Research Article

Bayesian Prediction of the Overhaul Effect on a Repairable System with Bounded Failure Intensity

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This paper deals with the Bayes prediction of the future failures of a deteriorating repairable mechanical system subject to minimal repairs and periodic overhauls. To model the effect of overhauls on the reliability of the system a proportional age reduction model is assumed and the 2-parameter Engelhardt-Bain process (2-EBP) is used to model the failure process between two successive overhauls. 2-EBP has an advantage over Power Law Process (PLP) models. It is found that the failure intensity of deteriorating repairable systems attains a finite bound when repeated minimal repair actions are combined with some overhauls. If such a data is analyzed through models with unbounded increasing failure intensity, such as the PLP, then pessimistic estimates of the system reliability will arise and incorrect preventive maintenance policy may be defined. On the basis of the observed data and of a number of suitable prior densities reflecting varied degrees of belief on the failure/repair process and effectiveness of overhauls, the prediction of the future failure times and the number of failures in a future time interval is found. Finally, a numerical application is used to illustrate the advantages from overhauls and sensitivity analysis of the improvement parameter carried out.

1. Introduction

A repairable system is a system that, after failing to perform one or more of its functions satisfactorily, can be restored to satisfactory performance.

Most repairable mechanical systems are subjected to degradation phenomena with operating time, so that the failures become increasingly frequent with time. Such systems often undergo a maintenance policy. Maintenance extends system's lifetime or at least the mean time to failure, and an effective maintenance policy can reduce the frequency of failures and the undesirable consequences of such failures. Maintenance can be categorized into two classes: corrective and preventive actions. Corrective maintenance, called repair, is all actions performed to restore the system to functioning condition when it fails. Preventive maintenance is all actions performed to prevent failures when the system is operating. Corrective and preventive maintenance actions are generally classified in terms of their effect on the operating conditions of the system. Pham and Wang [1] classified them as perfect maintenance, minimal maintenance, imper-

fect maintenance, and worse maintenance. At one extreme is the assumption of perfect maintenance, that a system is restored to good-as-new condition after maintenance. At the other extreme is the bad-as-old assumption that the failure rate of a system is not enhanced by maintenance. In real-world situations, maintenance generally enhances the condition of the equipment at a level between these two extremes, that is, imperfect maintenance.

We consider a system that deteriorates with age and receives two kinds of maintenance actions: minimal repair and overhaul. When a failure occurs, minimal repair is carried out. The minimal repair is a corrective maintenance action that brings the repaired equipment to the conditions it was just before the failure occurrence (bad-as-old). Hence, the reliability of the system decreases with operating time until it reaches unacceptable values. When it reaches unacceptable values or at prefixed epochs, preventive maintenance action (overhaul) is performed so as to improve the system condition and hence reduce the probability of failure occurrence in the following interval. However, overhaul cannot return the system to "good-as-new", and

thus it can be treated as imperfect repair. When the overhaul is effective, the reliability of the system improves significantly. An overhaul usually consists of a set of preventive maintenance actions such as oil change, cleaning, greasing, and replacing some worn components of the system.

Many imperfect repair models have already been proposed [1]. Malik [2] proposed a general approach to model the improvement effect of maintenance, where each maintenance reduces the age of the unit in the view of the rate of occurrences of failures. Each maintenance is assumed to reduce proportionally the operating time elapsed from the previous maintenance. Malik's proportional age reduction (PAR) model for imperfect maintenance is a generalization of good-as-new and bad-as-old. On the basis of this general model, Shin et al. [3] have proposed a PAR model which assumes that each major overhaul reduces proportionally the age of the equipment by a fraction of the epoch of the overhaul. Shin's paper proposes a method of data analysis to estimate the parameters of the failure process and the maintenance effect for a repairable unit. Two parametric families of intensity functions are considered, power law [4–11] and log linear [12]. To model different effects of rejuvenation of the preventive maintenance, two classes of models have been proposed by Jack [13].

