Research Article

Shrinking Projection Method for Fixed Point Problems of an Infinite Family of Strictly Pseudocontractive Mappings and the System of Cocoercive Quasivariational Inclusions Problems in Hilbert Spaces

Pattanapong Tianchai

Faculty of Science, Maejo University, Chiangmai 50290, Thailand

Correspondence should be addressed to Pattanapong Tianchai, pattana@mju.ac.th

Received 12 April 2011; Accepted 3 May 2011

Academic Editor: C. Zhu

Copyright © 2011 Pattanapong Tianchai. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is concerned with a common element of the set of common fixed points for an infinite family of strictly pseudocontractive mappings and the set of solutions of a system of cocoercive quasivariational inclusions problems in Hilbert spaces. The strong convergence theorem for the above two sets is obtained by a general iterative scheme based on the shrinking projection method, and the applicability of the results is shown to extend and improve some well-known results existing in the current literature.

1. Introduction

Throughout this paper, we always assume that C is a nonempty closed convex subset of a real Hilbert space H with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, 2^H denoting the family of all the nonempty subsets of H .

Let $B : H \rightarrow H$ be a single-valued nonlinear mapping and $M : H \rightarrow 2^H$ a set-valued mapping. We consider the following *quasivariational inclusion problem*, which is to find a point $x \in H$,

$$\theta \in Bx + Mx, \quad (1.1)$$

where θ is the zero vector in H . The set of solutions of problem (1.1) is denoted by $VI(H, B, M)$.

Recall that P_C is the *metric projection* of H onto C ; that is, for each $x \in H$, there exists the unique point in $P_C x \in C$ such that $\|x - P_C x\| = \min_{y \in C} \|x - y\|$. A mapping $T : C \rightarrow C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a *fixed point* of T provided $Tx = x$. We denote by $F(T)$ the set of fixed points of T ; that is, $F(T) = \{x \in C : Tx = x\}$. If C is nonempty bounded closed convex subset of H and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty (see [1]). Recall that a mapping $A : C \rightarrow C$ is said to be

(i) *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C, \quad (1.2)$$

(ii) *k-Lipschitz continuous* if there exists a constant $k > 0$ such that

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \forall x, y \in C, \quad (1.3)$$

if $k = 1$, then A is a nonexpansive,

(iii) *pseudocontractive* if

$$\|Ax - Ay\|^2 \leq \|x - y\|^2 + \|(I - A)x - (I - A)y\|^2, \quad \forall x, y \in C, \quad (1.4)$$

(iv) *k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$ such that

$$\|Ax - Ay\|^2 \leq \|x - y\|^2 + k\|(I - A)x - (I - A)y\|^2, \quad \forall x, y \in C, \quad (1.5)$$

it is obvious that A is a nonexpansive if and only if A is 0-strictly pseudocontractive,

(v) *α -strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C, \quad (1.6)$$

(vi) *α -inverse-strongly monotone (or α -cocoercive)* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C, \quad (1.7)$$

if $\alpha = 1$, then A is said to be *firmly nonexpansive*; it is obvious that any α -inverse-strongly monotone mapping A is monotone and $(1/\alpha)$ -Lipschitz continuous.

The existence of common fixed points for a finite family of nonexpansive mappings has been considered by many authors (see [2–5] and the references therein).

In this paper, we study the mapping W_n defined by

$$\begin{aligned}
 U_{n,n+1} &= I, \\
 U_{n,n} &= \mu_n S_n U_{n,n+1} + (1 - \mu_n)I, \\
 U_{n,n-1} &= \mu_{n-1} S_{n-1} U_{n,n} + (1 - \mu_{n-1})I, \\
 &\vdots \\
 U_{n,k} &= \mu_k S_k U_{n,k+1} + (1 - \mu_k)I, \\
 U_{n,k-1} &= \mu_{k-1} S_{k-1} U_{n,k} + (1 - \mu_{k-1})I, \\
 &\vdots \\
 U_{n,2} &= \mu_2 S_2 U_{n,3} + (1 - \mu_2)I, \\
 W_n &= U_{n,1} = \mu_1 S_1 U_{n,2} + (1 - \mu_1)I,
 \end{aligned} \tag{1.8}$$

where $\{\mu_i\}$ is nonnegative real sequence in $(0, 1)$, for all $i \in \mathbb{N}$, $S_1, S_2, \dots$ from a family of infinitely nonexpansive mappings of C into itself. It is obvious that W_n is a nonexpansive mapping of C into itself; such a mapping W_n is called a W -mapping generated by $S_1, S_2, \dots, S_n$ and $\mu_1, \mu_2, \dots, \mu_n$.

Definition 1.1 (see [6]). Let $M : H \rightarrow 2^H$ be a multivalued maximal monotone mapping. Then, the single-valued mapping $J_{M,\lambda} : H \rightarrow H$ defined by $J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u)$, for all $u \in H$, is called the *resolvent operator associated with M* , where λ is any positive number and I is the identity mapping.

Recently, Zhang et al. [6] considered the problem (1.1) and the problem of a fixed point of nonexpansive mapping. To be more precise, they proved the following theorem.

