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Research Article

k-Tuple Total Domination in Complementary Prisms

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Let k be a positive integer, and let G be a graph with minimum degree at least k. In their study (2010), Henning and Kazemi defined the k-tuple total domination number $\gamma_{\times k,t}(G)$ of G as the minimum cardinality of a k-tuple total dominating set of G, which is a vertex set such that every vertex of G is adjacent to at least k vertices in it. If $\overline{G}$ is the complement of G, the complementary prism $G\overline{G}$ of G is the graph formed from the disjoint union of G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. In this paper, we extend some of the results of Haynes et al. (2009) for the k-tuple total domination number and also obtain some other new results. Also we find the k-tuple total domination number of the complementary prism of a cycle, a path, or a complete multipartite graph.

1. Introduction

In this paper, $G = (V, E)$ is a simple graph with the *vertex set* V and the *edge set* E. The *order* $|V|$ of G is denoted by $n = n(G)$. The *open neighborhood* and the *closed neighborhood* of a vertex $v \in V$ are $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. Also the *degree* of v is $\deg_G(v) = |N_G(v)|$. Similarly, the *open neighborhood* and the *closed neighborhood* of a set $S \subseteq V$ are $N_G(S) = \bigcup_{v\in S} N(v)$ and $N_G[S] = N_G(S) \cup S$, respectively. The *complement* of G is the graph $\overline{G}$ with the vertex set $V(\overline{G}) = V(G)$ and the edge set $E(\overline{G}) = \{uv \mid uv \notin E(G)\}$. The *minimum* and *maximum degree* of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We also write K_n, C_n, and P_n for the *complete graph*, *cycle,* and *path* of order n, respectively, while $G[S]$ and $K_{n_1,n_2,\ldots,n_p}$ denote the *subgraph induced* on G by a vertex set S, and the complete p-partite graph, respectively.

Haynes et al. in [1] have defined complementary product of two graphs that generalizes the Cartesian product of two graphs. Let G and H be two graphs. For each

$R \subseteq V(G)$ and $S \subseteq V(H)$, the *complementary product* $G(R)\square H(S)$ is a graph with the vertex set $\{(u_i, v_j) \mid u_i \in V(G), v_i \in V(H)\}$ and $(u_i, v_j)(u_h, v_k)$ is an edge in $E(G(R)\square H(S))$

(1) if $i = h$, $u_i \in R$, and $v_j v_k \in E(H)$ or if $i = h$, $u_i \notin R$, and $v_j v_k \notin E(H)$, or

(2) if $j = k$, $v_j \in S$, and $u_i u_h \in E(G)$ or if $j = k$, $v_j \notin S$, and $u_i u_h \notin E(G)$.

In other words, for each $u_i \in V(G)$, we replace u_i by a copy of H if u_i is in R and by a copy of its complement $\overline{H}$ if u_i is not in R, and for each $v_j \in V(H)$, we replace each v_j by a copy of G if $v_j \in S$ and by a copy of $\overline{G}$ if $v_j \notin S$. If $R = V(G)$ (resp., $S = V(H)$), we write simply $G\square H(S)$ (resp., $G(R)\square H$). Thus, $G\square H(S)$ is the graph obtained by replacing each vertex v of H by a copy of G if $v \in S$ and by a copy of $\overline{G}$ if $v \notin S$ and replacing each vertex u of G by a copy of H. We recall that the *Cartesian product* $G\square H$ of two graphs G and H is the complementary product $G(V(G))\square H(V(H))$. The special complementary product $G\square K_2(S)$, where $|S| = 1$, is called the *complementary prism* of G and denoted by $G\overline{G}$. For example, the graph $C_5\overline{C_5}$ is the Petersen graph. Also, if $G = K_n$, the graph $K_n\overline{K_n}$ is the corona $K_n \circ K_1$, where the *corona* $G \circ K_1$ of a graph G is the graph obtained from G by attaching a pendant edge to each vertex of G. We notice that $\delta(G\overline{G}) = \min\{\delta(G), \delta(\overline{G})\} + 1$.

In [2], Henning and Kazemi introduced the k-tuple total domination number of graphs. Let k be a positive integer. A subset S of V is a *k-tuple total dominating set* of G, abbreviated kTDS, if for every vertex $v \in V$, $|N(v) \cap S| \geq k$, that is, S is a kTDS of G if every vertex of V has at least k neighbors in S. The *k-tuple total domination number* $\gamma_{\times k,t}(G)$ of G is the minimum cardinality of a kTDS of G. We remark that a 1-tuple total domination is the well-studied *total domination number*. Thus, $\gamma_t(G) = \gamma_{\times 1,t}(G)$. For a graph to have a k-tuple total dominating set, its minimum degree is at least k. Since every $(k+1)$-tuple total dominating set is also a k-tuple total dominating set, we note that $\gamma_{\times k,t}(G) \leq \gamma_{\times (k+1),t}(G)$ for all graphs with minimum degree at least $k+1$. A kTDS of cardinality $\gamma_{\times k,t}(G)$ is called a *$\gamma_{\times k,t}(G)$-set*. When $k = 2$, a 2-tuple total dominating set is called a *double total dominating set*, abbreviated DTDS, and the 2-tuple total domination number is called the *double total domination number*. The redundancy involved in k-tuple total domination makes it useful in many applications. The paper in [3] gives more information about the k-tuple total domination number of a graph.

In [4], Haynes et al. discussed the domination and total domination number of complementary prisms. In this paper, we extend some of their results for the k-tuple total domination number and obtain some other results. More exactly, we find some useful lower and upper bounds for the k-tuple total domination number of the complementary prism $G\overline{G}$ in terms on the order of G, $\gamma_{\times k,t}(G)$, $\gamma_{\times k,t}(\overline{G})$, $\gamma_{\times (k-1),t}(G)$, and $\gamma_{\times (k-1),t}(\overline{G})$, in which some of the bounds are sharp. Also we find this number for $G\overline{G}$, when G is a cycle, a path, or a complete multipartite graph.