In contrast to the classical approach used by them, Bayes approach has been used by several authors as it helps in incorporating prior information and/or technical knowledge on the failure mechanism and on the overhaul effectiveness into the inferential procedure. Pulcini [14] deals with the statistical analysis, from a Bayes viewpoint, of the failure data of repairable mechanical units subjected to minimal repairs and periodic overhauls. The effect of overhauls on the reliability of the system is modeled by PAR model and the power law process (PLP) is used to model the failure process between two successive overhauls. Pulcini provided Bayes point and interval estimation of model parameters (and functions thereof), as well as testing procedures on the effectiveness of the performed overhauls. Further, Pulcini [15] deals with the prediction, from a Bayes viewpoint, of future failures of repairable mechanical units subjected to minimal repairs and periodic overhauls. In [15], also PLP is used to model the failure process between two successive overhaul epochs, and the effect of overhauls on the reliability of the system is modeled by proportional age reduction model.

In PLP models, the increasing failure intensity tends to infinity as the system age increases. However, it is noted that the failure intensity of deteriorating repairable systems attains a finite bound when beginning from a given system age, repeated minimal repair actions are combined with some overhauls performed in order to oppose the growth of failure intensity with the operating time. The average behavior of the intensity function due to the consecutive steps with increasing intensity between two subsequent overhauls results in globally constant asymptotic intensity. If such data is analyzed through models with unbounded increasing failure intensity, such as the PLP, then pessimistic estimates of the system reliability will arise, and incorrect preventive maintenance policy may be defined.

Engelhardt and Bain [16] were the first to suggest NHPP with an increasing bounded intensity, called Engelhardt-Bain process (EBP). But the mathematical simplicity of their 1-parameter model makes it inadequate to analyze failure data in many cases. Most recently, Pulcini [17] proposed a 2-parameter NHPP called the bounded intensity process (BIP) to fit the failure data of deteriorating repairable systems showing bounded intensity function. Attardi and Pulcini [18] proposed a new bounded intensity process called 2-parameter Engelhardt-Bain process (2-EBP) whose failure intensity function is given by

$$\lambda(t) = \eta \left(\frac{t}{t + \theta} \right), \quad \eta, \theta > 0, t \geq 0 \quad (1)$$

The failure intensity in (1) tends to its asymptote more slowly than the BIP intensity and is in a way a compromise between the PLP and BIP models. When $\eta \equiv 1/\theta$, the intensity in (1) degenerates into the EBP intensity.

This paper deals with the Bayes prediction of the future failures of a deteriorating repairable mechanical system subjected to minimal repairs and periodic overhauls. To model the effect of overhauls on the reliability of the system, a proportional age reduction model is assumed, and the 2-parameter Engelhardt-Bain process (2-EBP) is used to model the failure process between two successive overhauls. On the basis of the observed data and of a number of suitable prior densities reflecting varied degrees of belief on the failure/repair process and effectiveness of overhauls, the prediction of the future failure times and the number of failures in a future time interval is found. Finally, a numerical application is used to illustrate the advantages of overhauls.

2. Basic Assumptions

- (i) Failure rate of the system is an increasing function of time that attains a finite bound as t tends to infinity.
- (ii) System is subjected to two kinds of maintenance actions: minimal repair and overhaul.
- (iii) The times to perform maintenance actions are ignored.
- (iv) Minimal repair will restore the failure rate only to bad-as-old condition. But overhauls will improve the system to a condition between bad-as-old and good-as-new.
- (v) The failure density function is not changed by overhauls.
- (vi) The quality of an overhaul is dependent on improvement factor ρ ($0 \leq \rho \leq 1$).
- (vii) The improvement parameter ρ has a uniform value over all the overhaul actions.
- (viii) The j th overhaul reduces the age t of the system by a fraction ρ of the epoch x_j of the overhaul.
- (ix) The effect of overhauls on the reliability of the system is modeled by proportional age reduction model, and the 2-parameter Engelhardt-Bain process (2-EBP)

is used to model the failure process between two successive overhauls, say (x_j, x_{j+1}) .

(x) $x_1, x_2, \dots, x_k$ are the k overhaul epochs, which may coincide with failure times.

3. Model Formulation

Let $t_1 < t_2 < \dots < t_n$ denote the n failure times of the repairable system observed till T . If $T \equiv t_n$, that is, the process is observed till n th failure, then it is failure-truncated sampling, and T is a random variable. If $T > t_n$, then it is time-truncated sampling, where T is a prefixed quantity, and n is a random variable.

