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Research Article

On New Wilker-Type Inequalities

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In this paper, we present and prove a new Wilker-type inequality for hyperbolic functions. We also give a simple proof of the counterpart of the above inequality for the circular functions.

1. Introduction

Wilker [1] proposed two open questions as the following statements.

Problem 1. If $0 < x < \pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1.1}$$

Problem 2. There exists a largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \tag{1.2}$$

for $0 < x < \pi/2$.

Sumner et al. [2] affirmed the truth of the problems above and obtained a further result as follows.

Theorem 1.1. *If* $0 < x < \pi/2$, *then*

$$\frac{16}{\pi^4}x^3\tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 < \frac{8}{45}x^3\tan x. \tag{1.3}$$

Furthermore, $16/\pi^4$ *and* $8/45$ *are the best constants in* (1.3).

Guo et al. [3] gave new proofs of inequalities (1.1) and (1.2). In recent years, Zhu [4] showed a new simple proof of inequality (1.1); Pinelis [5] got other proof of inequalities (1.3) by using L'Hospital rules for monotonicity; Zhang and Zhu [6] gave a new elementary proof of double inequalities (1.3).

Besides, in article [7], Wang offered another type of the inequality and proved it by the power series expansions of circular functions as follows

Theorem 1.2. *Let* $0 < x < \pi/2$. *Then,*

$$\frac{2}{45}x^3\sin x < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 < \left(\frac{2}{\pi} - \frac{16}{\pi^3}\right)x^3\sin x. \tag{1.4}$$

Furthermore, $2/45$ *and* $(2/\pi - 16/\pi^3)$ *are the best constants in* (1.4).

Zhu [8] established a new Wilker-type inequality involving hyperbolic functions as follows:

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2 > \frac{8}{45}x^3\tanh x, \quad x > 0, \tag{1.5}$$

where the constant $8/45$ is the optimum constant in (1.5).

In fact, we can show a new Wilker-type inequality involving hyperbolic functions referring to the result above.

Theorem 1.3. *Let* $x > 0$. *Then*

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 < \frac{2}{45}x^3\sinh x. \tag{1.6}$$

Furthermore, $2/45$ *is the best constant in* (1.6).

The purpose of this paper is to present a concise proof of inequality (1.6) by using the power series expansion of hyperbolic functions and to give an elementary proof of inequality (1.4) in another way.

2. The Proof of Theorem 1.3

Let $((x/\sinh x)^2 + x/\tanh x - 2)/(x^3\sinh x) = (x^2 + x\sinh x\cosh x - 2(\sinh x)^2)/(x\sinh x)^3 = A(x)/B(x)$, where $A(x) = x^2 + x\sinh x\cosh x - 2(\sinh x)^2$, and $B(x) = (x\sinh x)^3$. We get that

the existence of Theorem 1.3 can be ensured when proving the following two statements: the first is $A(x)/B(x) \le 2/45$ or $2B(x) \ge 45A(x)$, and the second is $\lim_{x\to 0^+}(A(x)/B(x)) = 2/45$.