Through of this paper, k is a positive integer, and for simplicity, we assume that $V(G\overline{G})$ is the disjoint union $V(G) \cup V(\overline{G})$ with $V(\overline{G}) = \{\overline{v} \mid v \in V(G)\}$ and $E(G\overline{G}) = E(G) \cup E(\overline{G}) \cup \{v\overline{v} \mid v \in V(G)\}$ such that $E(\overline{G}) = \{\overline{u}\,\overline{v} \mid uv \notin E(G)\}$. The vertices v and $\overline{v}$ are called the *corresponding vertices*. Also for a subset $X \subseteq V(G)$, we show its corresponding subset in $\overline{G}$ with $\overline{X}$. The next known results are useful for our investigations.

Proposition A (Haynes et al. [2]). *If G is a path or a cycle of order $n \geq 5$ such that $n \equiv 2 \pmod 4$ or is the corona graph $K_n \circ K_1$, where $n \geq 3$, then $\gamma_t(G\overline{G}) = \gamma_t(G)$.*

Proposition B (Henning and Kazemi [4]). *Let $p \geq 2$ be an integer, and let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph, where $n_1 \leq n_2 \leq \cdots \leq n_p$.*

(i) *If $k < p$, then $\gamma_{\times k,t}(G) = k+1$,*

(ii) *if $k = p$ and $\sum_{i=1}^{k-1} n_i \geq k$, then $\gamma_{\times k,t}(G) = k+2$,*

(iii) *if $2 \leq p < k$ and $\lceil k/(p-1)\rceil \leq n_1 \leq n_2 \leq \cdots \leq n_p$, then $\gamma_{\times k,t}(G) = \lceil kp/(p-1)\rceil$.*

Proposition C (Henning and Kazemi [5]). *Let G be a graph of order n with $\delta(G) \geq k$. Then*

$$\gamma_{\times k,t}(G) \geq \max\left\{k+1, \left\lceil \frac{kn}{\Delta(G)} \right\rceil\right\}. \tag{1.1}$$

Proposition D (Henning and Kazemi [5]). *Let G be a graph of order n with $\delta(G) \geq k$, and let S be a kTDS of G. Then for every vertex v of degree k in G, $N_G(v) \subseteq S$.*

2. Some Bounds

The next two theorems state some lower and upper bounds for $\gamma_{\times k,t}(G\overline{G})$.

Theorem 2.1. *If G is a graph of order n with $2 \leq k \leq \min\{\delta(G), \delta(\overline{G})\}$, then*

$$\gamma_{\times(k-1),t}(G) + \gamma_{\times(k-1),t}\left(\overline{G}\right) \leq \gamma_{\times k,t}\left(G\overline{G}\right) \leq \min\left\{\gamma_{\times(k-1),t}(G), \gamma_{\times(k-1),t}\left(\overline{G}\right)\right\} + n. \tag{2.1}$$

Proof. Since for every $\gamma_{\times(k-1),t}(G)$-set D the set $D \cup V(\overline{G})$ is a kTDS of $G\overline{G}$, we get $\gamma_{\times k,t}(G\overline{G}) \leq \gamma_{\times(k-1),t}(G) + n$. Similarly, we have $\gamma_{\times k,t}(G\overline{G}) \leq \gamma_{\times(k-1),t}(\overline{G}) + n$. Therefore

$$\gamma_{\times k,t}\left(G\overline{G}\right) \leq \min\left\{\gamma_{\times(k-1),t}(G), \gamma_{\times(k-1),t}\left(\overline{G}\right)\right\} + n. \tag{2.2}$$

For proving $\gamma_{\times(k-1),t}(G)+\gamma_{\times(k-1),t}(\overline{G}) \leq \gamma_{\times k,t}(G\overline{G})$, let D be a kTDS of $G\overline{G}$. Then $D \cap V(G)$ is a $(k-1)$TDS of G and $D \cap V(\overline{G})$ is a $(k-1)$TDS of $\overline{G}$. Since every vertex of $V(G)$ (resp., $V(\overline{G})$) is adjacent to only one vertex of $V(\overline{G})$ (resp., $V(G)$). Therefore

$$\gamma_{\times(k-1),t}(G) + \gamma_{\times(k-1),t}\left(\overline{G}\right) \leq |D \cap V(G)| + \left|D \cap V\left(\overline{G}\right)\right| = |D| = \gamma_{\times k,t}\left(G\overline{G}\right). \tag{2.3}$$

□

The given bounds in Theorem 2.1 are sharp. Let G be a $(k-1)$-regular graph of odd order $n = 2k-1$. Then $\overline{G}$ and $G\overline{G}$ are $(k-1)$- and k-regular, respectively, and Proposition D implies $\gamma_{\times k,t}(G\overline{G}) = 2n$ and $\gamma_{\times(k-1),t}(G) = \gamma_{\times(k-1),t}(\overline{G}) = n$. Therefore

$$\gamma_{\times(k-1),t}(G) + \gamma_{\times(k-1),t}\left(\overline{G}\right) = \gamma_{\times k,t}\left(G\overline{G}\right) = \min\left\{\gamma_{\times(k-1),t}(G), \gamma_{\times(k-1),t}\left(\overline{G}\right)\right\} + n. \tag{2.4}$$

The Harary graphs $H_{2m,4m+1}$ [6] are a family of this kind of graphs. We recall that the Harary graph $H_{2m,n}$ is a $2m$-regular graph with the vertex set $\{i \mid 1 \le i \le n\}$ and every vertex i is adjacent to the $2m$ vertices in the set

$$\left\{\sigma_j^i \mid \sigma_j^i \equiv i+j \pmod{n} \text{ or } \sigma_j^i \equiv i-j \pmod{n}, \text{ for } 1 \le j \le m\right\}. \tag{2.5}$$

Theorem 2.2. *If G is a graph of order n with $1 \le k \le \min\{\delta(G), \delta(\overline{G})\}$, then*

$$\max\left\{\gamma_{\times k,t}(G), \gamma_{\times k,t}\left(\overline{G}\right)\right\} \le \gamma_{\times k,t}\left(G\overline{G}\right) \le \gamma_{\times k,t}(G) + \gamma_{\times k,t}\left(\overline{G}\right), \tag{2.6}$$

and the lower bound is sharp for $k = 1$.