The 2-EBP model is an NHPP whose failure intensity is of the form

$$\lambda(t) = \eta \left(\frac{t}{t + \theta} \right), \quad \eta, \theta > 0, t \geq 0, \quad (2)$$

which is an increasing bounded intensity function with the operating time t , equal to 0 at $t = 0$ and approaching an asymptote of η as t increases. So, the parameter η is the asymptotic value of the intensity function: $\eta \equiv \lambda_\infty$, and the parameter θ is a measure of the initial increasing rate of $\lambda(t)$: the smaller θ is, the faster the failure intensity increases until it approaches η .

Thus, the initial failure intensity, that is, the intensity function till the first overhaul epoch x_1 , is

$$\lambda_1(t) = \eta \left(\frac{t}{t + \theta} \right), \quad t \leq x_1. \quad (3)$$

The conditional intensity function at a generic time t in the interval (x_j, x_{j+1}) , given x_j is,

$$\begin{aligned} \lambda_{j+1}(t | x_j) &= \lambda_1(t - \rho x_j) \\ &= \eta \left(\frac{t - \rho x_j}{t - \rho x_j + \theta} \right); \quad x_j < t \leq x_{j+1}. \end{aligned} \quad (4)$$

The expected number of failures between two successive overhaul epochs is

$$\begin{aligned} E\{N(x_j, x_{j+1})\} &= \int_{x_j}^{x_{j+1}} \lambda_{j+1}(t | x_j) dt \\ &= \eta \left[(x_{j+1} - x_j) - \theta \log \left(\frac{x_{j+1} - \rho x_j + \theta}{x_j - \rho x_j + \theta} \right) \right]. \end{aligned} \quad (5)$$

The cumulative value of expected number of failures from 0 up to a generic time τ in the interval (x_i, x_{i+1}) is

$$\begin{aligned} N(\tau) &= \left[\sum_{j=0}^{i-1} E\{N(x_j, x_{j+1})\} \right] \\ &+ E\{N(x_i, \tau)\}, \quad x_i \leq \tau \leq x_{i+1}, \quad x_0 = 0 \end{aligned}$$

$$\begin{aligned} &= \eta \left[\sum_{j=0}^{i-1} \left((x_{j+1} - x_j) - \theta \log \left(\frac{x_{j+1} - \rho x_j + \theta}{x_j - \rho x_j + \theta} \right) \right) \right] \\ &+ \eta \left[(\tau - x_i) - \theta \log \left(\frac{\tau - \rho x_i + \theta}{x_i - \rho x_i + \theta} \right) \right]. \end{aligned} \quad (6)$$

4. Likelihood Function

The likelihood function based on observed data: $t_1, t_2, \dots, t_n, T$, is

The conditional probability that the unit of age t_{i-1} will fail before t_i is

$$\begin{aligned} F(t_i | t_{i-1}) &= P[T \leq t_i | T > t_{i-1}] \\ &= \frac{F(t_i) - F(t_{i-1})}{\bar{F}(t_{i-1})} \\ &= \frac{-(1 - F(t_i)) + (1 - F(t_{i-1}))}{\bar{F}(t_{i-1})} \\ &= \frac{\bar{F}(t_{i-1}) - \bar{F}(t_i)}{\bar{F}(t_{i-1})} \\ &= 1 - \frac{\bar{F}(t_i)}{\bar{F}(t_{i-1})}. \end{aligned} \quad (7)$$

The conditional reliability just before the i th failure will be

$$\begin{aligned} \bar{F}(t_i | t_{i-1}) &= 1 - F(t_i | t_{i-1}) \\ &= \frac{\bar{F}(t_i)}{\bar{F}(t_{i-1})} \\ &= e^{-\int_{t_{i-1}}^{t_i} r(u) du}, \end{aligned} \quad (8)$$

and the conditional pdf is $f(t_i | t_{i-1})$.

If the data is time truncated, the joint pdf of the failure times $(t_1, t_2, \dots, t_n)$ is

$$\begin{aligned} f(t_1, t_2, \dots, t_n) &= f(t_1) f(t_2 | t_1) \cdots f(t_n | t_1, t_2, \dots, t_{n-1}) \\ &\times \bar{F}(T | t_n) \\ &= \lambda(t_1) \cdot e^{-\int_0^{t_1} \lambda(u) du} \cdot \lambda(t_2) \\ &\cdot e^{-[\int_0^{t_2} \lambda(u) du - \int_0^{t_1} \lambda(u) du]} \cdots \lambda(t_n) \\ &\cdot e^{-[\int_0^{t_n} \lambda(u) du - \int_0^{t_{n-1}} \lambda(u) du]} \\ &\cdot e^{-\int_{t_n}^T \lambda(u) du} \\ &= \left[\prod_{i=1}^n \lambda(t_i) \right] \cdot e^{-\int_0^T \lambda(u) du}, \end{aligned} \quad (9a)$$

where T is the truncation time.