Theorem ZLC. Let H be a real Hilbert space, $B : H \rightarrow H$ an α -inverse-strongly monotone mapping, $M : H \rightarrow 2^H$ a maximal monotone mapping, and $T : H \rightarrow H$ a nonexpansive mapping. Suppose that the set $F(T) \cap \text{VI}(H, B, M) \neq \emptyset$, where $\text{VI}(H, B, M)$ is the set of solutions of quasivariational inclusion (1.1). Suppose that $x_1 = x \in H$ and $\{x_n\}$ is the sequence defined by

$$\begin{aligned}
 y_n &= J_{M,\lambda}(x_n - \lambda Bx_n), \\
 x_{n+1} &= \alpha_n x + (1 - \alpha_n)Ty_n,
 \end{aligned} \tag{1.9}$$

for all $n \in \mathbb{N}$, where $\lambda \in (0, 2\alpha)$ and $\{\alpha_n\} \subset (0, 1)$ satisfying the following conditions:

(C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, $\{x_n\}$ converges strongly to $P_{F(T) \cap \text{VI}(H, B, M)}(x)$.

Nakajo and Takahashi [7] introduced an iterative scheme for finding a fixed point of a nonexpansive mapping by a hybrid method which is called that shrinking projection method (or CQ method) as in the following theorem.

Theorem NT. Let C be a nonempty closed convex subset of a real Hilbert space H . Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is the sequence defined by

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_1), \quad \forall n \in \mathbb{N}, \end{aligned} \tag{1.10}$$

where $0 \leq \alpha_n \leq \alpha < 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}(x_1)$.

In the same way, Kikkawa and Takahashi [8] introduced an iterative scheme for finding a common fixed point of an infinite family of nonexpansive mappings as follows:

$$\begin{aligned} y_n &= W_n x_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_1), \quad \forall n \in \mathbb{N}, \end{aligned} \tag{1.11}$$

where $x_1 = x \in C$ and W_n is a W -mapping of C into itself generated by $\{T_n : C \rightarrow C\}$ and $\{\mu_n\}$. They prove that, if $\Omega = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.11) converges strongly to $P_{\Omega}(x_1)$.

Recently, Su and Qin [9] modified the shrinking projection method for finding a fixed point of a nonexpansive mapping, for which the convergence rate of the iterative scheme is faster than that of the iterative scheme of Nakajo and Takahashi [7] as follows:

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \|y_n - z\| \leq \|x_n - z\|\}, \quad n \geq 1, \\ Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ C_0 &= \{z \in C : \|y_0 - z\| \leq \|x_0 - z\|\}, \\ Q_0 &= C, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{1.12}$$

where $x_0 = x \in C$ and T is a nonexpansive mapping of C into itself. They prove that, under the parameter $0 \leq \alpha_n \leq \alpha < 1$, if $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.12) converges strongly to $P_{F(T)}(x_0)$.

On the other hand, Tada and Takahashi [10] introduced an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of solutions of a fixed point problem of a nonexpansive mapping as follows:

$$\begin{aligned}
 u_n \in C \quad \text{such that } F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\
 y_n &= (1 - \alpha_n)x_n + \alpha_n T u_n, \\
 C_n &= \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\
 Q_n &= \{z \in H : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\
 x_{n+1} &= P_{C_n \cap Q_n}(x_1), \quad \forall n \in \mathbb{N},
 \end{aligned} \tag{1.13}$$

where $x_1 = x \in H$, T is a nonexpansive mapping of C into H and F is a bifunction from $C \times C$ into $\mathbb{R}$. They prove that, under the sequences $\{\alpha_n\} \subset [\alpha, 1]$ for some $\alpha \in (0, 1)$ and $\{r_n\} \subset [r, \infty)$ for some $r > 0$, if $\Omega = F(T) \cap \text{EP}(F) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.13) converges strongly to $P_\Omega(x_1)$ such that $\text{EP}(F)$ is the set of solutions of *equilibrium problem* defined by

$$\text{EP}(F) = \{x \in C : F(x, y) \geq 0, \quad \forall y \in C\}. \tag{1.14}$$

In this paper, we introduce an iterative scheme (1.15) for finding a common element of the set of common fixed points for an infinite family of strictly pseudocontractive mappings and the set of solutions of a system of cocoercive quasivariational inclusions problems by the shrinking projection method in Hilbert spaces as follows:

$$\begin{aligned}
 y_n &= \alpha_n W_n x_n + (1 - \alpha_n) \sum_{i=1}^N \beta_i J_{M_i, \lambda_i}(x_n - \lambda_i B_i x_n), \\
 \epsilon_n &= \alpha_n (1 - \alpha_n) \left\| W_n x_n - \sum_{i=1}^N \beta_i J_{M_i, \lambda_i}(x_n - \lambda_i B_i x_n) \right\|^2, \\
 C_{n+1} &= \left\{ z \in C_n \cap Q_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \epsilon_n \right\}, \\
 Q_{n+1} &= \{z \in C_n \cap Q_n : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\
 C_1 &= Q_1 = H, \\
 x_{n+1} &= P_{C_{n+1} \cap Q_{n+1}}(x_1), \quad \forall n \in \mathbb{N},
 \end{aligned} \tag{1.15}$$

where $x_1 = u \in H$ chosen arbitrarily, $M_i : H \rightarrow 2^H$ is a maximal monotone mapping, $B_i : H \rightarrow H$ is a ξ_i -cocoercive mapping for each $i = 1, 2, \dots, N$, and W_n is a W -mapping on H generated by $\{S_n\}$ and $\{\mu_n\}$ such that the mapping $S_n : H \rightarrow H$ defined by $S_n x = \alpha x + (1 - \alpha) T_n x$ for all $x \in H$, where $\{T_n : H \rightarrow H\}$ is an infinite family of k -strictly pseudocontractive mappings with a fixed point.

