Research Article

# On the Idempotent Solutions of a Kind of Operator Equations 

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This paper provides some relations between the idempotent operators and the solutions to operator equations $A B A=A^{2}$ and $B A B=B^{2}$.

## 1. Introduction

Let $\mathscr{L}$ be a complex Hilbert space. Denote by $\mathcal{B}(\mathscr{L})$ the Banach algebra of all bounded linear operators on $\mathscr{H}$. For $A, B \in B(\mathscr{L})$, if $A$ and $B$ satisfy the relations

$$
\begin{equation*}
A B A=A^{2}, \quad B A B=B^{2}, \tag{1.1}
\end{equation*}
$$

we say the pair of $(A, B)$ is the solution to (1.1). In [1], Vidav has investigated the self-adjoint solutions to (1.1) and showed that the pair of $(A, B)$ is self-adjoint solution to (1.1) if and only if there exists unique idempotent operator $P$ such that $A=P P^{*}$ and $B=P^{*} P$. In [2], Rakočević gave another proof of this result by using some properties of generalized inverses. In [3], Schmoeger generalized the Vidav's result concerning (1.1) by using some properties of Drazin inverses. The aim of this paper is to investigate some connections between idempotent operators and the solutions to (1.1). We prove main results as follows.
(1) $A$ and $B$ are idempotent solution to (1.1) if and only if there exist idempotent operators $P$ and $Q$ satisfying (1.1) such that $A=P Q$ and $B=Q P$.
(2) If $A$ is generalized Drazin invertible such that $A^{\pi} B\left(I-A^{\pi}\right)=0$. Then $A$ and $B$ satisfy (1.1) if and only if $A=P_{1}+N_{1}$ and $B=P_{2}+N_{2}$, where $N_{1}$ and
$N_{2}$ are arbitrary quasinilpotent elements satisfying (1.1), $P_{1}$ and $P_{2}$ are arbitrary idempotent elements satisfying $\mathcal{R}\left(P_{1}\right)=\mathcal{R}\left(P_{2}\right)$ and $P_{i} \perp^{c} N_{j}, i, j=1,2$.

Before proving the main results in this paper, let us introduce some notations and terminology which are used in the later. For $T \in \mathcal{B}(\mathscr{H})$, we denote by $\mathcal{R}(T), \mathcal{N}(T), \sigma_{p}(T)$ and $\sigma(T)$ the range, the null space, the point spectrum, and the spectrum of $T$, respectively. An operator $P \in \mathbb{B}(\mathscr{H})$ is said to be idempotent if $P^{2}=P . P$ is called an orthogonal projection if $P=P^{2}=P^{*}$, where $P^{*}$ denotes the adjoint of $P$. An operator $A \in B(\mathscr{H})$ is unitary if $A A^{*}=A^{*} A=I$. $A$ is positive if $(A x, x) \geq 0$ for all $x \in \mathscr{H}$ and its unique positive square root is denoted by $A^{1 / 2}$. For a closed subspace $\mathcal{K}$ of $\mathscr{H},\left.T\right|_{\mathcal{K}}$ denotes the restriction of $T$ on $\not \mathscr{K}$ and $P_{\nless}$ denotes the orthogonal projection onto $\mathcal{K}$. The generalized Drazin inverse (see $[4,5]$ ) is the element $T^{d} \in \mathcal{B}(\mathscr{H})$ such that

$$
\begin{equation*}
T T^{d}=T^{d} T, \quad T^{d} T T^{d}=T^{d}, \quad T-T^{2} T^{d} \text { is quasinilpotent. } \tag{1.2}
\end{equation*}
$$

It is clear $T^{d}=T^{-1}$ if $T \in \mathcal{B}(\mathscr{H})$ is invertible. If $T$ is generalized Drazin invertible, then the spectral idempotent $T^{\pi}$ of $T$ corresponding to $\{0\}$ is given by $T^{\pi}=I-T T^{d}$. The operator matrix form of $T$ with respect to the space decomposition $\mathscr{H}=\mathcal{N}\left(T^{\pi}\right) \oplus \mathcal{R}\left(T^{\pi}\right)$ is given by $T=T_{1} \oplus T_{2}$, where $T_{1}$ is invertible and $T_{2}$ is quasinilpotent.

## 2. Some Lemmas

To prove the main results, some lemmas are needed.
Lemma 2.1 (see [6, Lemma 1.1]). Let $P$ be an idempotent in $\mathcal{B}(\mathscr{H})$. Then there exists an invertible operator $S \in B(\mathscr{H})$ such that $S P S^{-1}$ is an orthogonal projection.