Proof. Trivially $\max\{\gamma_{\times k,t}(G), \gamma_{\times k,t}(\overline{G})\} \le \gamma_{\times k,t}(G\overline{G})$. Let S be a kTDS of G, and let S' be a kTDS of $\overline{G}$. Then $S \cup S'$ is a kTDS of $G\overline{G}$, and so

$$\gamma_{\times k,t}\left(G\overline{G}\right) \le \gamma_{\times k,t}(G) + \gamma_{\times k,t}\left(\overline{G}\right). \tag{2.7}$$

Proposition A implies that, if $k = 1$, then the lower bound is sharp for all paths and cycles of order $n \ge 5$, where $n \equiv 2 \pmod{4}$, and for the corona graph $K_n \circ K_1$, where $n \ge 3$. □

In special case $k = 1$, we get the following result in [1].

Corollary 2.3 (see [1]). *If G and $\overline{G}$ have no isolated vertices, then*

$$\max\left\{\gamma_t(G), \gamma_t\left(\overline{G}\right)\right\} \le \gamma_t\left(G\overline{G}\right) \le \gamma_t(G) + \gamma_t\left(\overline{G}\right). \tag{2.8}$$

3. The Complementary Prism of Some Graphs

In this section, we calculate the k-tuple total domination number of the complementary prism $G\overline{G}$, when G is a complete multipartite graph, a cycle, or a path. First let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with the vertex partition $V(G) = X_1 \cup X_2 \cup \cdots \cup X_p$ such that for each $1 \le i \le p$, $|X_i| = n_i$ and $n_1 \le n_2 \le \cdots \le n_p$. Then $V(G\overline{G}) = \bigcup_{1 \le i \le p}(X_i \cup \overline{X_i})$, where $\overline{X_i}$ denotes the corresponding set of X_i. Trivially for $G\overline{G}$ to have k-tuple total domination number we should have $k \le n_1 \le n_2 \le \cdots \le n_p$. In the next five propositions, we calculate this number for the complementary prism of the complete p-partite graph G. First we state the following key lemma which has an easy proof that is left to the reader.

Lemma 3.1. *Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with $V(G\overline{G}) = \bigcup_{1 \le i \le p}(X_i \cup \overline{X_i})$. If S is a kTDS of $G\overline{G}$, then for each $1 \le i \le p$, $|S \cap \overline{X_i}| \ge k$. Furthermore, if $|S \cap \overline{X_i}| = k$ for some i, then $|S \cap X_i| \ge k$.*

Proposition 3.2. *Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with $1 \le n_1 \le n_2 \le \cdots \le n_p$. Then*

$$\gamma_t\left(G\overline{G}\right) = 2p - \alpha, \tag{3.1}$$

where $\alpha = |\{i \mid 1 \le i \le p, \text{ and } n_i = 1\}|$.

Proof. Let S be an arbitrary kTDS of $G\overline{G}$, and let $n_1 = n_2 = \cdots = n_\alpha = 1 < n_{\alpha+1} \le \cdots \le n_p$. Proposition D implies that for every $1 \le i \le p$, $|S \cap \overline{X_i}| \ge 2$ or $|S \cap \overline{X_i}| = 1$ and $|S \cap X_i| \ge 1$. Also if $|\overline{X_i}| = 1$ and $|S \cap \overline{X_i}| = 0$, it implies $|S \cap X_i| = 1$. Therefore $|S| \ge \alpha + 2(p-\alpha) = 2p - \alpha$, and hence $\gamma_t(G\overline{G}) \ge 2p - \alpha$. Now we set A as a p-set such that $|A \cap X_i| = 1$, for each $1 \le i \le p$. Since $A \cup \{\overline{x_i} \mid x_i \in A \text{ and } \alpha+1 \le i \le p\}$ is a TDS of G of cardinality $2p-\alpha$, we get $\gamma_t(G\overline{G}) = 2p-\alpha$. □

Corollary 3.3 (see [1]). *If $n \ge 2$, then $\gamma_t(K_n\overline{K_n}) = n$.*

Proposition 3.4. *If $G = K_{n_1,n_2,\ldots,n_p}$ is a complete p-partite graph with $2 \le k = n_1 = \cdots = n_\alpha < n_{\alpha+1} \le \cdots \le n_p$, then*

$$\gamma_{\times k,t}\left(G\overline{G}\right) = \begin{cases} p(k+1) + 2k - 2 & \text{if } \alpha = 1, \\ p(k+1) + \alpha(k-1) & \text{otherwise.} \end{cases} \tag{3.2}$$

Proof. We discuss α.

Case 1 ($\alpha \ge 2$). It follows by $\alpha \ge 2$ and Lemma 3.1 that, for every k-tuple total dominating set S of $G\overline{G}$, $|S \cap X_i| \ge |S \cap \overline{X_i}| = k$ for $1 \le i \le \alpha$ and $|S \cap \overline{X_i}| \ge k+1$ for $\alpha+1 \le i \le p$. Then

$$\gamma_{\times k,t}\left(G\overline{G}\right) \ge p(k+1) + \alpha(k-1). \tag{3.3}$$

Now we set $D = (\bigcup_{1 \le i \le \alpha}(X_i \cup \overline{X_i})) \cup (\bigcup_{\alpha+1 \le i \le p} \overline{D_i})$ such that $\overline{D_i}$ is a $(k+1)$-subset of $\overline{X_i}$, for $\alpha+1 \le i \le p$. Since D is a kTDS of $G\overline{G}$ of cardinality $p(k+1)+\alpha(k-1)$, we have $\gamma_{\times k,t}(G\overline{G}) = p(k+1) + \alpha(k-1)$.