It is well known that the class of strictly pseudocontractive mappings contains the class of nonexpansive mappings, and it follows that, if $k = 0$, then the iterative scheme (1.15) is

reduced to find a common element of the set of common fixed points for an infinite family of nonexpansive mappings and the set of solutions of a system of cocoercive quasivariational inclusions problems in Hilbert spaces.

Furthermore, if $M_i \equiv B_i \equiv 0$ for all $i = 1, 2, \dots, N$ and $\sum_{i=1}^N \beta_i = 1$, then the iterative scheme (1.15) is reduced to extend and improve the results of Kikkawa and Takahashi [8] for finding a common fixed point of an infinite family of k -strictly pseudocontractive mappings as follows:

$$\begin{aligned}
 x_1 &= u \in C \text{ chosen arbitrarily,} \\
 y_n &= \alpha_n W_n x_n + (1 - \alpha_n) x_n, \\
 \epsilon_n &= \alpha_n (1 - \alpha_n) \|W_n x_n - x_n\|^2, \\
 C_{n+1} &= \left\{ z \in C_n \cap Q_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \epsilon_n \right\}, \\
 Q_{n+1} &= \{ z \in C_n \cap Q_n : \langle x_n - z, x_1 - x_n \rangle \geq 0 \}, \\
 C_1 &= Q_1 = C, \\
 x_{n+1} &= P_{C_{n+1} \cap Q_{n+1}}(x_1), \quad \forall n \in \mathbb{N},
 \end{aligned} \tag{1.16}$$

and if $k = \alpha = 0$ and setting $T_1 \equiv T$, $T_n \equiv I$ for all $n = 2, 3, \dots$, then the iterative scheme (1.16) is reduced to find a fixed point of a nonexpansive mapping, for which the convergence rate of the iterative scheme is faster than that of the iterative scheme of Su and Qin [9] as follows:

$$\begin{aligned}
 x_1 &= u \in C \text{ chosen arbitrarily,} \\
 y_n &= \sigma_n T x_n + (1 - \sigma_n) x_n, \\
 \delta_n &= \sigma_n (1 - \sigma_n) \|T x_n - x_n\|^2, \\
 D_{n+1} &= \left\{ z \in D_n \cap Q_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \delta_n \right\}, \\
 Q_{n+1} &= \{ z \in D_n \cap Q_n : \langle x_n - z, x_1 - x_n \rangle \geq 0 \}, \\
 D_1 &= Q_1 = C, \\
 x_{n+1} &= P_{D_{n+1} \cap Q_{n+1}}(x_1), \quad \forall n \in \mathbb{N}.
 \end{aligned} \tag{1.17}$$

We suggest and analyze the iterative scheme (1.15) under some appropriate conditions imposed on the parameters. The strong convergence theorem for the above two sets is obtained, and the applicability of the results is shown to extend and improve some well-known results existing in the current literature.

2. Preliminaries

We collect the following lemmas which will be used in the proof of the main results in the next section.

Lemma 2.1 (see [11]). Let H be a Hilbert space. For any $x, y \in H$ and $\lambda \in \mathbb{R}$, one has

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \quad (2.1)$$

Lemma 2.2 (see [1]). Let C be a nonempty closed convex subset of a Hilbert space H . Then the following inequality holds:

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \quad \forall x \in H, y \in C. \quad (2.2)$$

Lemma 2.3 (see [5]). Let C be a nonempty closed convex subset of a Hilbert space H , define mapping W_n as (1.8), let $S_i : C \rightarrow C$ be a family of infinitely nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, and let $\{\mu_i\}$ be a sequence such that $0 < \mu_i \leq \mu < 1$, for all $i \geq 1$. Then

- (1) W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^n F(S_i)$ for each $n \geq 1$,
- (2) for each $x \in C$ and for each positive integer k , $\lim_{n \rightarrow \infty} U_{n,k}x$ exists,
- (3) the mapping $W : C \rightarrow C$ defined by

$$Wx := \lim_{n \rightarrow \infty} W_n x = \lim_{n \rightarrow \infty} U_{n,1}x, \quad x \in C, \quad (2.3)$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(S_i)$ and it is called the W -mapping generated by $S_1, S_2, \dots$ and $\mu_1, \mu_2, \dots$.

Lemma 2.4 (see [6]). The resolvent operator $J_{M,\lambda}$ associated with M is single valued and nonexpansive for all $\lambda > 0$.

Lemma 2.5 (see [6]). $u \in H$ is a solution of quasivariational inclusion (1.1) if and only if $u = J_{M,\lambda}(u - \lambda Bu)$, for all $\lambda > 0$, that is,

$$VI(H, B, M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0. \quad (2.4)$$

Lemma 2.6 (see [12]). Let C be a nonempty closed convex subset of a strictly convex Banach space X . Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C . Suppose that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n = 1$. Then a mapping S on C defined by

$$Sx = \sum_{n=1}^{\infty} \alpha_n T_n x, \quad (2.5)$$

for $x \in C$, is well defined, nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.

Lemma 2.7 (see [13]). Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \rightarrow C$ a nonexpansive mapping. Then $I - S$ is demiclosed at zero. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y , it follows that $(I - S)x = y$.