As $x_1, x_2, \dots, x_k$ are the k overhaul epochs, they provide disjoint fixed intervals $(x_0, x_1), (x_1, x_2), \dots, (x_{k-1}, x_k)$. Since

the overhaul epochs are prescheduled, they are not random. Let the time interval (x_j, x_{j+1}) be the $(j+1)$ th period.

From (4) and (9a), for $t \in (x_j, x_{j+1})$, the joint pdf of failure times in (x_j, x_{j+1}) is

$$= \left[\prod_{\{i: t_i \in (x_j, x_{j+1})\}} \lambda_{j+1}(t_i | x_j) \right] \cdot e^{-\int_{x_j}^{x_{j+1}} \lambda_{j+1}(t | x_j) dt}. \quad (9b)$$

Since the overhaul epochs are prescheduled, the failure times in disjoint epoch periods are independent, and therefore, the joint pdf of the failure times $(t_1, t_2, \dots, t_n)$ is simply the product form of (9b), that is,

$$f(t_1, t_2, \dots, t_n) = L(\text{data} | \eta, \theta, \rho)$$

$$\begin{aligned} &= \left[\prod_{j=0}^k \prod_{t_i \in (x_j, x_{j+1})} \lambda_{j+1}(t_i | x_j) \right] \\ &\quad \cdot e^{-\sum_{j=0}^k \int_{x_j}^{x_{j+1}} \lambda_{j+1}(t | x_j) dt} \\ &= \eta^n \left[\prod_{j=0}^k \prod_{t_i \in (x_j, x_{j+1})} \left(\frac{t_i - \rho x_j}{t_i - \rho x_j + \theta} \right) \right] \\ &\quad \cdot e^{-\eta \sum_{j=0}^k [(x_{j+1} - x_j) - \theta \log((x_{j+1} - \rho x_j + \theta) / (x_j - \rho x_j + \theta))]}, \end{aligned} \quad (10)$$

where $x_{k+1} = T$.

5. Bayesian Procedure

5.1. Informative Prior. Suppose that the analyst is able to anticipate the following:

- (a) the asymptotic value $\lambda_\infty \equiv \eta$ of the intensity function,
- (b) the time t_r at which the failure intensity is r times the asymptotic value ($r < 1$), that is,

$$\begin{aligned} \frac{\lambda(t_r)}{\lambda_\infty} &= r; \quad x_i < t_r < x_{i+1} \\ \Rightarrow t_r &= \left(\frac{(1-r)\rho x_i + r\theta}{1-r} \right). \end{aligned} \quad (11)$$

Now, we assume a Gamma density for η and t_r , both for the mathematical tractability and flexibility. Therefore,

$$\begin{aligned} g(\eta) &= \frac{b^a}{\Gamma(a)} \eta^{a-1} \exp(-b\eta), \quad a, b > 0, \\ g(t_r) &= \frac{d^c}{\Gamma(c)} (t_r)^{c-1} \exp(-dt_r), \quad c, d > 0, \end{aligned} \quad (12)$$

where the gamma parameters are related to the prior mean and standard deviation by

$$\begin{aligned} a &= \left(\frac{\mu_\eta}{\sigma_\eta} \right)^2, & b &= \frac{\mu_\eta}{\sigma_\eta^2}, \\ c &= \left(\frac{\mu_t}{\sigma_t} \right)^2, & d &= \frac{\mu_t}{\sigma_t^2}. \end{aligned} \quad (13)$$

