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Research Article

The Fundamental Groups of *m*-Quasi-Einstein Manifolds

Hee Kwon Lee

School of Mathematics, KIAS, 207-43 Cheongnyangni 2-dong, Dongdaemun-gu, Seoul 130–722, Republic of Korea

Correspondence should be addressed to Hee Kwon Lee, heekwon@kias.re.kr

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In Ricci flow theory, the topology of Ricci soliton is important. We call a metric quasi-Einstein if the m-Bakry-Emery Ricci tensor is a constant multiple of the metric tensor. This is a generalization of gradient shrinking Ricci soliton. In this paper, we will prove the finiteness of the fundamental group of m-quasi-Einstein with a positive constant multiple.

1. Introduction and Main Results

Ricci flow is introduced in 1982 and developed by Hamilton (cf. [1]):

$$\begin{aligned} &\frac{\partial}{\partial t} g = -2\mathrm{Ric}, \\ &g(0) = g_0. \end{aligned} \tag{1.1}$$

Recently, Perelman supplemented Hamilton's result and solved the Poincaré Conjecture and the Geometrization Conjecture by using a Ricci flow theory. But in higher dimension greater than 4 classification using Ricci flow is still far-off. Most above all the classification of Ricci solitons, which are singularity models, is not completed. But there exist many properties of Ricci solitons. Here we say g is a Ricci soliton if (M,g) is a Riemannian manifold such that the identity

$$\mathrm{Ric} + L_X g = cg \tag{1.2}$$

holds for some constant c and some complete vector field X on M. If $c > 0$, $c = 0$, or $c < 0$, then we call it shrinking, steady, or expanding. Moreover, if the vector field X appearing in (1.2) is the gradient field of a potential function $(1/2)f$, one has $\mathrm{Ric} + \nabla\nabla f = cg$ and says g is a gradient Ricci soliton. In 2008, Lōpez and Río have shown that if (M, g) is a complete manifold with $\mathrm{Ric} + L_X g \geq cg$ and some positive constant c, then M is compact if and only if $\|X\|$ is bounded. Moreover, under these assumptions if M is compact, then $\pi_1(M)$ is finite. Furthermore, Wylie [2] has shown that under these conditions if M is complete, then $\pi_1(M)$ is finite. Moreover, in 2008, Fang et al. (cf. [3]) have shown that a gradient shrinking Ricci soliton with a bounded scalar curvature has finite topological type. By [4, Proposition 1.5.6], Cao and Zhu have shown that compact steady or expanding Ricci solitons are Einstein manifolds. In addition by [4, Corollary 1.5.9 (ii)] note that compact shrinking Ricci solitons are gradient Ricci solitons. So we are interested in shrinking gradient Ricci solitons. In [6, page 354], Eminenti et al. have shown that compact shrinking Ricci solitons have positive scalar curvatures. In [6] Case et al. have shown that an m-quasi-Einstein with $1 \leq m < \infty$ and $c > 0$ has a positive scalar curvature. Let me introduce the definition of m-quasi-Einstein.

Definition 1.1. The triple (M, g, f) is an m-quasi-Einstein manifold if it satisfies the equation

$$\mathrm{Ric} + \mathrm{Hess} f - \frac{1}{m} df \otimes df = cg \tag{1.3}$$

for some $c \in R$.

Here m-Bakry-Emery Ricci tensor $\mathrm{Ric}_f^m \doteq \mathrm{Ric} + \mathrm{Hess} f - (1/m) df \otimes df$ for $0 < m \leq \infty$ is a natural extension of the Ricci tensor to smooth metric measure spaces (cf. [6, Section 1]). Note that if $m = \infty$, then it reduces to a gradient Ricci soliton. In this paper, we will prove the finiteness of the fundamental group of an m-quasi-Einstein with $c > 0$.

Theorem 1.2. *Let (M, g, f) be a complete manifold with $c > 0$ and $\mathrm{Ric} + \mathrm{Hess} f - (1/m) df \otimes df \geq cg$. Then it has a finite fundamental group.*

2. The Proof of Theorem 1.2

The proof of Theorem 1.2 is similar to the proofs of [2, 7].