Case 2 ($\alpha = 1$). It follows by $\alpha = 1$ and Lemma 3.1 that, for every kTDS S of $G\overline{G}$, $X_1 \cup \overline{X_1}$ is a subset of S and also every vertex of $\overline{X_1} \cup X_2 \cup \cdots \cup X_p$ is adjacent to at least k vertices of $S \cap (\overline{X_1} \cup X_1)$. Thus either $|S \cap \overline{X_i}| = k+1$ for each $2 \le i \le p$ and $\sum_{2 \le i \le p} |S \cap X_i| \ge k-1$ or

$$\left|S \cap \overline{X_2}\right| = \cdots = \left|S \cap \overline{X_\beta}\right| = k, \qquad \left|S \cap \overline{X_{\beta+1}}\right| = \cdots = \left|S \cap \overline{X_p}\right| = k+1, \tag{3.4}$$

for some $2 \le \beta \le p$. Therefore

$$\begin{aligned} |S| &\ge \min\{2k + (k-1) + (p-1)(k+1), 2k + 2(\beta-1)k + (p-\beta)(k+1)\} \\ &= p(k+1) + 2(k-1). \end{aligned} \tag{3.5}$$

Now we set $D = (X_1 \cup \overline{X_1}) \cup (\bigcup_{2\le i\le p} \overline{D_i}) \cup D_0$ such that $\overline{D_i}$ is a $(k+1)$-subset of $\overline{X_i}$ for $2 \le i \le p$ and D_0 is a $(k-1)$-subset of $V(G)$ such that $|D_0 \cap X_2| = \cdots = |D_0 \cap X_k| = 1$. Since D is a kTDS of $G\overline{G}$ of cardinality $p(k+1) + 2k - 2$, we get $\gamma_{\times k,t}(G\overline{G}) = p(k+1) + 2k - 2$. □

Now let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with $3 \le k+1 = n_1 = \cdots = n_\alpha < n_{\alpha+1} \le \cdots \le n_p$, and let S be a minimal kTDS of $G\overline{G}$. Then $|S \cap \overline{X_i}| \ge k$, by Lemma 3.1. We notice that if $|S \cap \overline{X_i}| \ge k+2$, for some i, then we may improve S and obtain another kTDS S' of cardinality $|S|$ such that $|S' \cap \overline{X_i}| = k+1$ (since every vertex in $\overline{X_i}$ (respectively X_i) is adjacent to only one vertex in X_i (respectively $\overline{X_i}$)). Therefore, we may assume that for every minimal kTDS S of $G\overline{G}$, we have $k \le |S \cap \overline{X_i}| \le k+1$.

Now let S be a minimal kTDS of $G\overline{G}$, and let $B = \{i \mid 1 \le i \le p, |S \cap \overline{X_i}| = k\}$ be a set of cardinality β. We consider the following two cases.

Case 1 ($\beta \ne 0$). In this case, if $i \in B$, we have $|S \cap \overline{X_i}| = |S \cap X_i| = k$ such that $x \in S \cap X_i$ if and only if $\overline{x} \in S \cap \overline{X_i}$, and $|S \cap \overline{X_i}| = k+1$ otherwise. If $\beta \ge 2$, then

$$|S| = p(k+1) + \beta(k-1), \tag{3.6}$$

and if $\beta = 1$ and $B = \{i\}$, then we have also $|S \cap (V(G) - X_i)| = k$. Hence

$$|S| = p(k+1) + 2k - 1. \tag{3.7}$$

Comparing (3.6), (3.7) shows that for $\beta \ne 0$ if S is a set of vertices such that $S \cap X_i = \{x_j^i \mid 1 \le j \le k\}$ and $S \cap \overline{X_i} = \{\overline{x_j^i} \mid x_j^i \in S \cap X_i\}$ for $i = 1, 2$ and $|S \cap \overline{X_i}| = k+1$ for $3 \le i \le p$, then S is a minimum kTDS of $G\overline{G}$ and

$$|S| = p(k+1) + 2k - 2. \tag{3.8}$$

Case 2 ($\beta = 0$). In this case, for each $1 \le i \le p$ we have $|S \cap \overline{X_i}| = k+1$. We continue our discussion in the next subcases.

Subcase 1 ($\alpha \ge k+1$ or $\alpha = k \le p$). Then obviously $|S \cap V(G)| \ge k$. If for $1 \le i \le k$ we consider $|S \cap X_i| = 1$, then S is a minimum kTDS of $G\overline{G}$ and

$$|S| = p(k+1) + k. \tag{3.9}$$

Subcase 2 ($\alpha < k \le p$). Then obviously $|S \cap V(G)| \ge k+1$. If we set S such that $|S \cap X_1| = 2$, and $|S \cap X_i| = 1$ when $2 \le i \le k$, then S is a minimum kTDS of $G\overline{G}$ and

$$|S| = p(k+1) + k. \tag{3.10}$$

Subcase 3 ($\alpha = p \le k-1$). Then obviously $|S \cap V(G)| \ge \gamma_{\times(k-1),t}(G)$. If $S \cap V(G)$ is a $\gamma_{\times(k-1),t}(G)$-set, then S is a minimum kTDS of $G\overline{G}$, and Proposition B implies

$$|S| = \begin{cases} (p+1)(k+1) & \text{if } \alpha = p = k-1, \\ p(k+1) + \left\lceil \dfrac{(k-1)p}{p-1} \right\rceil & \text{if } \alpha = p < k-1. \end{cases} \tag{3.11}$$