Lemma 2.2 (see [7, Theorem 2.1]). Let $P, Q \in B(\mathscr{H})$ with $P=P^{2}$ and $Q=Q^{2}=Q^{*}$. If $\mathcal{R}(P)=$ $\mathcal{R}(Q)$, then $P+P^{*}-I$ is always invertible and

$$
\begin{equation*}
Q=P\left(P+P^{*}-I\right)^{-1}=\left(P+P^{*}-I\right)^{-1} P^{*} \tag{2.1}
\end{equation*}
$$

Lemma 2.3 (see [8, Remark 1.2.1]). Let $A, B \in B(\mathscr{H})$. Then $\sigma(A B) \backslash\{0\}=\sigma(B A) \backslash\{0\}$.
Lemma 2.4 (see $[9,10]$ ). Let $A \in B(H)$ have the matrix form $A=\left(\begin{array}{ll}A_{11} & A_{12} \\ A_{21} & A_{22}\end{array}\right)$. Then $A \geq 0$ if and only if $A_{i i} \geq 0, i=1,2, A_{21}=A_{12}^{*}$ and there exists a contraction operator $D$ such that $A_{12}=$ $A_{11}^{1 / 2} D A_{22}^{1 / 2}$.

Lemma 2.5. Let $A, B \in B(\mathscr{H})$ with $A B A=A^{2}$ and $B A B=B^{2}$. Then

$$
\begin{equation*}
(A B)^{k}=A^{k} B=A B^{k}, \quad A^{k} B^{l}=A^{k+l-1} B=A B^{k+l-1} \tag{2.2}
\end{equation*}
$$

for all nonnegative integer $k, l \geq 1$.

Proof. The conditions $A B A=A^{2}$ and $B A B=B^{2}$ imply that $(A B)^{2}=A B A B=A^{2} B=A B^{2}$. Now suppose $(A B)^{k}=A^{k} B=A B^{k}$ holds for nonnegative integer $2 \leq k \leq m$. Then, for $k=m+1$, we have

$$
\begin{align*}
(A B)^{m+1} & =(A B)^{2}(A B)^{m-1}=A^{2} B(A B)^{m-1}=A(A B)^{m}=A^{m+1} B \\
& =A A^{m} B=A^{2} B^{m}=A B^{2} B^{m-1}=A B^{m+1} \tag{2.3}
\end{align*}
$$

Hence $(A B)^{k}=A^{k} B=A B^{k}$ and $A^{k} B^{l}=A^{k-1} A B^{l}=A^{k-1} A^{l} B=A^{k+l-1} B=A B^{k+l-1}$ for all nonnegative integer $k, l \geq 1$.

An element $T \in \mathbb{B}(\mathscr{H})$ whose spectrum $\sigma(T)$ consists of the set $\{0\}$ is said to be quasinilpotent [8]. It is clear that $T$ is quasi-nilpotent if and only if the spectral radius $\gamma(T)=$ $\sup \{|\lambda|: \lambda \in \sigma(T)\}=0$. In particular, if there exists a positive integer $m$ such that $A^{m}=0$, then $A$ is $m$-nilpotent element. For the quasi-nilpotent operator, we have the following results.

Lemma 2.6. Let $A$ and $B$ satisfy (1.1). Then $A$ is quasinilpotent if and only if $B$ is quasinilpotent. In particular, $A$ is nilpotent if and only if $B$ is nilpotent; if $A$ is quasinilpotent and $A B=B A$, then $A^{2}=B^{2}=0$.

Proof. Because $\sigma\left(A^{2}\right) \cup\{0\}=\sigma(A B A) \cup\{0\}=\sigma\left(A^{2} B\right) \cup\{0\}=\sigma\left(A B^{2}\right) \cup\{0\}=\sigma(B A B) \cup\{0\}=$ $\sigma\left(B^{2}\right) \cup\{0\}$, it follows that $A$ is quasinilpotent if and only if $B$ is quasinilpotent.

By Lemma 2.5, if there is a nonnegative integer $m \geq 1$ such that $A^{m}=0$, then

$$
\begin{equation*}
B^{m+1}=B^{m-1} B A B=B A^{m} B=0 \tag{2.4}
\end{equation*}
$$

Similarly we can show that $A^{n+1}=0$ if $B^{n}=0$. Hence $A$ is nilpotent if and only if $B$ is nilpotent.
If $A$ is quasinilpotent, then $B$ is quasinilpotent, so $I-A$ and $I-B$ are invertible. From the condition $A B=B A$, we obtain $A^{2}(I-B)=A^{2}-A B A=0, B^{2}(I-B)=B^{2}-B A B=0$. It follows $A^{2}=0$ and $B^{2}=0$.

