$

That is, N is the toric manifold M_Δ determined by the trapezium Δ .

As the conormals to the facets of Δ are the vectors $\mathbf{n}_1 = (-1, 0)$, $\mathbf{n}_2 = (0, -1)$, $\mathbf{n}_3 = (0, 1)$, and $\mathbf{n}_4 = (1, r)$, the facets of a generic polytope $\Delta(k)$ in $\mathcal{C}_\Delta$ are on the straights

$$-x = k_1, \quad -y = k_2, \quad y = k_3, \quad x + ry = k_4. \quad (4.5)$$

The vertices of $\Delta(k)$ are the points

$$(-k_1, -k_2), \quad (-k_1, k_3), \quad (k_4 - rk_3, k_3), \quad (k_4 + rk_2, -k_2). \quad (4.6)$$

Thus, the translation in the plane x, y defined by $(-k_1, -k_2)$ transforms the trapezium determined by the vertices (4.4) in $\Delta(k)$ if

$$\tau = k_4 + rk_2 + k_1, \quad \lambda = k_3 + k_2. \quad (4.7)$$

So,

$$\text{Cm}(\Delta(k)) = \text{Cm}(\Delta) + (-k_1, -k_2). \quad (4.8)$$

Moreover, the mass center of Δ is

$$\text{Cm}(\Delta) = \left(\frac{3\tau^2 - 3r\tau\lambda + r^2\lambda^2}{3(2\tau - r\lambda)}, \frac{3\lambda\tau - 2r\lambda^2}{3(2\tau - r\lambda)} \right). \quad (4.9)$$

The chamber $\mathcal{C}_\Delta$ consists of the points $(k_1, \dots, k_4)$ such that $\tau - r\lambda > 0$, with τ and λ given by (4.7). So, the point $k = 0$ belongs to the closure of $\mathcal{C}_\Delta$. From (4.8), together with (4.7) and (4.9), it follows

$$\text{Cm}(\Delta(\check{\epsilon})) = \left(\frac{r^2\epsilon}{12}, \frac{-r\epsilon}{6} \right), \quad (4.10)$$

where $\check{\epsilon} = (\epsilon, \epsilon, \epsilon, \epsilon)$. By Proposition 3.6, if $I(k; \mathbf{b})$ with $\mathbf{b} = (b_1, b_2) \in \mathbb{Z}^2$ vanishes on the chamber $\mathcal{C}_\Delta$, then $rb_1 - 2b_2 = 0$.

On the other hand, from (4.9) and (4.8), it follows

$$\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \frac{(3\tau^2 - 3r\tau\lambda + r^2\lambda^2)b_1 + (3\lambda\tau - 2r\lambda^2)b_2}{3(2\tau - r\lambda)} - k_1b_1 - k_2b_2. \quad (4.11)$$

By (4.7), expression (4.11) is linear in the k_i if and only if

$$\frac{(3\tau^2 - 3r\tau\lambda + r^2\lambda^2)b_1 + (3\lambda\tau - 2r\lambda^2)b_2}{3(2\tau - r\lambda)} \quad (4.12)$$

is linear in τ, λ . That is, if and only if there exist constants A, B such that for all τ, λ

$$(3\tau^2 - 3r\tau\lambda + r^2\lambda^2)b_1 + (3\lambda\tau - 2r\lambda^2)b_2 = 3(2\tau - r\lambda)(A\tau + B\lambda). \quad (4.13)$$

From this relation, it follows the above condition $rb_1 = 2b_2$. In this case (4.11) reduces to

$$\langle \text{Cm}(\Delta(k), \mathbf{b}) \rangle = \frac{-b_1}{2}k_1 + \frac{b_1}{2}k_4. \quad (4.14)$$

Comparing (1.5) with (4.14), we obtain $R_1 = -R_4 = -b_1/2$, $R_2 = R_3 = 0$; so, $\sum_j R_j = 0$. That is, the condition $\sum_j R_j = 0$ holds for all the mass pairs $(\Delta, \mathbf{b})$ when Δ is the polytope associated to a Hirzebruch surface. Hence, we have following proposition.

Proposition 4.1. *$(\Delta, \mathbf{b})$ is a mass linear pair if and only if $rb_1 = 2b_2$. Moreover, in this case $\sum_j R_j = 0$.*

By Theorem 1.2, we have the following corollary.