The analyst can use the fact that a ρ value closer to 0 (to 1) indicates less (more) effectiveness of the overhaul actions to formulate an informative prior on ρ . We assume that the analyst is able to anticipate both a prior mean and a standard deviation on ρ , say μ_ρ and σ_ρ , then we formalize this information through a beta density

$$g(\rho) = \frac{\rho^{p-1} (1-\rho)^{q-1}}{\beta(p, q)}, \quad 0 \leq \rho \leq 1, \quad (14)$$

where the values of the parameters related to the prior mean and standard deviation are

$$\begin{aligned} p &= \frac{(1-\mu_\rho)\mu_\rho^2}{\sigma_\rho^2} - \mu_\rho, \\ q &= p \frac{(1-\mu_\rho)}{\mu_\rho}. \end{aligned} \quad (15)$$

5.2. Joint Prior Density. Assuming the prior independence of the parameters (η, θ, ρ) , and using (11) to find density of θ , the joint prior density $g(\eta, \theta, \rho)$ is given by

$$\begin{aligned} g(\eta, \theta, \rho) &\propto \eta^{a-1} e^{-b\eta} \left(\frac{(1-r)\rho x_i + r\theta}{1-r} \right)^{c-1} \\ &\quad \times e^{-d(((1-r)\rho x_i + r\theta)/(1-r))} \rho^{p-1} (1-\rho)^{q-1}. \end{aligned} \quad (16)$$

5.3. Joint Posterior Density on Model Parameters. By combining the likelihood (10) with the joint prior density (16), the joint posterior density on the model parameters is given by

$$\begin{aligned} \pi(\eta, \theta, \rho | \text{data}) &= \frac{1}{D} \eta^n \cdot \left[\prod_{j=0}^k \prod_{t_i \in (x_j, x_{j+1})} \left(\frac{t_i - \rho x_j}{t_i - \rho x_j + \theta} \right) \right] \\ &\quad \cdot e^{-\eta \sum_{j=0}^k [(x_{j+1} - x_j) - \theta \log((x_{j+1} - \rho x_j + \theta) / (x_j - \rho x_j + \theta))]} \\ &\quad \cdot \eta^{a-1} \cdot e^{-b\eta} \cdot \left(\frac{(1-r)\rho x_i + r\theta}{1-r} \right)^{c-1} \\ &\quad \cdot e^{-d(((1-r)\rho x_i + r\theta)/(1-r))} \\ &\quad \cdot \rho^{p-1} \cdot (1-\rho)^{q-1}, \end{aligned} \quad (17)$$

where

$$D = \Gamma(n+a) \int_0^\infty \int_0^1 \frac{f(\theta, \rho \mid \text{data})}{\left(b + \sum_{j=0}^k [(x_{j+1} - x_j) - \theta \log((x_{j+1} - \rho x_j + \theta)/(x_j - \rho x_j + \theta))]\right)^{n+a}} d\rho d\theta, \quad (18)$$

$$f(\theta, \rho \mid \text{data}) = \left[\prod_{j=0}^k \prod_{t_i \in (x_j, x_{j+1})} \left(\frac{t_i - \rho x_j}{t_i - \rho x_j + \theta} \right) \right] \cdot \left(\frac{(1-r)\rho x_i + r\theta}{1-r} \right)^{c-1} \cdot e^{-d(((1-r)\rho x_i + r\theta)/(1-r))} \cdot \rho^{p-1} \cdot (1-\rho)^{q-1}.$$

6. Posterior Inference on the Expected Number of Failures in a Future Time Interval

Let M_τ be the expected number of failures in the future time interval (T, τ) ,

$$M(\tau) = E\{N(T, \tau)\} = \eta \left[(\tau - T) - \theta \log \left(\frac{\tau - \rho s + \theta}{T - \rho s + \theta} \right) \right], \quad (19)$$

where

$$s = \begin{cases} T & \text{if overhaul is performed at } T, \\ x_k, & \text{otherwise.} \end{cases} \quad (20)$$

If no overhaul is performed after T , the conditional distribution of M_τ is Poisson with mean $M(\tau)$ equal to the expected number of failures in the time interval (T, τ) ,