Lemma 2.7. Let $A$ and $B$ satisfy (1.1). Then for every integer $k \geq 1$,

$$
\begin{equation*}
\sigma\left(A^{2}\right)=\sigma\left(B^{2}\right), \quad \sigma\left(A^{k} B\right)=\sigma\left(B^{k} A\right) \tag{2.5}
\end{equation*}
$$

Proof. Since $A^{2} B=A B^{2}$ by Lemma 2.5,

$$
\begin{align*}
& \sigma\left(A^{2} B\right) \cup\{0\}=\sigma(A B A) \cup\{0\}=\sigma\left(A^{2}\right) \cup\{0\}  \tag{2.6}\\
& \sigma\left(A B^{2}\right) \cup\{0\}=\sigma(B A B) \cup\{0\}=\sigma\left(B^{2}\right) \cup\{0\}
\end{align*}
$$

Note that $A$ is invertible if and only if $B$ is invertible. We get $\sigma\left(A^{2}\right)=\sigma\left(B^{2}\right)$. Next, if $0 \notin$ $\sigma(A B)$, then from $(A B)^{2}=A^{2} B$ we obtain $A B=A$, that is, $A$ is invertible. It follows that $A=B=I$ because $A B A=A^{2}$ and $B A B=B^{2}$, so $\sigma(A B)=\sigma(B A)$. Now,

$$
\begin{align*}
& \sigma\left(A^{k} B\right)=\sigma\left((A B)^{k}\right)=\left\{\lambda^{k}: \lambda \in \sigma(A B)\right\} \\
& \sigma\left(B^{k} A\right)=\sigma\left((B A)^{k}\right)=\left\{\mu^{k}: \mu \in \sigma(B A)\right\} \tag{2.7}
\end{align*}
$$

Hence, $\sigma\left(A^{k} B\right)=\sigma\left(B^{k} A\right)$ for every integer $k \geq 1$.

## 3. Idempotent Solutions

In this section, we will show that the solutions to (1.1) have a closed connection with the idempotent operators. Our main results are as follows.

Theorem 3.1. The following assertions are equivalent.
(a) A and B are idempotent solution to (1.1).
(b) There exist idempotent operators $P$ and $Q$ satisfying (1.1) such that

$$
\begin{equation*}
A=P Q, \quad B=Q P \tag{3.1}
\end{equation*}
$$

Proof. Clearly, we only needs prove that (a) implies (b). Since $A$ and $B$ are idempotent operators, $A B A=A^{2}$ and $B A B=B^{2}$, without loss of generality, we can assume that one of $A$ and $B$ is orthogonal projection by Lemma 2.1. For example, assume that $B$ is an orthogonal projection. From $A B A=A$, we obtain $\mathcal{N}\left(\left.B\right|_{\mathcal{R}(A)}\right)=0$. Since $B$ is an orthogonal projection and $B A B=B$, we have $B A^{*} B=B$ and $\mathcal{N}\left(\left.I\right|_{\mathcal{R}(A)^{\perp}}-\left.B\right|_{\mathcal{R}(A)^{\perp}}\right)=0$. By Lemma 2.4, $A$ and $B$ can be written in the forms of

$$
A=\left(\begin{array}{cccc}
I & 0 & P_{13} & P_{14}  \tag{3.2}\\
& I & P_{23} & P_{24} \\
& & 0 & \\
& & & \\
& & & \\
& Q_{1} & Q_{1}^{1 / 2} D Q_{2}^{1 / 2} & \\
& Q_{2}^{1 / 2} D^{*} Q_{1}^{1 / 2} & Q_{2} & \\
& & &
\end{array}\right)
$$