Corollary 4.2. *If $rb_1 \neq 2b_2$, then $\psi_{\mathbf{b}}$ generates an infinite cyclic subgroup in $\pi_1(\text{Ham}(M_{\Delta}, \omega_{\Delta}))$.*

Remark 4.3. We denote by ϕ_t the following isotopy of M_{Δ}

$$\phi_t[z] = \left[e^{2\pi it} z_1, z_2, z_3, z_4 \right], \quad (4.15)$$

where ϕ is a loop in the Hamiltonian group of M_{Δ} . By ϕ' , we denote the Hamiltonian loop

$$\phi'_t[z] = \left[z_1, e^{2\pi it} z_2, z_3, z_4 \right]. \quad (4.16)$$

In Theorem 8 of [11], we proved that $I(\phi') = (-2/r)I(\phi)$. If $\mathbf{b} = (b_1, b_2) \in \mathbb{Z}^2$, then

$$I(\psi_{\mathbf{b}}) = b_1 I(\phi) + b_2 I(\phi') = \left(b_1 - \left(\frac{2}{r} \right) b_2 \right) I(\phi). \quad (4.17)$$

That is, $I(\psi_{\mathbf{b}}) = 0$ if and only if $rb_1 = 2b_2$, which is in agreement with Proposition 4.1 and Theorem 1.1.

4.2. Δ_p Bundle over Δ_1

Given the integer $p > 1$, as McDuff and Tolman in [3], we consider the following vectors in $\mathbb{R}^{p+1}$:

$$\mathbf{n}_i = -e_i, \quad i = 1, \dots, p, \quad \mathbf{n}_{p+1} = \sum_{i=1}^p e_i, \quad \mathbf{n}_{p+2} = -e_{p+1}, \quad \mathbf{n}_{p+3} = e_{p+1} - \sum_{i=1}^p a_i e_i, \quad (4.18)$$

where $e_1, \dots, e_{p+1}$ is the standard basis of $\mathbb{R}^{p+1}$ and $a_i \in \mathbb{Z}$. We write

$$\mathbf{a} := (a_1, \dots, a_p) \in \mathbb{Z}^p, \quad A := \sum_{i=1}^p a_i, \quad \mathbf{a} \cdot \mathbf{a} = \sum_{i=1}^p a_i^2. \quad (4.19)$$

Figure 1: Δ_2 bundle over Δ_1 .

Let λ, τ be real positive numbers with $\lambda + a_i > 0$, for $i = 1, \dots, p$. In this subsection, we will consider the polytope Δ in $(\mathbb{R}^{p+1})^*$ defined by the above conormals $\mathbf{n}_j$ and the following k_j :

$$k_1 = \dots = k_p = k_{p+2} = 0, \quad k_{p+1} = \tau, \quad k_{p+3} = \lambda. \quad (4.20)$$

This polytope will be also denoted by $\Delta_0(\lambda, \tau)$. It is a Δ_p bundle on Δ_1 (see [3]). When $p = 2$, $\Delta = \Delta_0(\lambda, \tau)$ is the prism whose base is the triangle of vertices $(0, 0, 0)$, $(\tau, 0, 0)$, and $(0, \tau, 0)$ and whose ceiling is the triangle determined by $(0, 0, \lambda)$, $(\tau, 0, \lambda + a_1\tau)$, and $(0, \tau, \lambda + a_2\tau)$ (see Figure 1).

We assume that the above polytope Δ is a Delzant polytope. The manifold (2.2) is in this case

$$M_\Delta = \frac{\left\{ z \in \mathbb{C}^{p+3} : \sum_{i=1}^{p+1} |z_i|^2 = \tau/\pi, \quad -\sum_{j=1}^p a_j |z_j|^2 + |z_{p+2}|^2 + |z_{p+3}|^2 = \lambda/\pi \right\}}{\simeq}, \quad (4.21)$$

where $(z_j) \simeq (z'_j)$ if and only if there are $\alpha, \beta \in U(1)$ such that

$$z'_j = \alpha \beta^{-a_j} z_j, \quad j = 1, \dots, p, \quad z'_{p+1} = \alpha z_{p+1}, \quad z'_k = \beta z_k, \quad k = p+2, p+3. \quad (4.22)$$

Thus, M_Δ is the total space of the fibre bundle $\mathbb{P}(L_1 \oplus \dots \oplus L_p \oplus \mathbb{C}) \rightarrow \mathbb{C}P^1$, where L_j is the holomorphic line bundle over $\mathbb{C}P^1$ with Chern number a_j .