$$\begin{aligned} \Pr\{M_\tau = m \mid \eta, \theta, \rho; \text{data}\} &= \frac{[M(\tau)]^m \exp[-M(\tau)]}{m!}, \\ \Pr\{M_\tau = m \mid \text{data}\} &= \int_0^\infty \int_0^1 \int_0^1 \pi(\eta, \theta, \rho) \cdot \frac{[M(\tau)]^m \exp[-M(\tau)]}{m!} d\rho d\theta d\eta \\ &= \frac{1}{D} \cdot \frac{\Gamma(n+a+m)}{m!} \int_0^\infty \int_0^1 f(\theta, \rho \mid \text{data}) \\ &\quad \cdot \frac{[X_\tau^{(\theta)}]^m}{\left(b + \sum_{j=0}^k [(x_{j+1} - x_j) - \theta \log((x_{j+1} - \rho x_j + \theta)/(x_j - \rho x_j + \theta))]\right)^{n+a+m}} d\rho d\theta, \end{aligned} \quad (21)$$

where

$$X_\tau^{(\theta)} = (\tau - T) - \theta \log \left(\frac{\tau - \rho s + \theta}{T - \rho s + \theta} \right). \quad (22)$$

The posterior mean of M_τ is given by the posterior expecta-

tion of $M(\tau)$,

$$E\{M_\tau \mid \text{data}\} = \sum_{m=0}^\infty m \cdot \Pr\{M_\tau = m \mid \text{data}\}. \quad (23)$$

After the transformation of variables, we have the following:

$$E\{M_\tau \mid \text{data}\} = \frac{(n+a)!}{D} \int_0^\infty \int_0^1 f(\theta, \rho \mid \text{data}) \cdot \frac{X_\tau^{(\theta)}}{\left(b + \sum_{j=0}^k [(x_{j+1} - x_j) - \theta \log((x_{j+1} - \rho x_j + \theta)/(x_j - \rho x_j + \theta))]\right)^{n+a+1}} d\rho d\theta, \quad (24)$$

A conservative γ Bayes upper credibility limit for M_τ is obtained by finding the smallest value of which m_u satisfies

$$\Pr\{M_\tau \leq m_u \mid \text{data}\} \geq \gamma. \quad (25)$$

7. Posterior Inference on the Future Failure Times

Given the observed data, we are interested in predicting the m th future failure time t_{n+m} , under the assumption that no

overhaul is performed after T ,

$$M_m = E\{N(T, t_{n+m})\} \\ = \eta \left[(t_{n+m} - T) - \theta \log \left(\frac{t_{n+m} - \rho s + \theta}{T - \rho s + \theta} \right) \right], \quad (26)$$

where

$$s = \begin{cases} T & \text{if overhaul is performed at } T, \\ x_k, & \text{otherwise.} \end{cases} \quad (27)$$

M_m is gamma distributed with shape parameter m and unit scale parameter. Thus, the conditional pdf of t_{n+m} , with

$t_{n+m} > T$, is given by

$$p(t_{n+m} | \eta, \theta, \rho; \text{data}) = \frac{1}{\Gamma m} \eta^m [X_m^{(\theta)}]^{m-1} \exp[-\eta X_m^{(\theta)}] \\ \times \left[\frac{t_{n+m} - \rho s}{t_{n+m} - \rho s + \theta} \right], \quad (28)$$

where

$$X_m^{(\theta)} = (t_{n+m} - T) - \theta \log \left(\frac{t_{n+m} - \rho s + \theta}{T - \rho s + \theta} \right). \quad (29)$$

The predictive density function of t_{n+m} is given by

$$h\{t_{n+m} | \text{data}\} = \int_0^\infty \int_0^\infty \int_0^1 \pi(\eta, \theta, \rho) \cdot p(t_{n+m} | \eta, \theta, \rho; \text{data}) d\rho d\theta d\eta \\ = \frac{1}{D} \cdot \frac{\Gamma(n+a+m)}{\Gamma(m)} \\ \times \int_0^\infty \int_0^1 f(\theta, \rho | \text{data}) \cdot \frac{[X_m^{(\theta)}]^{m-1} \cdot [(t_{n+m} - \rho s)/(t_{n+m} - \rho s + \theta)]}{\left(b + \sum_{j=0}^k [(x_{j+1} - x_j) - \theta \log((x_{j+1} - \rho x_j + \theta)/(x_j - \rho x_j + \theta))] + X_m^{(\theta)} \right)^{n+a+m}} d\rho d\theta. \quad (30)$$

A Bayes prediction interval of a given probability content γ is defined as the interval I such that

$$\Pr\{I | \text{data}\} = \int_I h(t_{n+m} | \text{data}) dt_{n+m} + \gamma. \quad (31)$$

8. Sensitivity Analysis

Sensitivity analysis is carried out with respect to the prior information on ρ , namely the prior mean μ_ρ and the standard deviation σ_ρ , by evaluating M_τ : expected number of failures in the future time interval (T, τ) when overhaul is performed at T and not performed at T , over a reasonable range of values for μ_ρ and σ_ρ . If the slight change in values of μ_ρ and σ_ρ does not have much effect on M_τ then the proposed prior is robust.