Proof. We will prove it by dividing into two cases.

Case 1. $\|\nabla f\|$ is bounded. We claim that the bounded $\|\nabla f\|$ implies the compactness of M. Let q be a point in M, and consider any geodesic $\gamma : [0, \infty) \to M$ emanating from q and parametrized by arc length t. Then we have

$$\int_0^T \mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) \geq cT + \frac{1}{m}\int_0^T (df(\dot{\gamma}))^2 - \int_0^T \dot{\gamma}(g(\nabla f, \dot{\gamma})) \geq cT - g(\nabla f, \dot{\gamma})|_0^T. \tag{2.1}$$

Since $g(\nabla f, \dot{\gamma})|_0^T$ is bounded we have that $\int_0^\infty \mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) = \infty$. Hence, the claim is followed by the proof of [4, Theorem 1]. Let $(\widetilde{M}, \widetilde{g})$ be the Riemannian universal cover of (M, g), let $p : (\widetilde{M}, \widetilde{g}) \to (M, g)$ be a projection map, and let $\widetilde{f}$ be a map $f \circ p$. Since p is a local isometry, then the same inequality holds, that is, $\mathrm{Ric}(\widetilde{g}) + \mathrm{Hesse}_{\widetilde{g}} \widetilde{f} - (1/m) d\widetilde{f} \otimes d\widetilde{f} \geq c\widetilde{g}$. Now, since

$\|\tilde{\nabla}\tilde{f}\|$ is bounded, it is followed from the above argument that $\widetilde{M}$ is compact. So $\pi_1(M)$ is finite.

Case 2. $\|\nabla f\|$ is unbounded. We will prove this case by following the proof of [2]. By Case 1, M is noncompact. For any $p \in M$, define

$$H_p \doteq \max\{0, \sup\{\mathrm{Ric}_y(v,v) : y \in B(p,1),\ \|v\| = 1\}\}. \tag{2.2}$$

Note that by [7, Lemma 2.2] we have

$$\int_0^r \mathrm{Ric}(\dot{\gamma},\dot{\gamma})ds \le 2(n-1) + H_p + H_q. \tag{2.3}$$

Assume that $d(p,q) > 1$. On the other hand, from the inequality of Theorem 1.2, we have

$$\begin{aligned}\int_0^r \mathrm{Ric}(\dot{\gamma},\dot{\gamma})ds &\ge cd(p,q) + \frac{1}{m}\int_0^r (df(\dot{\gamma}))^2 - \int_0^r \dot{\gamma}(g(\nabla f,\dot{\gamma})) \\ &\ge cd(p,q) - \|\nabla f\|_p - \|\nabla f\|_q,\end{aligned} \tag{2.4}$$

since $g(\nabla f,\dot{\gamma}) \le \|\nabla f\|\|\dot{\gamma}\|$. Hence, we have that for any $p, q \in M$

$$d(p,q) \le \max\left\{1, \frac{1}{c}\left(2(n-1) + H_p + H_q + \|\nabla f\|_p + \|\nabla f\|_q\right)\right\}. \tag{2.5}$$

Now we will apply a similar argument like Case 1. Fix $\tilde{p} \in \widetilde{M}$, and let $h \in \pi_1(M)$ identified as a deck transformation on $\widetilde{M}$. Note that $B(\tilde{p},1)$ and $B(h(\tilde{p}),1)$ are isometric, and thus $H_{\tilde{p}} = H_{h(\tilde{p})}$. Also $\|\tilde{\nabla}\tilde{f}\|_{\tilde{p}} = \|\tilde{\nabla}\tilde{f}\|_{h(\tilde{p})}$. So we conclude that

$$d(\tilde{p},h(\tilde{p})) \le \max\left\{1, \frac{2}{c}\left(\mathrm{n} - 1 + H_{\tilde{p}} + \left\|\tilde{\nabla}\tilde{f}\right\|_{\tilde{p}}\right)\right\} \tag{2.6}$$

for any $h \in \pi_1(M)$. Since the right-hand side is independent of h, this proves this case. □

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