The symplectic form (2.4) is

$$\omega_\Delta = \frac{1}{2}(\sigma_1 + \dots + \sigma_p + \sigma_{p+2}), \quad (4.23)$$

where $\sigma_k = d\rho_k^2 \wedge d\varphi_k$.

And the moment map

$$\mu_\Delta([z]) = (x_1, \dots, x_p, x_{p+2}), \quad (4.24)$$

where $x_i := \pi \rho_i^2$.

Proposition 4.4. *The coordinates $\bar{x}_j$ of $\text{Cm}(\Delta_0(\lambda, \tau))$ are given by*

$$\bar{x}_k = \frac{\tau}{p+2} \frac{\lambda(p+2) + \tau(A + a_k)}{\lambda(p+1) + \tau A}, \quad \text{for } k = 1, \dots, p, \quad (4.25)$$

$$\bar{x}_{p+2} = \frac{1}{2} \frac{(p+1)(p+2)\lambda^2 + 2(p+2)A\lambda\tau + (\mathbf{a} \cdot \mathbf{a} + A^2)\tau^2}{(p+2)((p+1)\lambda + A\tau)}. \quad (4.26)$$

Proof. Since the points $[z] \in M_\Delta$ satisfy $|z_{p+2}|^2 \leq \lambda/\pi + \sum_{j=1}^p a_j |z_j|^2$, by (2.9) and Lemma 2.2, we have

$$\int_{M_\Delta} (\omega_\Delta)^{p+1} = (p+1)! \int_{S_p(\tau)} \left(\lambda + \sum_{j=1}^p a_j x_j \right) = (p+1)! \left(\frac{\lambda \tau^p}{p!} + \frac{\tau^{p+1} A}{(p+1)!} \right). \quad (4.27)$$

Similarly, for $k = 1, \dots, p$

$$\int_{M_\Delta} x_k (\omega_\Delta)^{p+1} = (p+1)! \left(\frac{\lambda \tau^{p+1}}{(p+1)!} + \frac{\tau^{p+2}}{(p+2)!} \sum_{j \neq k} a_j + \frac{2\tau^{p+2} a_k}{(p+2)!} \right). \quad (4.28)$$

The k th coordinate of $\text{Cm}(\Delta)$, $\bar{x}_k$, is the quotient of (4.28) by (4.27); that is,

$$\bar{x}_k = \frac{\tau}{p+2} \frac{\lambda(p+2) + \tau(A + a_k)}{\lambda(p+1) + \tau A}. \quad (4.29)$$

For the $p+2$ -coordinate of $\text{Cm}(\Delta)$, we need to calculate $\int_M x_{p+2} (\omega_\Delta)^{p+1}$. By Lemma 2.2,

$$\begin{aligned} \frac{1}{(p+1)!} \int_M x_{p+2} (\omega_\Delta)^{p+1} &= \frac{1}{2} \int_{S_p(\tau)} \left(\lambda + \sum_{j=1}^p a_j x_j \right)^2 \\ &= \frac{1}{2} \left(\frac{\lambda^2 \tau^p}{p!} + \frac{2A\lambda \tau^{p+1}}{(p+1)!} + \frac{(\mathbf{a} \cdot \mathbf{a} + A^2) \tau^{p+2}}{(p+2)!} \right). \end{aligned} \quad (4.30)$$

Formula (4.26) is a consequence of (4.27) together with (4.30). $\square$

The translation in $(\mathbb{R}^{p+1})^*$ defined by the vector $(-k_1, \dots, -k_p, -k_{p+2})$ transforms the hyperplanes $\langle x, \mathbf{n}_{p+3} \rangle = \lambda$ and $\langle x, \mathbf{n}_{p+1} \rangle = \tau$ in

$$\langle x, \mathbf{n}_{p+3} \rangle = \lambda - k_{p+2} + \sum_{j=1}^p a_j k_j, \quad \langle x, \mathbf{n}_{p+1} \rangle = \tau - \sum_{j=1}^p k_j, \quad (4.31)$$

respectively.