9. Numerical Application

Consider the following hypothetical data for illustrative purpose.

The failure times ($n = 18$) and overhaul epochs are given in Table 1, major overhauls marked with*. We have assumed it to be a time-truncated sample with failures observed for 1500 units. Four major overhauls are assumed to be performed at times different from failure times.

The proportional age reduction 2-EBP model is adequate for this hypothetical data set in contrast to pre-existing proportional age reduction power law process model as its log-likelihood value obtained using (10) is -95.4606 which

is greater than the log-likelihood value -96.1537 of power law process model.

Suppose that analyst is able to anticipate a prior mean $\mu_\eta = 0.133$ and $\sigma_\eta = 0.004$ ($a = 4, b = 30$).

In addition, from previous experiences, the analyst possesses a vague belief that the failure intensity, at the time $t_r = 753$ units, is nearly half its asymptotic value: $\lambda(753)/\lambda_\infty = 0.5$. Then, he formalizes his prior knowledge on t_r through the exponential density having mean $\mu_t = 753$, so that $c = 1$ and $d = 0.001$. As $x_i < t_r < x_{i+1}$, therefore $600 < 753 < 900$. Hence, $x_i = 600$.

Again, the analyst possesses a vague belief that the overhaul actions are quite effective, and then he chooses the beta density for the improvement parameter with prior mean $\mu_\rho = 0.6$ and standard deviation $\sigma_\rho = 0.26$ ($p = 1.5$ and $q = 1$).

Now, suppose that the above failure process is time truncated at $T = 1200$ units. On the basis of the observed data, we predict the future failure times under two hypotheses:

- (a) major overhaul is performed at T ,
- (b) no overhaul is performed at T .

All further repair actions are assumed to be minimal repairs. The remaining observed values, that is, $\{1230, 1268, 1330, \text{ and } 1447\}$ have been used for comparison with the predicted values.

In Table 2, we compare the occurred failure times t_{13+m} ($m = 1, 2, 3, 4, 5$) with the Bayes prediction of the m th future failure times under both the above hypotheses; in particular, the lower (t_l) and upper (t_u) limits of the

TABLE 1: Failure times and overhaul epochs.

202	265	300*	363	508	571	600*	755	770	818	868
900*	999	1054	1068	1108	1200*	1230	1268	1330	1376	1447

TABLE 2: Comparison of the occurred failure times t_{13+m} ($m = 1, 2, 3, 4, 5$) with the Bayes prediction under the hypothesis that overhaul is or is not performed at T .

m	Actual	Overhaul at T		No overhaul at T	
	t_{13+m}	t_l	t_u	t_l	t_u
1	1230	1204.3	1421.0	1202.4	1342.0
2	1268	1226.1	1514.0	1216.4	1420.0
3	1330	1255.2	1588.4	1236.6	1486.0
4	1376	1286.5	1653.6	1259.5	1546.8
5	1447	1318.4	1713.5	1283.5	1603.7

TABLE 3: Comparison of occurred failure times with bayes prediction.

n	T	Actual t_{n+1}	90% Lower Limit	Median Value	90% Upper Limit
13	$x_4 = 1200$	1230	1204.3	1258.1	1421.0
14	$t_{14} = 1230$	1268	1233.4	1275.7	1417.0
15	$t_{15} = 1268$	1330	1271.0	1308.0	1435.2
16	$t_{16} = 1330$	1376	1332.7	1366.1	1482.0
17	$t_{17} = 1376$	1447	1378.5	1409.4	1518.1

TABLE 4: Posterior mean and 0.95 upper credibility limit of the number of failures in the future time interval (1200, 1500).

	Posterior Mean	0.95 Upper credibility limit
Overhaul at T	5.327	11
No Overhaul at T	7.612	14

TABLE 5: Sensitivity analysis with respect to the prior mean of ρ .