Since

$$\begin{aligned}
A(x) &= x^2 + \frac{1}{2}x\sinh x - (\cosh 2x - 1) = x^2 + \frac{1}{2}x\sum_{n=0}^{\infty}\frac{(2x)^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty}\frac{(2x)^{2n}}{(2n)!} + 1 \\
&= x^2 + \frac{1}{2}\sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}x^{2n+2} - \sum_{n=0}^{\infty}\frac{(2x)^{2n}}{(2n)!} + 1 \\
&= \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n+1)!}x^{2n+2} - \sum_{n=1}^{\infty}\frac{2^{2n+2}}{(2n+2)!}x^{2n+2} \\
&= \sum_{n=1}^{\infty}\left(\frac{2^{2n}}{(2n+1)!} - \frac{2^{2n+2}}{(2n+2)!}\right)x^{2n+2} = \sum_{n=2}^{\infty}\left(\frac{2^{2n}}{(2n+1)!} - \frac{2^{2n+2}}{(2n+2)!}\right)x^{2n+2} \\
&= \sum_{n=2}^{\infty}a_n x^{2n+2}, \\
B(x) &= \frac{1}{4}x^3(\sinh 3x - 3\sinh x) = \frac{1}{4}x^3\sum_{n=0}^{\infty}\frac{3^{2n+1}-3}{(2n+1)!}x^{2n+1} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{3^{2n+1}-3}{(2n+1)!}x^{2n+4} \\
&= \frac{1}{4}\sum_{n=1}^{\infty}\frac{3^{2n+1}-3}{(2n+1)!}x^{2n+4} = \sum_{n=2}^{\infty}\frac{3^{2n-1}-3}{4(2n-1)!}x^{2n+2} \\
&= \sum_{n=2}^{\infty}b_n x^{2n+2},
\end{aligned} \tag{2.1}$$

where $a_n = 2^{2n}/(2n+1)! - 2^{2n+2}/(2n+2)!$, $b_n = (3^{2n-1} - 3)/4(2n-1)!$, $n \ge 2$, and $n \in N^+$, according to the analysis above, the proof of Theorem 1.3 can be completed when proving that

$$2b_n \ge 45a_n, \tag{2.2}$$

that is,

$$n(n+1)(2n+1)(9^n - 9) \ge 135(n-1)4^n \tag{2.3}$$

for $n \ge 2$.

Let $f(x) = x(x+1)(2x+1)(9^x - 9) - 135(x-1)4^x$ for $x \in [2, +\infty)$. We compute

$$\begin{aligned}
f'(x) &= \left(6x^2 + 6x + 1\right)9^x - 9\left(6x^2 + 6x + 1\right) \\
&\quad + \left(2x^3 + 3x^2 + x\right)9^x \log 9 - 135 \cdot 4^x - 135(x-1)4^x \log 4, \\
f''(x) &= (12x+6)9^x + 2\left(6x^2 + 6x + 1\right)9^x \log 9 - 9(12x+6) \\
&\quad + \left(2x^3 + 3x^2 + x\right)9^x (\log 9)^2 - 270 \cdot 4^x \log 4 - 135(x-1)4^x (\log 4)^2, \\
f^{(3)}(x) &= 12 \cdot 9^x + (36x+18)9^x \log 9 + 3\left(6x^2 + 6x + 1\right)9^x (\log 9)^2 - 9 \cdot 12 \\
&\quad + \left(2x^3 + 3x^2 + x\right)9^x (\log 9)^3 - 405 \cdot 4^x (\log 4)^2 - 135(x-1)4^x (\log 4)^3, \\
f^{(4)}(x) &= \left(ax^3 + bx^2 + cx + d\right)9^x - (ex+m)4^x,
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
a &= 2(\log 9)^4 > 0, \\
b &= 3(\log 9)^4 + 24(\log 9)^3 > 0, \\
c &= (\log 9)^4 + 24(\log 9)^3 + 72(\log 9)^2 > 0, \\
d &= 4(\log 9)^3 + 36(\log 9)^2 + 48(\log 9) > 0, \\
e &= 135(\log 4)^4 > 0, \\
m &= 540(\log 4)^3 - 135(\log 4)^4 \approx 940.059 > 0.
\end{aligned} \tag{2.5}$$

Then let $H(x) = (ax^3 + bx^2 + cx + d)(9/4)^x/(ex+m)$; we compute $H'(x) = (l(x)/(ex+m)^2)(9/4)^x$, where

$$\begin{aligned}
l(x) &= a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \\
a_4 &= ae \log \frac{9}{4} > 0, \\
a_3 &= 2ae + (eb + am) \log \frac{9}{4} > 0, \\
a_2 &= be + 3am + (ec + mb) \log \frac{9}{4} > 0, \\
a_1 &= 2bm + (mc + de) \log \frac{9}{4} > 0, \\
a_0 &= m\left(c + d \log \frac{9}{4}\right) - ed \approx 671800 > 0.
\end{aligned} \tag{2.6}$$