Lemma 2.8 (see [14]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \rightarrow C$ a k -strict pseudocontraction. Define $S : C \rightarrow C$ by $Sx = \alpha x + (1 - \alpha)Tx$ for each $x \in C$. Then, as $\alpha \in [k, 1)$, S is nonexpansive such that $F(S) = F(T)$.*

Lemma 2.9 (see [1]). *Every Hilbert space H has Radon-Riesz property or Kadec-Klee property, that is, for a sequence $\{x_n\} \subset H$ with $x_n \rightarrow x$ and $\|x_n\| \rightarrow \|x\|$ then $x_n \rightarrow x$.*

3. Main Results

Theorem 3.1. *Let H be a real Hilbert space, $M_i : H \rightarrow 2^H$ a maximal monotone mapping, and $B_i : H \rightarrow H$ a ξ_i -cocoercive mapping for each $i = 1, 2, \dots, N$. Let $\{T_n : H \rightarrow H\}$ be an infinite family of k -strictly pseudocontractive mappings with a fixed point such that $k \in [0, 1)$. Define a mapping $S_n : H \rightarrow H$ by*

$$S_n x = \alpha x + (1 - \alpha)T_n x, \quad \forall x \in H, \quad (3.1)$$

for all $n \in \mathbb{N}$, where $\alpha \in [k, 1)$. Let $W_n : H \rightarrow H$ be a W -mapping generated by $\{S_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0, \mu]$, for some $\mu \in (0, 1)$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{i=1}^N \text{VI}(H, B_i, M_i)) \neq \emptyset$. For $x_1 = u \in H$ chosen arbitrarily, suppose that $\{x_n\}$ is generated iteratively by

$$\begin{aligned} y_n &= \alpha_n W_n x_n + (1 - \alpha_n) \sum_{i=1}^N \beta_i J_{M_i, \lambda_i}(x_n - \lambda_i B_i x_n), \\ \epsilon_n &= \alpha_n (1 - \alpha_n) \left\| W_n x_n - \sum_{i=1}^N \beta_i J_{M_i, \lambda_i}(x_n - \lambda_i B_i x_n) \right\|^2, \\ C_{n+1} &= \left\{ z \in C_n \cap Q_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \epsilon_n \right\}, \\ Q_{n+1} &= \{z \in C_n \cap Q_n : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ C_1 &= Q_1 = H, \\ x_{n+1} &= P_{C_{n+1} \cap Q_{n+1}}(x_1), \quad \forall n \in \mathbb{N}, \end{aligned} \quad (3.2)$$

where

- (C1) $\{\alpha_n\} \subset [a, b]$ such that $0 < a < b < 1$,
- (C2) $\beta_i \in (0, 1)$ and $\lambda_i \in (0, 2\xi_i]$ for each $i = 1, 2, \dots, N$,
- (C3) $\sum_{i=1}^N \beta_i = 1$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $w = P_{\Omega}(x_1)$.

Proof. For any $x, y \in H$ and for each $i = 1, 2, \dots, N$, by the ξ_i -cocoercivity of B_i , we have

$$\begin{aligned} \|(I - \lambda_i B_i)x - (I - \lambda_i B_i)y\|^2 &= \|(x - y) - \lambda_i(B_i x - B_i y)\|^2 \\ &= \|x - y\|^2 - 2\lambda_i \langle x - y, B_i x - B_i y \rangle \end{aligned}$$

$$\begin{aligned}
& + \lambda_i^2 \|B_i x - B_i y\|^2 \\
& \leq \|x - y\|^2 - (2\xi_i - \lambda_i)\lambda_i \|B_i x - B_i y\|^2 \\
& \leq \|x - y\|^2,
\end{aligned} \tag{3.3}$$

which implies that $I - \lambda_i B_i$ is nonexpansive. Pick $p \in \Omega$. Therefore, by Lemma 2.5, we have

$$p = J_{M_i, \lambda_i}(I - \lambda_i B_i)p, \tag{3.4}$$

for each $i = 1, 2, \dots, N$. Since $S_n x = \alpha x + (1 - \alpha)T_n x$, where $\alpha \in [k, 1)$ and $\{T_n\}$ is a family of k -strict pseudocontraction, therefore, by Lemma 2.8, we have that S_n is nonexpansive and $F(S_n) = F(T_n)$. It follows from Lemma 2.3(1) that $F(W_n) = \bigcap_{i=1}^n F(S_i) = \bigcap_{i=1}^n F(T_i)$, which implies that $W_n p = p$. Therefore, by (C3), (3.4), Lemma 2.1, and the nonexpansiveness of W_n , J_{M_i, λ_i} , and $I - \lambda_i B_i$, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \left\| \alpha_n W_n x_n + (1 - \alpha_n) \sum_{i=1}^N \beta_i J_{M_i, \lambda_i}(x_n - \lambda_i B_i x_n) - p \right\|^2 \\
&= \left\| \alpha_n (W_n x_n - p) + (1 - \alpha_n) \sum_{i=1}^N \beta_i (J_{M_i, \lambda_i}(x_n - \lambda_i B_i x_n) - p) \right\|^2 \\
&= \alpha_n \|W_n x_n - W_n p\|^2 \\
&\quad + (1 - \alpha_n) \left\| \sum_{i=1}^N \beta_i (J_{M_i, \lambda_i}(x_n - \lambda_i B_i x_n) - J_{M_i, \lambda_i}(p - \lambda_i B_i p)) \right\|^2 \\
&\quad - \alpha_n (1 - \alpha_n) \left\| W_n x_n - \sum_{i=1}^N \beta_i J_{M_i, \lambda_i}(x_n - \lambda_i B_i x_n) \right\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 \\
&\quad + (1 - \alpha_n) \left(\sum_{i=1}^N \beta_i \|x_n - \lambda_i B_i x_n - (p - \lambda_i B_i p)\| \right)^2 - \epsilon_n \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 - \epsilon_n \\
&= \|x_n - p\|^2 - \epsilon_n,
\end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N}$. Firstly, we prove that $C_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. It is obvious that $C_1 \cap Q_1$ is closed and, by mathematical induction, that $C_n \cap Q_n$ is closed for all $n \geq 2$, that is $C_n \cap Q_n$ is closed for all $n \in \mathbb{N}$. Since $\|y_n - z\|^2 \leq \|x_n - z\|^2 - \epsilon_n$ is equivalent to