Let $\Delta(k)$ be a polytope with $k = (k_1, \dots, k_{p+3})$ generic in the chamber $\mathcal{C}_\Delta$. From (4.31), it follows that $\Delta(k)$ is the image of the polytope $\Delta_0(\lambda, \tau)$ by the translation determined by $(-k_1, \dots, -k_p, -k_{p+2})$, whenever

$$k_{p+2} - \sum_{j=1}^p a_j k_j + k_{p+3} = \lambda, \quad \sum_{j=1}^p k_j + k_{p+1} = \tau. \quad (4.32)$$

In this case,

$$\text{Cm}(\Delta(k)) = \text{Cm}(\Delta_0(\lambda, \tau)) - (k_1, \dots, k_p, k_{p+2}). \quad (4.33)$$

According to (4.32), the coordinates of the mass center $\text{Cm}(\Delta(\check{\epsilon}))$, with $\check{\epsilon} = (\epsilon, \dots, \epsilon)$, can be obtained substituting in (4.25) and in (4.26) λ by

$$\epsilon - \sum_{j=1}^p a_j \epsilon + \epsilon = (2 - A)\epsilon \quad (4.34)$$

and τ by $(p+1)\epsilon$, and finally take into account (4.33). These operations give

$$\begin{aligned} \bar{x}_j(\Delta(\check{\epsilon})) &= \frac{\epsilon}{2(p+2)}((p+1)a_j - A), \quad j = 1, \dots, p, \\ \bar{x}_{p+2}(\Delta(\check{\epsilon})) &= \frac{\epsilon}{4(p+2)}(-A^2 + (p+1)(\mathbf{a} \cdot \mathbf{a})). \end{aligned} \quad (4.35)$$

Given $\mathbf{b} = (b_1, \dots, b_p, b) \equiv (\hat{\mathbf{b}}, \dot{\mathbf{b}})$, with $\hat{\mathbf{b}} = (b_1, \dots, b_p, 0)$ and $\dot{\mathbf{b}} = (0, \dots, 0, b)$,

$$\left\langle \frac{d}{d\epsilon} \Big|_{\epsilon=0} \text{Cm}(\Delta(\check{\epsilon})), \mathbf{b} \right\rangle = \frac{1}{4(p+2)} \left((p+1)(2\mathbf{a} \cdot \hat{\mathbf{b}} - \mathbf{b} \mathbf{a} \cdot \mathbf{a}) - A(2B + bA) \right), \quad (4.36)$$

where $\mathbf{a} \cdot \hat{\mathbf{b}} = \sum_{j=1}^p a_j b_j$ and $B = \sum_{j=1}^p b_j$.

By Proposition 3.6, we have the following theorem.

Theorem 4.5. *Let Δ be the Δ_p bundle over Δ_1 defined by (4.18) and (4.20). Given $\mathbf{b} = (\hat{\mathbf{b}}, \dot{\mathbf{b}}) \in \mathbb{Z}^{p+1}$, if*

$$(p+1)(2\mathbf{a} \cdot \hat{\mathbf{b}} - \mathbf{b} \mathbf{a} \cdot \mathbf{a}) - A(2B + bA) \neq 0, \quad (4.37)$$

then $\varphi_{\mathbf{b}}$ defines an infinite cyclic subgroup in the fundamental group $\pi_1(\text{Ham}(M_\Delta, \omega_\Delta))$.

It is straightforward to check that

$$(p+1)(2\mathbf{a} \cdot \widehat{\mathbf{b}} - b\mathbf{a} \cdot \mathbf{a}) - A(2B + bA) = 0 \quad (4.38)$$

is also a sufficient condition for $(\Delta, \mathbf{b})$ to be a mass linear pair.

Since

$$\langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle = \langle \text{Cm}(\Delta_0(\lambda, \tau)), \widehat{\mathbf{b}} \rangle + \langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle, \quad (4.39)$$

if (4.38) holds, using (4.25) and (4.26), one obtains

$$\langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle = \frac{b\lambda}{2} + \left(\frac{b}{2} \frac{(\mathbf{a} \cdot \mathbf{a} + A^2)}{(p+2)A} + \frac{(\mathbf{a} \cdot \widehat{\mathbf{b}} + AB)}{(p+2)A} \right) \tau. \quad (4.40)$$

By (4.33), for $k \in \mathcal{C}_\Delta$,

$$\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle - \sum_{j=1}^p b_j k_j - b k_{p+2}, \quad (4.41)$$

with λ and τ given by (4.32).