Subcase 4 ($\alpha < p < k$). Then obviously $|S \cap V(G)| \ge \gamma_{\times k,t}(G)$. If $S \cap V(G)$ is a $\gamma_{\times k,t}(G)$-set, then S is a minimum kTDS of $G\overline{G}$, and Proposition B implies

$$|S| = p(k+1) + \left\lceil \frac{kp}{p-1} \right\rceil. \tag{3.12}$$

Now let $G = K_{n_1,n_2,\dots,n_p}$ be a complete p-partite graph with $4 \le k+2 \le n_1 \le n_2 \le \cdots \le n_p$, and let S is a minimal kTDS of $G\overline{G}$. In this case, we may similarly assume that $k \le |S \cap \overline{X_i}| \le k+1$. Also it can be easily seen that if $|S \cap \overline{X_i}| = k$ for some i, then equality (3.8) holds. Thus let $\{i \mid 1 \le i \le p, |S \cap \overline{X_i}| = k\} = \emptyset$. Then obviously $|S \cap V(G)| \ge \gamma_{\times k,t}(G)$. If we choose a set S such that $S \cap V(G)$ is a $\gamma_{\times k,t}(G)$-set and $|S \cap \overline{X_i}| = k+1$ for $1 \le i \le p$, then S is a minimum kTDS of $G\overline{G}$, and Proposition B implies

$$|S| = \begin{cases} (p+1)(k+1) & \text{if } p \ge k+1, \\ (p+1)(k+1)+1 & \text{if } p = k, \\ p(k+1) + \left\lceil \dfrac{kp}{p-1} \right\rceil & \text{if } p < k. \end{cases} \tag{3.13}$$

Comparing (3.9), (3.10), (3.11), (3.12), and (3.13) with (3.8) shows that we have proved the following propositions.

Proposition 3.5. *Let $G = K_{n_1,n_2,\dots,n_p}$ be a complete p-partite graph with $3 \le n_1 \le n_2 \le \cdots \le n_p$. Then $\gamma_{\times 2,t}(G\overline{G}) = 3p+2$.*

Proposition 3.6. *Let $G = K_{n_1,n_2,\dots,n_p}$ be a complete p-partite graph with $4 \le k+1 = n_1 = \cdots = n_\alpha < n_{\alpha+1} \le \cdots \le n_p$. Then*

$$\gamma_{\times k,t}\left(G\overline{G}\right) = \begin{cases} p(k+1)+k & \text{if } \alpha = k \le p \text{ or } \alpha \ge k+1 \\ (p+1)(k+1) & \text{if } \alpha < k \le p \text{ or } \alpha = p = k-1, \\ p(k+1) + \min\left\{2k-2, \left\lceil \dfrac{(k-1)p}{p-1} \right\rceil\right\} & \text{if } \alpha = p < k-1, \\ p(k+1) + \min\left\{2k-2, \left\lceil \dfrac{kp}{p-1} \right\rceil\right\} & \text{if } \alpha < p < k. \end{cases} \tag{3.14}$$

Proposition 3.7. *Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with $5 \leq k+2 \leq n_1 \leq \cdots \leq n_p$. Then*

$$\gamma_{\times k,t}\left(G\overline{G}\right) = \begin{cases} (p+1)(k+1) & \text{if } p \geq k+1 \\ (p+1)(k+1)+1 & \text{if } p = k \geq 4, \\ 16 & \text{if } p = k = 3, \\ p(k+1) + \min\left\{2k-2, \left\lceil \dfrac{kp}{p-1} \right\rceil \right\} & \text{if } p < k. \end{cases} \tag{3.15}$$

We now determine the k-tuple total domination number of the complementary prism $C_n\overline{C_n}$, where $1 \leq k \leq 3 = \delta(C_n\overline{C_n})$. Here we assume that $V(C_n\overline{C_n}) = V(C_n) \cup V(\overline{C_n})$, $V(C_n) = \{i \mid 1 \leq i \leq n\}$, and $E(C_n) = \{(i, i+1) \mid 1 \leq i \leq n\}$. Proposition D implies that $\gamma_{\times 3,t}(C_n\overline{C_n}) = 2n$. In many references, for example, in [1], it can be seen that, for $n \geq 3$,

$$\gamma_t(C_n) = \begin{cases} 2\left\lceil \dfrac{n}{4} \right\rceil & \text{if } n \not\equiv 1 \pmod 4, \\ 2\left\lceil \dfrac{n}{4} \right\rceil - 1 & \text{if } n \equiv 1 \pmod 4, \end{cases} \tag{3.16}$$

and trivially we can prove

$$\gamma_t\left(\overline{C_n}\right) = \begin{cases} 4 & \text{if } n = 4, \\ 3 & \text{if } n = 5, \\ 2 & \text{if } n \geq 6. \end{cases} \tag{3.17}$$

Hence Theorem 2.1 implies that

$$\gamma_t(C_n) + 2 \leq \gamma_{\times 2,t}\left(C_n\overline{C_n}\right) \leq n+2, \tag{3.18}$$

where $n \geq 6$, and also Theorem 2.2 implies that

$$n \leq \gamma_{\times 2,t}\left(C_n\overline{C_n}\right) \leq n + \gamma_{\times 2,t}\left(\overline{C_n}\right), \tag{3.19}$$

where $n \geq 5$. In chain (3.19) we need to calculate $\gamma_{\times 2,t}(\overline{C_n})$, which is done by the next proposition.

Proposition 3.8. *If C_n is a cycle of order $n \geq 5$, then*

$$\gamma_{\times 2,t}\left(\overline{C_n}\right) = \begin{cases} 5 & \text{if } n = 5, \\ 4 & \text{if } 6 \leq n \leq 8, \\ 3 & \text{if } n \geq 9. \end{cases} \tag{3.20}$$

Proof. Proposition C implies that $\gamma_{\times 2,t}(\overline{C_n}) \geq 3$. If $n \geq 9$, then, for each $1 \leq i \leq n$, the set $\{\overline{i}, \overline{i+3}, \overline{i+6}\}$ is a DTDS of $\overline{C_n}$ and so $\gamma_{\times 2,t}(\overline{C_n}) = 3$. If $6 \leq n \leq 8$, then it can be easily verified that $\gamma_{\times 2,t}(\overline{C_n}) \geq 4$. Now since $\{\overline{1}, \overline{3}, \overline{4}, \overline{6}\}$ and $\{\overline{1}, \overline{2}, \overline{4}, \overline{6}\}$ are double total dominating sets of $\overline{C_n}$, where $n = 6$ and $n = 7, 8$, respectively, we get $\gamma_{\times 2,t}(\overline{C_n}) = 4$. Finally if $n = 5$, then $\overline{C_5}$ is 2-regular and Proposition D implies $\gamma_{\times 2,t}(\overline{C_5}) = 5$. □