$$\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z \rangle + \epsilon_n \leq 0, \tag{3.6}$$

for all $n \in \mathbb{N}$, therefore, for any $z_1, z_2 \in C_{n+1} \cap Q_{n+1} \subset C_n \cap Q_n$ and $\epsilon \in (0, 1)$, we have

$$\begin{aligned}
 & \|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - (\epsilon z_1 + (1 - \epsilon)z_2) \rangle + \epsilon_n \\
 &= \epsilon \left(\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z_1 \rangle + \epsilon_n \right) \\
 &+ (1 - \epsilon) \left(\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z_2 \rangle + \epsilon_n \right) \\
 &\leq 0,
 \end{aligned} \tag{3.7}$$

for all $n \in \mathbb{N}$, and we have

$$\langle x_n - (\epsilon z_1 + (1 - \epsilon)z_2), x_1 - x_n \rangle = \epsilon \langle x_n - z_1, x_1 - x_n \rangle + (1 - \epsilon) \langle x_n - z_2, x_1 - x_n \rangle \geq 0, \tag{3.8}$$

for all $n \in \mathbb{N}$. Since $C_1 \cap Q_1$ is convex, and by putting $n = 1$ in (3.6), (3.7), and (3.8), we have that $C_2 \cap Q_2$ is convex. Suppose that x_k is given and $C_k \cap Q_k$ is convex for some $k \geq 2$. It follows by putting $n = k$ in (3.6), (3.7), and (3.8) that $C_{k+1} \cap Q_{k+1}$ is convex. Therefore, by mathematical induction, we have that $C_n \cap Q_n$ is convex for all $n \geq 2$, that is, $C_n \cap Q_n$ is convex for all $n \in \mathbb{N}$. Hence, we obtain that $C_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$.

Next, we prove that $\Omega \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. It is obvious that $p \in \Omega \subset H = C_1 \cap Q_1$. Therefore, by (3.2) and (3.5), we have $p \in C_2$ and note that $p \in H = Q_2$, and so $p \in C_2 \cap Q_2$. Hence, we have $\Omega \subset C_2 \cap Q_2$. Since $C_2 \cap Q_2$ is a nonempty closed convex subset of H , there exists a unique element $x_2 \in C_2 \cap Q_2$ such that $x_2 = P_{C_2 \cap Q_2}(x_1)$. Suppose that $x_k \in C_k \cap Q_k$ is given such that $x_k = P_{C_k \cap Q_k}(x_1)$, and $p \in \Omega \subset C_k \cap Q_k$ for some $k \geq 2$. Therefore, by (3.2) and (3.5), we have $p \in C_{k+1}$. Since $x_k = P_{C_k \cap Q_k}(x_1)$, therefore, by Lemma 2.2, we have

$$\langle x_k - z, x_1 - x_k \rangle \geq 0 \tag{3.9}$$

for all $z \in C_k \cap Q_k$. Thus, by (3.2), we have $p \in Q_{k+1}$, and so $p \in C_{k+1} \cap Q_{k+1}$. Hence, we have $\Omega \subset C_{k+1} \cap Q_{k+1}$. Since $C_{k+1} \cap Q_{k+1}$ is a nonempty closed convex subset of H , there exists a unique element $x_{k+1} \in C_{k+1} \cap Q_{k+1}$ such that $x_{k+1} = P_{C_{k+1} \cap Q_{k+1}}(x_1)$. Therefore, by mathematical induction, we obtain $\Omega \subset C_n \cap Q_n$ for all $n \geq 2$, and so $\Omega \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$, and we can define $x_{n+1} = P_{C_{n+1} \cap Q_{n+1}}(x_1)$ for all $n \in \mathbb{N}$. Hence, we obtain that the iteration (3.2) is well defined.

Next, we prove that $\{x_n\}$ is bounded. Since $x_n = P_{C_n \cap Q_n}(x_1)$ for all $n \in \mathbb{N}$, we have

$$\|x_n - x_1\| \leq \|z - x_1\|, \tag{3.10}$$

for all $z \in C_n \cap Q_n$. It follows by $p \in \Omega \subset C_n \cap Q_n$ that $\|x_n - x_1\| \leq \|p - x_1\|$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded, and so is $\{y_n\}$.