As we can see, all coefficients of the polynomial $l(x)$ are positive integers. When $x > 2$, we have $l(x) > 0$ and $H'(x) > 0$. In view of that $H(2) \approx 8.4763 > 1$, $f^{(4)}(2) \approx 231740 > 0$, $f^{(3)}(2) \approx 67857 > 0$, $f''(2) \approx 16922 > 0$, $f'(2) \approx 2849.9 > 0$, we have that $H(x) > 1$ and $f^{(4)}(x) > 0$ for $x > 2$. We can also conclude that $f^{(i)}(x)$ $(i = 3, 2, 1, 0)$ is increasing on $[2, +\infty)$, respectively. Since $f(2) = 0$, we have $f(x) \geq 0$ for $x \geq 2$. So the proof of inequality (2.3) is completed. Furthermore, $\lim_{x\to 0^+}(A(x)/B(x)) = a_2/b_2 = (2^4/5! - 2^6/6!)/((3^3 - 3)/4 \cdot 3!) = 2/45$; the proof of Theorem 1.3 is completed.

3. Another Proof of Theorem 1.2

We simplify the double inequality (1.4) into another form:

$$\frac{2}{45} < \frac{(x/\sin x)^2 + x/\tan x - 2}{x^3 \sin x} < \frac{2}{\pi} - \frac{16}{\pi^3}, \quad 0 < x < \frac{\pi}{2}. \tag{3.1}$$

Here we will discuss the monotonicity of the function $g(x) = ((x/\sin x)^2 + x/\tan x - 2)/x^3 \sin x = ((1/\sin x)(x/\sin x)^2 + x\cos x/\sin^2 x - 2/\sin x)/x^3$. If the function $g(x)$ is increasing for $0 < x < \pi/2$ and both $\lim_{x\to 0^+} g(x) = 2/45$ and $\lim_{x\to(\pi/2)^-} g(x) = 2/\pi - 16/\pi^3$ are right, the proof of equality (1.4) is completed.

From [9] or [10, page 75], we know that for $0 < |x| < \pi$ the equality

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1} \tag{3.2}$$

holds. So we can also know that for $0 < |x| < \pi$

$$\begin{aligned}
\frac{\cos x}{\sin^2 x} &= \left(-\frac{1}{\sin x}\right)' = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| (2n-1) x^{2n-2}, \\
\frac{1}{\sin^3 x} &= \left(-\frac{\cos x}{2\sin^2 x}\right)' + \frac{1}{2\sin x} \\
&= \frac{1}{x^3} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| (2n-1)(2n-2) x^{2n-3} \\
&\quad + \frac{1}{2x} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1}
\end{aligned} \tag{3.3}$$

hold. Then we compute $g(x) = \sum_{n=2}^{\infty} a_n x^{2n-4}$ by using the results above, where

$$\begin{aligned} a_n &= \frac{1}{2}\frac{2^{2n}-2}{(2n)!}|B_{2n}|(2n-1)(2n-2) + \frac{1}{2}\frac{2^{2n-2}-2}{(2n-2)!}|B_{2n-2}| \\ &\quad - \frac{2^{2n}-2}{(2n)!}|B_{2n}|(2n-1) - 2\frac{2^{2n}-2}{(2n)!}|B_{2n}| \\ &= \frac{1}{2}\frac{2^{2n-2}-2}{(2n-2)!}|B_{2n-2}| + \frac{(2n-1)(2n-2)-2(2n-1)-4}{2(2n)!}\left(2^{2n}-2\right)|B_{2n}| \\ &= \frac{1}{2}\frac{2^{2n-2}-2}{(2n-2)!}|B_{2n-2}| + \frac{2n(2n-5)}{2(2n)!}\left(2^{2n}-2\right)|B_{2n}|. \end{aligned} \tag{3.4}$$