Proposition 3.9. *If $n \geq 5$, then $\gamma_{\times 2,t}(C_n\overline{C_n}) = n + 2$.*

Proof. Let $n \geq 5$. equalities (3.18), (3.19) and Propositions C and 3.8 imply

$$\max\left\{n, \left\lceil \frac{4n}{n-2} \right\rceil\right\} \leq \gamma_{\times 2,t}\left(C_n\overline{C_n}\right) \leq n + 2. \tag{3.21}$$

If $n = 5$, then $\max\{n, \lceil 4n/(n-2) \rceil\} = \lceil 4n/(n-2) \rceil = 7 = n + 2$, and so $\gamma_{\times 2,t}(C_n\overline{C_n}) = n + 2$. Thus we assume $n \geq 6$. Then $\max\{n, \lceil 4n/(n-2) \rceil\} = n$ and hence $n \leq \gamma_{\times 2,t}(C_n\overline{C_n}) \leq n + 2$. Now let S be a $\gamma_{\times 2,t}(C_n\overline{C_n})$-set. If $V(C_n) \subseteq S$, then $S = V(C_n) \cup \{\overline{x}, \overline{y}\}$, for some two adjacent vertices $\overline{x}, \overline{y} \in V(\overline{C_n})$, and so $\gamma_{\times 2,t}(C_n\overline{C_n}) = n + 2$. Thus we assume $V(C_n) \nsubseteq S$. Without loss of generality, let $3 \notin S$. Since $|S \cap \{2, 4, \overline{3}\}| \geq 2$, we continue our proof in the following two cases.

Case 1 ($\{2, 4\} \subseteq S$). Then $1, 5, \overline{2}, \overline{4} \in S$. We note that, for every $5 \leq i \leq n-1$, $S \cap \{\overline{i}, i+1\} \neq \emptyset$. This implies $|S| \geq (n-1-4)+6 = n+1$, and since $\overline{3}$ must be dominated by $S \cap V(\overline{C_n})$, we have $|S \cap V(\overline{C_n})| \geq 4$. If $n \notin S$, then $\overline{1} \in S$ and so $|S| \geq n + 1 + |\{\overline{1}\}| = n + 2$. Let $n \in S$. If $\overline{n} \in S$, again $|S| \geq n + 1 + |\{\overline{n}\}| = n + 2$. But $\overline{n} \notin S$ implies $n - 1 \in S$. Let ß $= \{i \in S \mid 5 \leq i \leq n-1 \text{ and } \overline{i} \in S\}$. The condition $|S \cap V(\overline{C_n})| \geq 4$ implies $|ß| \geq 2$. Therefore for at least one vertex $5 \leq x \leq n-1$, $\{\overline{x}, x+1\} \subseteq S$ and hence $|S| \geq n + |\{\overline{x}, x+1\}| = n + 2$.

Case 2 ($\{4, \overline{3}\} \subseteq S$ (similarly $\{2, \overline{3}\} \subseteq S$)). Case 1 implies $2 \notin S$. Then $\overline{1}, \overline{2}, \overline{4}, 1, 4, 5 \in S$. Again we see that, for every $5 \leq i \leq n-2$, $S \cap \{\overline{i}, i+1\} \neq \emptyset$ and so $|S| \geq (n-2-4) + 8 = n + 2$.

Therefore, in the previous all cases, we proved that $\gamma_{\times 2,t}(C_n\overline{C_n}) \geq n+2$ and chain (3.21) implies $\gamma_{\times 2,t}(C_n\overline{C_n}) = n + 2$. □

Corollary 3.10. *If $n \geq 5$, then*

$$\gamma_{\times 2,t}\left(C_n\overline{C_n}\right) = \begin{cases} \gamma_{\times 2,t}(C_n) + \gamma_{\times 2,t}\left(\overline{C_n}\right) - 1 & \text{if } n \geq 9, \\ \gamma_{\times 2,t}(C_n) + \gamma_{\times 2,t}\left(\overline{C_n}\right) - 2 & \text{if } 6 \leq n \leq 8, \\ \gamma_{\times 2,t}(C_n) + \gamma_{\times 2,t}\left(\overline{C_n}\right) - 3 & \text{if } n = 5. \end{cases} \tag{3.22}$$

Now we determine the exact amount of $\gamma_t(C_n\overline{C_n})$ for $n \geq 3$. Obviously $\gamma_t(C_3\overline{C_3}) = |V(C_3)| = 3$. In the next proposition we calculate it when $n \geq 4$.

Proposition 3.11. *Let $n \geq 4$. Then*

$$\gamma_t\left(C_n\overline{C_n}\right) = \begin{cases} 2\left\lceil \frac{n}{4} \right\rceil + 2 & \text{if } n \equiv 0 \pmod 4, \\ 2\left\lceil \frac{n}{4} \right\rceil + 1 & \text{if } n \equiv 3 \pmod 4, \\ 2\left\lceil \frac{n}{4} \right\rceil & \text{otherwise.} \end{cases} \tag{3.23}$$

Proof. Theorem 2.2 with equalities (3.16) and (3.17) implies

$$4 \leq \gamma_t\left(C_4\overline{C_4}\right) \leq 6, \qquad 4 \leq \gamma_t\left(C_5\overline{C_5}\right) \leq 8, \tag{3.24}$$

and if $n \geq 6$ and $n \not\equiv 1 \pmod 4$, then

$$2\left\lceil \frac{n}{4} \right\rceil \leq \gamma_t\left(C_n\overline{C_n}\right) \leq 2\left\lceil \frac{n}{4} \right\rceil + 2, \tag{3.25}$$

and if $n \geq 6$ and $n \equiv 1 \pmod 4$, then

$$2\left\lceil \frac{n}{4} \right\rceil - 1 \leq \gamma_t\left(C_n\overline{C_n}\right) \leq 2\left\lceil \frac{n}{4} \right\rceil + 1. \tag{3.26}$$