Next, we prove that $\|y_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. Since $x_{n+1} = P_{C_{n+1} \cap Q_{n+1}}(x_1) \in C_{n+1} \cap Q_{n+1} \subset C_n \cap Q_n$, therefore, by (3.10), we have $\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$ for all $n \in \mathbb{N}$. This implies that $\{\|x_n - x_1\|\}$ is a bounded nondecreasing sequence and there exists the limit of $\|x_n - x_1\|$, that is,

$$\lim_{n \rightarrow \infty} \|x_n - x_1\| = m, \tag{3.11}$$

for some $m \geq 0$. Since $x_{n+1} \in Q_{n+1}$, therefore, by (3.2), we have

$$\langle x_n - x_{n+1}, x_1 - x_n \rangle \geq 0. \quad (3.12)$$

It follows by (3.12) that

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_1) + (x_1 - x_{n+1})\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n \rangle \\ &\quad + 2\langle x_n - x_1, x_n - x_{n+1} \rangle + \|x_{n+1} - x_1\|^2 \\ &\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2. \end{aligned} \quad (3.13)$$

Therefore, by (3.11), we have

$$\|x_n - x_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (3.14)$$

Since $x_{n+1} \in C_{n+1}$, therefore, by (3.2), we have

$$\begin{aligned} \|y_n - x_{n+1}\|^2 &\leq \|x_n - x_{n+1}\|^2 - \epsilon_n \\ &\leq \|x_n - x_{n+1}\|^2. \end{aligned} \quad (3.15)$$

It follows by (3.15) that

$$\begin{aligned} \|y_n - x_n\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &= 2\|x_n - x_{n+1}\|. \end{aligned} \quad (3.16)$$

Therefore, by (3.14), we obtain

$$\|y_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (3.17)$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $\bar{w}$. Next, we prove that $\bar{w} \in \Omega$. Define the sequence of mappings $\{Q_n : H \rightarrow H\}$ and the mapping $Q : H \rightarrow H$ by

$$\begin{aligned} Q_n x &= \alpha_n W_n x + (1 - \alpha_n) \sum_{i=1}^N \beta_i J_{M_i, \lambda_i} (I - \lambda_i B_i) x, \quad \forall x \in H, \\ Q x &= \lim_{n \rightarrow \infty} Q_n x, \end{aligned} \quad (3.18)$$

for all $n \in \mathbb{N}$. Therefore, by (C1) and Lemma 2.3(3), we have

$$Qx = cWx + (1 - c) \sum_{i=1}^N \beta_i J_{M_i, \lambda_i} (I - \lambda_i B_i)x, \quad \forall x \in H, \quad (3.19)$$

where $a \leq c = \lim_{n \rightarrow \infty} \alpha_n \leq b$. From (C3) and Lemma 2.3(3), we have that W and $\sum_{i=1}^N \beta_i J_{M_i, \lambda_i} (I - \lambda_i B_i)$ are nonexpansive. Therefore, by (C2), (C3), Lemmas 2.3(3), 2.5, 2.6, and 2.8, we have

$$\begin{aligned} F(Q) &= F(W) \cap F\left(\sum_{i=1}^N \beta_i J_{M_i, \lambda_i} (I - \lambda_i B_i)\right) \\ &= \left(\bigcap_{i=1}^{\infty} F(S_i)\right) \cap \left(\bigcap_{i=1}^N F(J_{M_i, \lambda_i} (I - \lambda_i B_i))\right) \\ &= \left(\bigcap_{i=1}^{\infty} F(T_i)\right) \cap \left(\bigcap_{i=1}^N \text{VI}(H, B_i, M_i)\right), \end{aligned} \quad (3.20)$$

that is, $F(Q) = \Omega$. From (3.17), we have $\|y_{n_i} - x_{n_i}\| \rightarrow 0$ as $i \rightarrow \infty$. Thus, from (3.2) and (3.18), we get $\|Qx_{n_i} - x_{n_i}\| \rightarrow 0$ as $i \rightarrow \infty$. It follows from $x_{n_i} \rightharpoonup \bar{w}$ and Lemma 2.7 that $\bar{w} \in F(Q)$, that is, $\bar{w} \in \Omega$.

Since Ω is a nonempty closed convex subset of H , there exists a unique $w \in \Omega$ such that $w = P_{\Omega}(x_1)$. Next, we prove that $x_n \rightarrow w$ as $n \rightarrow \infty$. Since $w = P_{\Omega}(x_1)$, we have $\|x_1 - w\| \leq \|x_1 - z\|$ for all $z \in \Omega$, and it follows that

$$\|x_1 - w\| \leq \|x_1 - \bar{w}\|. \quad (3.21)$$

Since $w \in \Omega \subset C_n \cap Q_n$, therefore, by (3.10), we have

$$\|x_1 - x_n\| \leq \|x_1 - w\|. \quad (3.22)$$

Therefore, by (3.21), (3.22), and the weak lower semicontinuity of norm, we have

$$\begin{aligned} \|x_1 - w\| &\leq \|x_1 - \bar{w}\| \\ &\leq \liminf_{i \rightarrow \infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i \rightarrow \infty} \|x_1 - x_{n_i}\| \\ &\leq \|x_1 - w\|. \end{aligned} \quad (3.23)$$

It follows that

$$\|x_1 - w\| = \lim_{i \rightarrow \infty} \|x_1 - x_{n_i}\| = \|x_1 - \bar{w}\|. \quad (3.24)$$

Since $x_{n_i} \rightharpoonup \bar{w}$ as $i \rightarrow \infty$, therefore, we have

$$(x_1 - x_{n_i}) \rightharpoonup (x_1 - \bar{w}) \quad \text{as } i \rightarrow \infty. \quad (3.25)$$

Hence, from (3.24), (3.25), the Kadec-Klee property, and the uniqueness of $w = P_\Omega(x_1)$, we obtain

$$x_{n_i} \longrightarrow \bar{w} = w \quad \text{as } i \longrightarrow \infty. \quad (3.26)$$

It follows that $\{x_n\}$ converges strongly to w , and so is $\{y_n\}$. This completes the proof. $\square$

Remark 3.2. The iteration (3.2) is the difference with some well known results as the following.