Percentage Deviation	μ_ρ	M_τ with no overhaul at T	M_τ with overhaul at T
-1%	0.594	7.599	5.359
+1%	0.606	7.618	5.312
-2%	0.588	7.588	5.386
+2%	0.612	7.628	5.287
-3%	0.582	7.578	5.410
+3%	0.618	7.639	5.261

0.90 Bayes equal-tail credibility interval are shown. It clearly shows the advantages of performing overhauls as the future failure times are delayed considerably by overhaul.

In Table 3, we compare the observed values of t_{n+1} ($n = 13, 14, 15, 16, 17$) with the lower and upper limits of the Bayes 0.90 equal-tail credibility interval and with the posterior median obtained using (30) and (31). This allows the analyst to assess the adequacy of the selected model and the correctness of prior information. Table 3 shows that all occurred failures fall well within the Bayes prediction interval and are quite close to the respective posterior median, thus

TABLE 6: Sensitivity analysis with respect to the standard deviation of ρ .

Percentage deviation	σ_ρ	M_τ with no overhaul at T	M_τ with overhaul at T
-1%	0.2574	7.610	5.352
+1%	0.2626	7.614	5.322
-2%	0.2548	7.596	5.367
+2%	0.2652	7.619	5.308
-3%	0.2522	7.591	5.381
+3%	0.2678	7.625	5.292

indicating the adequacy of the 2-EBP to model the given data set.

Table 4 clearly shows that the, 95% upper credibility limit for the number of failures in future time interval (1200, 1500) when major overhaul is performed at $T = 1200$ is equal to 11 failures and the posterior mean = 5.327 failures, very close to the number of failures, namely 5, actually occurred in (1200, 1500).

On the contrary in the case of no overhaul 95% upper credibility limit for the number of failures in future time interval (1200, 1500) is equal to 14 failures which is more than that in major overhaul case, and the posterior mean is 7.612 which is again more than the value obtained in the major overhaul case. These results show the considerable advantages arising from performing a major overhaul at T .

Tables 5 and 6 show that the change in values of μ_ρ and σ_ρ does not have much effect on M_τ , so the proposed prior is robust.

10. Conclusion

In this paper, the prediction of future failures of a deteriorating repairable mechanical system subject to minimal repairs and periodic overhauls has been done using Bayesian approach. The effect of overhauls on the reliability of the system has been modeled using a proportional age reduction model, and the failure process between two successive overhauls has been modeled using 2-parameter Engelhardt-Bain process (2-EBP). The prediction of the future failure times and the number of failures in a future time interval has been done on the basis of the observed data and of a number of suitable prior densities reflecting varied degrees of belief on the failure/repair process and effectiveness of overhauls. The advantages of overhauls have been highlighted using a numerical application and sensitivity analysis of the improvement parameter carried out.

Nomenclature

n :	Total no. of failures
k :	Total no. of overhauls
x_i :	i th overhaul epoch, $i = 1 \dots k$
η, θ :	Parameters of 2-EBP
ρ :	Improvement factor
t_r :	Time at which the failure intensity is r times the asymptotic value
$\lambda_1(t)$:	Initial failure intensity
$\lambda_{j+1}(t x_j)$:	Conditional intensity function at a generic time t in the interval (x_j, x_{j+1})
$E\{N(x_j, x_{j+1})\}$:	Expected number of failures between two successive overhaul epochs
$g(\eta)$:	Prior pdf on η
$g(t_r)$:	Prior pdf on t_r
$g(\rho)$:	Prior pdf on ρ
a, b :	Prior gamma parameters of η
c, d :	Prior gamma parameters of t_r
p, q :	Prior beta parameters of ρ
μ_η, σ_η^2 :	Prior mean and variance of η
$\mu_{t_r}, \sigma_{t_r}^2$:	Prior mean and variance of t_r
μ_ρ, σ_ρ^2 :	Prior mean and variance of ρ
$g(\eta, t_r, \rho)$:	Joint prior pdf of η, t_r, ρ
$\pi(\eta, \theta, \rho \text{data})$:	Joint posterior pdf of η, θ, ρ
M_τ :	Expected number of failures in the future time interval (T, τ)
t_{n+m} :	m th future failure time
NHPP:	Nonhomogeneous poisson process
2-EBP:	2-parameter Engelhardt-Bain process
PLP:	Power law process.

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