If $\mathbf{b} = \widehat{\mathbf{b}}$, the condition (4.38) reduces to $(p+1)\mathbf{a} \cdot \widehat{\mathbf{b}} = AB$ and

$$\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \frac{(\mathbf{a} \cdot \widehat{\mathbf{b}} + AB)}{(p+2)A} \left(\sum_{j=1}^p k_j + k_{p+1} \right) - \sum_{j=1}^p b_j k_j. \quad (4.42)$$

Hence, $\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \sum R_j k_j$, where

$$R_j = \frac{(\mathbf{a} \cdot \widehat{\mathbf{b}} + AB)}{(p+2)A} - b_j, \quad j = 1, \dots, p, \quad R_{p+1} = \frac{(\mathbf{a} \cdot \widehat{\mathbf{b}} + AB)}{(p+2)A}, \quad R_{p+2} = R_{p+3} = 0. \quad (4.43)$$

So,

$$\sum_{j=1}^{p+3} R_j = \frac{(p+1)\mathbf{a} \cdot \widehat{\mathbf{b}} - AB}{(p+2)A} = 0. \quad (4.44)$$

A similar calculation for the case $\mathbf{b} = \mathbf{b}$ shows that the corresponding $\sum_j R_j$ vanishes. That is, we have the following proposition.

Proposition 4.6. *Let Δ be a Δ_p bundle over Δ_1 . If $(\Delta, \mathbf{b})$ is a mass linear pair, then $\sum_j R_j = 0$.*

For $p = 2$, let $\mathbf{b}$ be the following linear combination of the conormal vectors $\mathbf{b} = \gamma_1 \mathbf{n}_1 + \gamma_2 \mathbf{n}_2 + \gamma_3 \mathbf{n}_3$ with $\gamma_1 + \gamma_2 + \gamma_3 = 0$. By (4.18), $\mathbf{b} = (b_1, b_2, 0)$ with $b_1 = \gamma_3 - \gamma_1$, $b_2 = \gamma_3 - \gamma_2$. In this case, condition (4.38) reduces to

$$3(a_1 b_1 + a_2 b_2) = (a_1 + a_2)(b_1 + b_2). \quad (4.45)$$

Or in terms of the γ_i

$$a_1 \gamma_1 + a_2 \gamma_2 = 0. \quad (4.46)$$

This is a necessary and sufficient condition for $(\Delta, \mathbf{b})$ to be mass linear. This result is the statement of Lemma 4.8 in [3].

4.3. One Point Blowup of $\mathbb{C}P^n$

In this subsection, $\Delta \equiv \Delta_0(\lambda, \tau)$ will be

$$\Delta = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \leq \tau, \ 0 \leq x_i, \ x_n \leq \lambda \right\}, \quad (4.47)$$

where $\tau, \lambda \in \mathbb{R}_{>0}$ and $\sigma := \tau - \lambda > 0$. That is, Δ is the polytope obtained truncating the simplex $S_n(\tau)$, defined in Lemma 2.2, by a “horizontal” hyperplane through the point $(0, \dots, 0, \lambda)$. The manifold M_Δ associated with Δ is the one point blowup of $\mathbb{C}P^n$.

The mass center of the simplex $S_n(\tau)$ is the point

$$\text{Cm}(S_n(\tau)) = \frac{\tau}{n+1} w, \quad (4.48)$$

with $w = (1, \dots, 1)$.

As the volume of $S_n(\tau)$ is $\tau^n/n!$, it follows from (4.48)

$$(\tau^n - \sigma^n) \text{Cm}(\Delta) = \tau^n \frac{\tau}{n+1} w - \sigma^n \left(\frac{\sigma}{n+1} w + \lambda e_n \right). \quad (4.49)$$

That is,

$$\text{Cm}(\Delta) = \frac{1}{\tau^n - \sigma^n} \left(\left(\frac{\tau^{n+1} - \sigma^{n+1}}{n+1} \right) w - \lambda \sigma^n e_n \right). \quad (4.50)$$

Given $k = (k_1, \dots, k_{n+2}) \in \mathcal{C}_\Delta$, the facets of $\Delta(k)$ are in the following hyperplanes:

$$-x_j = k_j, \quad j = 1, \dots, n; \quad \sum_{i=1}^p x_k = k_{n+1}; \quad x_{n+1} = k_{n+2}. \quad (4.51)$$

As in the preceding subsections,

$$\Delta(k) = \Delta_0(\lambda, \tau) - (k_1, \dots, k_n), \quad (4.52)$$

provided that $\lambda = k_n + k_{n+2}$ and $\tau = \sum_{i=1}^{n+1} k_i$.