- (1) The sequence $\{x_n\}$ is the projection sequence of x_1 onto $C_n \cap Q_n$ for all $n \in \mathbb{N}$ such that

$$C_1 \cap Q_1 \supset C_2 \cap Q_2 \supset \cdots \supset C_n \cap Q_n \supset \cdots \supset \Omega. \quad (3.27)$$

- (2) The proof of $\bar{w} \in \Omega$ is simple by the demiclosedness principle because the sequence $\{y_n\}$ is a linear nonexpansive mapping form of the mappings W_n and $J_{M_i, \lambda_i}(I - \lambda_i B_i)$.
- (3) Solving a common fixed point for an infinite family of strictly pseudocontractive mappings and a system of cocoercive quasivariational inclusions problems by iteration is obtained.

4. Applications

Theorem 4.1. Let H be a real Hilbert space, $M_i : H \rightarrow 2^H$ a maximal monotone mapping, and $B_i : H \rightarrow H$ a ξ_i -cocoercive mapping for each $i = 1, 2, \dots, N$. Let $\{T_n : H \rightarrow H\}$ be an infinite family of nonexpansive mappings. Define a mapping $S_n : H \rightarrow H$ by

$$S_n x = \alpha x + (1 - \alpha)T_n x, \quad \forall x \in H, \quad (4.1)$$

for all $n \in \mathbb{N}$, where $\alpha \in [0, 1)$. Let $W_n : H \rightarrow H$ be a W -mapping generated by $\{S_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0, \mu]$, for some $\mu \in (0, 1)$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{i=1}^N \text{VI}(H, B_i, M_i)) \neq \emptyset$. For $x_1 = u \in H$ chosen arbitrarily, suppose that $\{x_n\}$ is generated iteratively by

$$\begin{aligned} y_n &= \alpha_n W_n x_n + (1 - \alpha_n) \sum_{i=1}^N \beta_i J_{M_i, \lambda_i}(x_n - \lambda_i B_i x_n), \\ \epsilon_n &= \alpha_n (1 - \alpha_n) \left\| W_n x_n - \sum_{i=1}^N \beta_i J_{M_i, \lambda_i}(x_n - \lambda_i B_i x_n) \right\|^2, \\ C_{n+1} &= \left\{ z \in C_n \cap Q_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \epsilon_n \right\}, \\ Q_{n+1} &= \{z \in C_n \cap Q_n : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ C_1 &= Q_1 = H, \\ x_{n+1} &= P_{C_{n+1} \cap Q_{n+1}}(x_1), \quad \forall n \in \mathbb{N}, \end{aligned} \quad (4.2)$$

where

- (C1) $\{\alpha_n\} \subset [a, b]$ such that $0 < a < b < 1$,
- (C2) $\beta_i \in (0, 1)$ and $\lambda_i \in (0, 2\xi_i]$ for each $i = 1, 2, \dots, N$,
- (C3) $\sum_{i=1}^N \beta_i = 1$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $w = P_\Omega(x_1)$.

Proof. It is concluded from Theorem 3.1 immediately, by putting $k = 0$. $\square$

Theorem 4.2. Let C be a nonempty closed convex subset of a real Hilbert space H and $\{T_n : C \rightarrow C\}$ an infinite family of k -strictly pseudocontractive mappings with a fixed point such that $k \in [0, 1)$. Define a mapping $S_n : C \rightarrow C$ by

$$S_n x = \alpha x + (1 - \alpha)T_n x, \quad \forall x \in C, \quad (4.3)$$

for all $n \in \mathbb{N}$, where $\alpha \in [k, 1)$. Let $W_n : C \rightarrow C$ be a W -mapping generated by $\{S_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0, \mu]$, for some $\mu \in (0, 1)$. Assume that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For $x_1 = u \in C$ chosen arbitrarily, suppose that $\{x_n\}$ is generated iteratively by

$$\begin{aligned} y_n &= \alpha_n W_n x_n + (1 - \alpha_n)x_n, \\ \epsilon_n &= \alpha_n(1 - \alpha_n)\|W_n x_n - x_n\|^2, \\ C_{n+1} &= \left\{ z \in C_n \cap Q_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \epsilon_n \right\}, \\ Q_{n+1} &= \{z \in C_n \cap Q_n : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ C_1 &= Q_1 = C, \\ x_{n+1} &= P_{C_{n+1} \cap Q_{n+1}}(x_1), \quad \forall n \in \mathbb{N}, \end{aligned} \quad (4.4)$$

where $\{\alpha_n\} \subset [a, b]$ such that $0 < a < b < 1$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $w = P_\Omega(x_1)$.