When $n = 2$, we can compute $a_2 = (1/2)(2/2!)|B_2| + (-4/2 \cdot 4!) \cdot 14 \cdot |B_4| = (1/2) \cdot (1/6) - (7/6) \cdot (1/30) = 2/45 > 0$ by using $|B_2| = 1/6$ and $|B_4| = 1/30$. Furthermore, there is an obvious fact that $a_n > 0$ for $n \geq 3$. So far, we can demonstrate that $a_n > 0$ for all $n \geq 2$. So the function $g(x)$ is increasing on $(0, \pi/2)$. Evidently, $\lim_{x \to 0^+} g(x) = 2/45$ and $\lim_{x \to (\pi/2)^-} g(x) = 2/\pi - 16/\pi^3$ are true. So the proof of the double inequality of (1.4) is completed.

Remark 3.1. Wilker's inequalities (1.1) and (1.2) have been further refined by many scholars in the past few years; the readers can refer to [11–16].

References

[1] J. B. Wilker, "E3306," *The American Mathematical Monthly*, vol. 96, p. 55, 1989.

[2] J. S. Sumner, A. A. Jagers, M. Vowe, and J. Anglesio, "Inequalities involving trigonometric functions," *The American Mathematical Monthly*, vol. 98, no. 3, pp. 264–267, 1991.

[3] B.-N. Guo, B.-M. Qiao, F. Qi, and W. Li, "On new proofs of Wilker's inequalities involving trigonometric functions," *Mathematical Inequalities and Applications*, vol. 6, no. 1, pp. 19–22, 2003.

[4] L. Zhu, "A new simple proof of Wilker's inequality," *Mathematical Inequalities and Applications*, vol. 8, no. 4, pp. 749–750, 2005.

[5] I. Pinelis, "l'Hospital rules for monotonicity and the Wilker-Anglesio inequality," *The American Mathematical Monthly*, vol. 111, no. 10, pp. 905–909, 2004.

[6] L. Zhang and L. Zhu, "A new elementary proof of Wilker's inequalities," *Mathematical Inequalities and Applications*, vol. 11, no. 1, pp. 149–151, 2008.

[7] Z.-H. Wang, "A new Wilker-type inequality," *Journal of Yibing University*, vol. 6, pp. 21–22, 2007.

[8] L. Zhu, "On Wilker-type inequalities," *Mathematical Inequalities and Applications*, vol. 10, no. 4, pp. 727–731, 2007.

[9] J.-L. Li, "An identity related to Jordan's inequality," *International Journal of Mathematics and Mathematical Sciences*, Article ID 76782, 2006.

[10] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, U.S. National Bureau of Standards, Washington, DC, USA, 1964.

[11] L. Zhu, "Some new Wilker-type inequalities for circular and hyperbolic functions," *Abstract and Applied Analysis*, vol. 2009, Article ID 485842, 2009.

[12] S.-H. Wu and H. M. Srivastava, "A further refinement of Wilker's inequality," *Integral Transforms and Special Functions*, vol. 19, no. 9-10, pp. 757–765, 2008.

[13] E. Neuman, "One- and two-sided inequalities for Jacobian elliptic functions and related results," *Integral Transforms and Special Functions*, vol. 21, no. 5-6, pp. 399–407, 2010.

[14] E. Neuman, "Inequalities involving inverse circular and inverse hyperbolic functions II," *Journal of Mathematical Inequalities*, vol. 4, no. 1, pp. 11–14, 2010.

[15] E. Neuman and J. Sandor, "On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities," *Mathematical Inequalities and Applications*, vol. 13, no. 4, pp. 715–723, 2010.

[16] S.-H. Wu and H. M. Srivastava, "A weighted and exponential generalization of Wilker's inequality and its applications," *Integral Transforms and Special Functions*, vol. 18, no. 7-8, pp. 529–535, 2007.