The pair $(\Delta, \mathbf{b} = (b_1, \dots, b_n))$ is mass linear if and only if there exist $A, B, C \in \mathbb{R}$ such that

$$\sum_{j=1}^{n-1} b_j \frac{\tau^{n+1} - \sigma^{n+1}}{n+1} + b_n \left(\frac{\tau^{n+1} - \sigma^{n+1}}{n+1} - (\tau - \sigma)\sigma^n \right) = (A\tau + B\sigma + C)(\tau^n - \sigma^n), \quad (4.53)$$

for all τ, σ “admissible”. A simple calculation proves the following proposition.

Proposition 4.7. *The pair $(\Delta, \mathbf{b})$ is mass linear if and only if*

$$b_n = \frac{1}{n} \sum_{j=1}^{n-1} b_j. \quad (4.54)$$

From Theorem 1.2 together with Proposition 4.7, it follows the following theorem.

Theorem 4.8. *If $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n$ and $\sum_{j=1}^{n-1} b_j \neq nb_n$, then $\varphi_{\mathbf{b}}$ generates an infinite cyclic subgroup in $\pi_1(\text{Ham}(M_{\Delta}, \omega_{\Delta}))$.*

For $k \in \mathcal{C}_{\Delta}$, by (4.52)

$$\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle - \sum_{j=1}^n b_j k_j. \quad (4.55)$$

If $(\Delta, \mathbf{b})$ is a mass linear pair, by (4.50) and Proposition 4.7, we have $\langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle = b_n \tau$. Thus,

$$\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \sum_{j=1}^{n+1} R_j k_j, \quad (4.56)$$

where $R_j = b_n - b_j$, for $j = 1, \dots, n$ and $R_{n+1} = b_n$. Hence, we have the following proposition.

Proposition 4.9. *Let Δ be the polytope obtained by truncating the standard n -simplex $S_n(\tau)$ by a horizontal hyperplane. If $(\Delta, \mathbf{b})$ is a mass linear pair, then $\sum_j R_j = 0$.*

Remark 4.10. When $n = 3$, the toric manifold M_{Δ} is

$$M_{\Delta} = \frac{\left\{ z \in \mathbb{C}^5 : |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_5|^2 = \tau/\pi, \ |z_3|^2 + |z_4|^2 = \lambda/\pi \right\}}{\mathbb{T}}, \quad (4.57)$$

where the action of $\mathbb{T} = (\mathrm{U}(1))^2$ is defined by

$$(a, b)(z_1, z_2, z_3, z_4, z_5) = (az_1, az_2, abz_3, bz_4, az_5), \quad (4.58)$$

for $a, b \in \mathrm{U}(1)$.

We consider the following loops in the Hamiltonian group of $(M_\Delta, \omega_\Delta)$:

$$\begin{aligned} \psi_t[z] &= [z_1 e^{2\pi i t}, z_2, z_3, z_4, z_5], & \psi'_t[z] &= [z_1, z_2 e^{2\pi i t}, z_3, z_4, z_5], \\ \tilde{\psi}_t[z] &= [z_1, z_2, z_3 e^{2\pi i t}, z_4, z_5]. \end{aligned} \quad (4.59)$$

In [9] (Remark 4.10), we gave formulas that relate the characteristic numbers associated with these loops

$$I(\psi) = I(\psi') = \left(-\frac{1}{3}\right) I(\tilde{\psi}). \quad (4.60)$$

So, for $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{Z}^3$,

$$I(\psi_{\mathbf{b}}) = (b_1 + b_2 - 3b_3) I(\psi). \quad (4.61)$$

By Proposition 4.7, the vanishing of $I(\psi_{\mathbf{b}})$ in (4.61) is equivalent to the fact that $(\Delta, \mathbf{b})$ is a mass linear pair. This equivalence is a new checking of Theorem 1.1.

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