Proof. It is concluded from Theorem 3.1 immediately, by putting $M_i \equiv B_i \equiv 0$ for all $i = 1, 2, \dots, N$. $\square$

Theorem 4.3. Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \rightarrow C$ a nonexpansive mapping. Assume that $F(T) \neq \emptyset$. For $x_1 = u \in C$ chosen arbitrarily, suppose that $\{x_n\}$ is generated iteratively by

$$\begin{aligned} y_n &= \sigma_n T x_n + (1 - \sigma_n)x_n \\ \delta_n &= \sigma_n(1 - \sigma_n)\|T x_n - x_n\|^2, \\ D_{n+1} &= \left\{ z \in D_n \cap Q_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \delta_n \right\}, \\ Q_{n+1} &= \{z \in D_n \cap Q_n : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \end{aligned}$$

$$\begin{aligned}
D_1 &= Q_1 = C, \\
x_{n+1} &= P_{D_{n+1} \cap Q_{n+1}}(x_1), \quad \forall n \in \mathbb{N},
\end{aligned} \tag{4.5}$$

where $\{\sigma_n\} \subset [a, b]$ such that $0 < a < b < 1$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $w = P_{F(T)}(x_1)$.

Proof. It is concluded from Theorem 4.2, by putting $\alpha = 0$. Setting $T_1 \equiv T$, $T_n \equiv I$ for all $n = 2, 3, \dots$ and letting $\mu_n \in (0, \mu]$ for some $\mu \in (0, 1)$, therefore, from the definition of S_n in Theorem 4.2, we have $S_1 = T_1 = T$ and $S_n = I$ for all $n = 2, 3, \dots$. Since W_n is a W -mapping generated by $\{S_n\}$ and $\{\mu_n\}$, therefore, by the definition of $U_{n,i}$ and W_n in (1.8), we have $U_{n,i} = I$ for all $i = 2, 3, \dots$ and $W_n = U_{n,1} = \mu_1 S_1 U_{n,2} + (1 - \mu_1)I = \mu_1 T + (1 - \mu_1)I$. Hence, by Theorem 4.2, we obtain

$$\begin{aligned}
y_n &= \alpha_n W_n x_n + (1 - \alpha_n) x_n \\
&= \alpha_n (\mu_1 T x_n + (1 - \mu_1) x_n) + (1 - \alpha_n) x_n \\
&= \sigma_n T x_n + (1 - \sigma_n) x_n,
\end{aligned} \tag{4.6}$$

where $\sigma_n := \alpha_n \mu_1$. Since, the same as in the proof of Theorem 3.1, we have that $D_n \cap Q_n$ is a nonempty closed convex subset of C for all $n \in \mathbb{N}$ and by Theorem 4.2, we have

$$\begin{aligned}
\epsilon_n &= \alpha_n (1 - \alpha_n) \|W_n x_n - x_n\|^2 \\
&= \alpha_n (1 - \alpha_n) \|\mu_1 T x_n + (1 - \mu_1) x_n - x_n\|^2 \\
&= (\alpha_n \mu_1) (\mu_1 - \mu_1 \alpha_n) \|T x_n - x_n\|^2 \\
&= \sigma_n (\mu_1 - \sigma_n) \|T x_n - x_n\|^2 \\
&\leq \sigma_n (1 - \sigma_n) \|T x_n - x_n\|^2 \\
&= \delta_n,
\end{aligned} \tag{4.7}$$

for all $n \in \mathbb{N}$. It follows that $D_n \subset C_n$ for all $n \in \mathbb{N}$, where C_n is defined as in Theorem 4.2. Hence, by Theorem 4.2, we obtain the desired result. This completes the proof. $\square$

References

- [1] W. Takahashi, *Nonlinear Functional Analysis, Fixed Point Theory and its Application*, Yokohama Publishers, Yokohama, Japan, 2000.
- [2] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, "Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space," *Nonlinear Analysis. Theory, Methods & Applications Series A*, vol. 67, no. 8, pp. 2350–2360, 2007.
- [3] H. H. Bauschke, "The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 202, no. 1, pp. 150–159, 1996.
- [4] M. Shang, Y. Su, and X. Qin, "Strong convergence theorems for a finite family of nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2007, Article ID 76971, 9 pages, 2007.

- [5] K. Shimoji and W. Takahashi, "Strong convergence to common fixed points of infinite nonexpansive mappings and applications," *Taiwanese Journal of Mathematics*, vol. 5, no. 2, pp. 387–404, 2001.
- [6] S.-S. Zhang, J. H. W. Lee, and C. K. Chan, "Algorithms of common solutions to quasi variational inclusion and fixed point problems," *Applied Mathematics and Mechanics*, vol. 29, no. 5, pp. 571–581, 2008.
- [7] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.
- [8] M. Kikkawa and W. Takahashi, "Approximating fixed points of infinite nonexpansive mappings by the hybrid method," *Journal of Optimization Theory and Applications*, vol. 117, no. 1, pp. 93–101, 2003.
- [9] Y. Su and X. Qin, "Strong convergence of monotone hybrid method for fixed point iteration processes," *Journal of Systems Science & Complexity*, vol. 21, no. 3, pp. 474–482, 2008.
- [10] A. Tada and W. Takahashi, "Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem," *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 359–370, 2007.
- [11] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, Japan, 2009.
- [12] R. E. Bruck, Jr., "Properties of fixed-point sets of nonexpansive mappings in Banach spaces," *Transactions of the American Mathematical Society*, vol. 179, pp. 251–262, 1973.
- [13] F. E. Browder, "Nonlinear monotone operators and convex sets in Banach spaces," *Bulletin of the American Mathematical Society*, vol. 71, pp. 780–785, 1965.
- [14] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.