If $n = 4$ and $n = 5$, then the sets $\{1, 2, \overline{1}, \overline{2}\}$ and $\{1, \overline{1}, 4, \overline{4}\}$ are total dominating sets of $C_n\overline{C_n}$, respectively. Hence chain (3.24) implies $\gamma_t(C_n\overline{C_n}) = 4$ for $n = 4, 5$. Now we assume $n \geq 6$. For $n \equiv 2 \pmod 4$, since the sets $\{1, \overline{1}, 4, \overline{4}\}$ and $\{1, \overline{1}, 4, \overline{4}\} \cup \{7+4i, 8+4i \mid 0 \leq i \leq \lceil n/4 \rceil - 3\}$ are two total dominating sets of $C_n\overline{C_n}$ of cardinality $2\lceil n/4 \rceil$, where $n = 6$ and $n > 6$, respectively, we have $\gamma_t(C_n\overline{C_n}) = 2\lceil n/4 \rceil$, by chain (3.25). Now let $n \not\equiv 2 \pmod 4$. We assume that S is a TDS of $C_n\overline{C_n}$. Obviously $S \cap V(\overline{C_n}) \neq \emptyset$. If $|S \cap V(\overline{C_n})| = 1$ and $S \cap V(\overline{C_n}) = \{\overline{1}\}$, then $1, 2, n \in S$, and hence $|S \cap X| \geq 2\lceil |X|/4 \rceil = 2\lceil (n-5)/4 \rceil$, where $X = V(C_n) - \{1, 2, 3, n-1, n\}$. This implies

$$|S| = |S \cap X| + 4 \geq \begin{cases} 2\left\lceil \frac{n}{4} \right\rceil + 2 & \text{if } n \equiv 0 \pmod 4, \\ 2\left\lceil \frac{n}{4} \right\rceil + 1 & \text{if } n \equiv 3 \pmod 4, \\ 2\left\lceil \frac{n}{4} \right\rceil & \text{if } n \equiv 1 \pmod 4. \end{cases} \tag{3.27}$$

Now let $|S \cap V(\overline{C_n})| = \alpha \geq 2$. If $n \equiv 0, 1 \pmod 4$, then

$$|S \cap V(C_n)| \geq \begin{cases} 2\left\lceil \frac{n-\alpha}{4} \right\rceil & \text{if } n \equiv \alpha \pmod 4, \\ 2\left\lceil \frac{n-\alpha}{4} \right\rceil + 1 & \text{otherwise,} \end{cases} \tag{3.28}$$

and if $n \equiv 3 \pmod 4$, then

$$|S \cap V(C_n)| \geq \begin{cases} 2\left\lceil \dfrac{n-\alpha}{4} \right\rceil - 1 & \text{if } n \equiv \alpha + 1 \pmod 4, \\ 2\left\lceil \dfrac{n-\alpha}{4} \right\rceil & \text{otherwise.} \end{cases} \tag{3.29}$$

It can be calculated that

$$|S| = |S \cap V(C_n)| + \alpha \geq \begin{cases} 2\left\lceil \dfrac{n}{4} \right\rceil + 2 & \text{if } n \equiv 0 \pmod 4, \\ 2\left\lceil \dfrac{n}{4} \right\rceil + 1 & \text{if } n \equiv 3 \pmod 4, \\ 2\left\lceil \dfrac{n}{4} \right\rceil & \text{if } n \equiv 1 \pmod 4. \end{cases} \tag{3.30}$$

Then by chains (3.25) and (3.26) we have

$$\gamma_t\left(C_n\overline{C_n}\right) = 2\left\lceil \frac{n}{4} \right\rceil + 2 \quad \text{if } n \equiv 0 \pmod 4, \tag{3.31}$$

$$2\left\lceil \frac{n}{4} \right\rceil \leq \gamma_t\left(C_n\overline{C_n}\right) \leq 2\left\lceil \frac{n}{4} \right\rceil + 1 \quad \text{if } n \equiv 1 \pmod 4, \tag{3.32}$$

$$2\left\lceil \frac{n}{4} \right\rceil + 1 \leq \gamma_t\left(C_n\overline{C_n}\right) \leq 2\left\lceil \frac{n}{4} \right\rceil + 2 \quad \text{if } n \equiv 3 \pmod 4. \tag{3.33}$$

If $n \equiv 1 \pmod 4$, then the sets $\{1,\overline{1},4,\overline{4},7,\overline{7}\}$ and $\{1,\overline{1},4,\overline{4},7,\overline{7}\} \cup \{10+4i, 11+4i \mid 0 \leq i \leq \lceil n/4 \rceil - 4\}$ are total dominating sets of $C_n\overline{C_n}$ of cardinality $2\lceil n/4 \rceil$ when $n = 9$ and $n > 9$, respectively. Hence $\gamma_t(C_n\overline{C_n}) = 2\lceil n/4 \rceil$, by chain (3.32). If also $n \equiv 3 \pmod 4$, the sets $\{1,\overline{1},4,\overline{4},\overline{n-1}\}$ and $\{1,\overline{1},4,\overline{4},\overline{n-1}\} \cup \{7+4i, 8+4i \mid 0 \leq i \leq \lceil n/4 \rceil - 3\}$ are total dominating sets of $C_n\overline{C_n}$ of cardinality $2\lceil n/4 \rceil + 1$ when $n = 7$ and $n > 7$, respectively. Hence $\gamma_t(C_n\overline{C_n}) = 2\lceil n/4 \rceil + 1$, by chain (3.33). □

Finally we determine the k-tuple total domination number of the complementary prism $P_n\overline{P_n}$, where $1 \leq k < 2 = \delta(P_n\overline{P_n})$. We recall that $V(P_n\overline{P_n}) = V(P_n) \cup V(\overline{P_n})$, $V(P_n) = \{i \mid 1 \leq i \leq n\}$, and $E(P_n) = \{ij \mid 1 \leq i \leq n-1,\ j = i+1\}$. In many references, for example, in [1], it can be seen that, for $n \geq 2$,

$$\gamma_t(P_n) = \begin{cases} 2\left\lceil \dfrac{n}{4} \right\rceil & \text{if } n \not\equiv 1 \pmod 4, \\ 2\left\lceil \dfrac{n}{4} \right\rceil - 1 & \text{if } n \equiv 1 \pmod 4, \end{cases} \tag{3.34}$$

and trivially $\gamma_t(\overline{P_n}) = |\{\overline{1},\overline{n}\}| = 2$, where $n \geq 4$. Therefore, by Theorems 2.1 and 2.2, for $n \geq 4$, we have the following chain:

$$\gamma_t(P_n) \leq \gamma_t\left(P_n\overline{P_n}\right) \leq \gamma_t(P_n) + 2 \leq \gamma_{\times 2,t}\left(P_n\overline{P_n}\right) \leq n + 2. \tag{3.35}$$

It can be easily proved that $\gamma_t(P_n\overline{P_n}) = n$, where $n = 2, 3$. Next proposition calculates $\gamma_t(P_n\overline{P_n})$ when $n \geq 4$.

Proposition 3.12. *Let $n \geq 4$. Then*

$$\gamma_t\left(P_n\overline{P_n}\right) = \begin{cases} 2\left\lceil \dfrac{n-2}{4} \right\rceil + 1 & \text{if } n \equiv 3 \pmod 4, \\ 2\left\lceil \dfrac{n-2}{4} \right\rceil + 2 & \text{otherwise.} \end{cases} \tag{3.36}$$

Proof. Let D be a γ_t-set of the induced path $P_n[V(P_n) - \{1, n\}]$ of P_n. Since $D \cup \{\overline{1}, \overline{n}\}$ is a TDS of $P_n\overline{P_n}$, we have

$$\gamma_t\left(P_n\overline{P_n}\right) \leq \left|D \cup \left\{\overline{1}, \overline{n}\right\}\right| = \begin{cases} 2\left\lceil \dfrac{n-2}{4} \right\rceil + 1 & \text{if } n \equiv 3 \pmod 4, \\ 2\left\lceil \dfrac{n-2}{4} \right\rceil + 2 & \text{otherwise.} \end{cases} \tag{3.37}$$

Let $n \equiv 2 \pmod 4$. Then chains (3.34), (3.35), (3.37) imply $\gamma_t(P_n\overline{P_n}) = 2\lceil (n-2)/4 \rceil + 2$. Since $2\lceil n/4 \rceil = 2\lceil (n-2)/4 \rceil + 2$. Now let $n \not\equiv 2 \pmod 4$, and let S be a TDS of $P_n\overline{P_n}$. Obviously $S \cap V(\overline{P_n}) \neq \emptyset$. In all cases, (i) $|S \cap V(\overline{P_n})| = 1$ and $S \cap \{\overline{1}, \overline{n}\} \neq \emptyset$, (ii) $|S \cap V(\overline{P_n})| = 1$ and $S \cap \{\overline{1}, \overline{n}\} = \emptyset$, and (iii) $|S \cap V(\overline{P_n})| \geq 2$, then similar to the proof of Proposition 3.11, it can be verified that

$$|S| \geq \begin{cases} 2\left\lceil \dfrac{n-2}{4} \right\rceil + 1 & \text{if } n \equiv 3 \pmod 4, \\ 2\left\lceil \dfrac{n-2}{4} \right\rceil + 2 & \text{otherwise.} \end{cases} \tag{3.38}$$

Hence chain (3.37) completes the proof of our proposition. □

Propositions 3.11 and 3.12 imply the next result in [1].

Corollary 3.13 (see [1]). *If $G \in \{P_n, C_n\}$ with order $n \geq 5$, then*

$$\gamma_t\left(G\overline{G}\right) = \begin{cases} \gamma_t(G) & \text{if } n \equiv 2 \pmod 4, \\ \gamma_t(G) + 2 & \text{if } n \equiv 0 \pmod 4, \\ \gamma_t(G) + 1 & \text{otherwise.} \end{cases} \tag{3.39}$$

4. Problems

If we look carefully at the propositions of Section 3, we obtain the following result.

Proposition 4.1. (i) *Let G be a cycle or a path of order $n \geq 4$. Then* $\max\{\gamma_t(G), \gamma_t(\overline{G})\} = \gamma_t(G\overline{G})$ *if and only if* $n \equiv 2 \pmod 4$.

(ii) *Let G be a cycle of order $n \geq 5$ or a path of order $n \geq 4$. Then* $\gamma_t(G\overline{G}) = \gamma_t(G) + \gamma_t(\overline{G})$ *if and only if* $n \equiv 0 \pmod 4$.

(iii) *Let C_n be a cycle of order $n \geq 5$. Then*

$$\max\left\{\gamma_{\times 2,t}(C_n), \gamma_{\times 2,t}\left(\overline{C_n}\right)\right\} < \gamma_{\times 2,t}\left(C_n\overline{C_n}\right) < \gamma_{\times 2,t}(C_n) + \gamma_{\times 2,t}\left(\overline{C_n}\right). \quad (4.1)$$

(iv) *Let C_n be a cycle of order $n \geq 5$. Then*

$$\gamma_t(C_n) + \gamma_t\left(\overline{C_n}\right) < \gamma_{\times 2,t}\left(C_n\overline{C_n}\right) = n + \min\left\{\gamma_t(C_n), \gamma_t\left(\overline{C_n}\right)\right\}. \quad (4.2)$$

Therefore it is natural that we state the following problem.

Problem 1. Characterize graphs G with

(1) $\gamma_{\times k,t}(G\overline{G}) = \gamma_{\times k,t}(G) + \gamma_{\times k,t}(\overline{G})$,

(2) $\gamma_{\times k,t}(\overline{G}) = \max\{\gamma_{\times k,t}(G), \gamma_{\times k,t}(\overline{G})\}$.

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