with respect to the space decomposition $\mathscr{H}=\sum_{i=1}^{4} \oplus \mathscr{H}_{i}$, respectively, where $\mathscr{H}_{1}=\mathcal{N}\left(I_{\mathcal{R}(A)}-\right.$ $\left.\left.B\right|_{\mathcal{R}(A)}\right), \mathscr{H}_{2}=\mathcal{R}(A) \ominus \mathscr{H}_{1}, \mathscr{H}_{4}=\mathcal{N}\left(\left.B\right|_{\mathcal{R}(A)^{\perp}}\right), \mathscr{H}_{3}=\mathcal{R}(A)^{\perp} \ominus \mathscr{H}_{4}$ and the entries omitted are zero. It is easy to see that $Q_{i}$ as operators on $\mathscr{H}_{1+i}, i=1,2$, are injective positive contractions, and $D$ is a contraction from $\mathscr{H}_{3}$ into $\mathscr{H}_{2}$ by Lemma 2.4. Since $B$ is an orthogonal projection,

$$
\left(\begin{array}{cc}
Q_{1} & Q_{1}^{1 / 2} D Q_{2}^{1 / 2}  \tag{3.3}\\
Q_{2}^{1 / 2} D^{*} Q_{1}^{1 / 2} & Q_{2}
\end{array}\right)^{2}=\left(\begin{array}{cc}
Q_{1} & Q_{1}^{1 / 2} D Q_{2}^{1 / 2} \\
Q_{2}^{1 / 2} D^{*} Q_{1}^{1 / 2} & Q_{2}
\end{array}\right)
$$

that is,

$$
\left(\begin{array}{cc}
Q_{1}^{2}+Q_{1}^{1 / 2} D Q_{2} D^{*} Q_{1}^{1 / 2} & Q_{1}^{3 / 2} D Q_{2}^{1 / 2}+Q_{1}^{1 / 2} D Q_{2}^{3 / 2}  \tag{3.4}\\
Q_{2}^{1 / 2} D^{*} Q_{1}^{3 / 2}+Q_{2}^{3 / 2} D^{*} Q_{1}^{1 / 2} & Q_{2}^{2}+Q_{2}^{1 / 2} D^{*} Q_{1} D Q_{2}^{1 / 2}
\end{array}\right)=\left(\begin{array}{cc}
Q_{1} & Q_{1}^{1 / 2} D Q_{2}^{1 / 2} \\
Q_{2}^{1 / 2} D^{*} Q_{1}^{1 / 2} & Q_{2}
\end{array}\right)
$$

Comparing both sides of the above equation and observing that self-adjoint operators $Q_{i}$, $I-Q_{i}, i=1,2$ are injective, by a straightforward computation we obtain

$$
\begin{equation*}
Q_{2}=D^{*}\left(I-Q_{1}\right) D, \quad D D^{*}=I, \quad D^{*} D=I \tag{3.5}
\end{equation*}
$$

Hence

$$
B=I \oplus\left(\begin{array}{cc}
Q_{1} & Q_{1}^{1 / 2}\left(I-Q_{1}\right)^{1 / 2} D  \tag{3.6}\\
D^{*}\left(I-Q_{1}\right)^{1 / 2} Q_{1}^{1 / 2} & D^{*}\left(I-Q_{1}\right) D
\end{array}\right) \oplus 0
$$

where 0 and 1 are not in $\sigma_{p}\left(Q_{1}\right), D$ is unitary from $\mathscr{H}_{3}$ onto $\mathscr{H}_{2}$ (see [11] and Lemma 1 in [12]). Denote by $A B A=\left(T_{i j}\right)_{1 \leq i, j \leq 4}$. A direct computation shows that

$$
\begin{equation*}
T_{12}=P_{13} D^{*} Q_{1}^{1 / 2}\left(I-Q_{1}\right)^{1 / 2}, \quad T_{22}=Q_{1}+P_{23} D^{*} Q_{1}^{1 / 2}\left(I-Q_{1}\right)^{1 / 2} \tag{3.7}
\end{equation*}
$$

and $A B A=A$ if and only if $T_{12}=0$ and $T_{22}=I$. Since $Q_{1}$ and $I-Q_{1}$ injective self-adjoint operators, we obtain $P_{13}=0$ and $P_{23} D^{*} Q_{1}^{1 / 2}=\left(I-Q_{1}\right)^{1 / 2}$. Moreover, we can show $B A B=B$ when $P_{13}=0$ and $P_{23} D^{*} Q_{1}^{1 / 2}=\left(I-Q_{1}\right)^{1 / 2}$. Hence,

$$
A=\left(\begin{array}{cccc}
I & 0 & 0 & P_{14}  \tag{3.8}\\
& I & P_{23} & P_{24} \\
& & 0 & \\
& & & 0
\end{array}\right), \quad B=\left(\begin{array}{ccc}
I & & \\
& Q_{1} & Q_{1}^{1 / 2}\left(I-Q_{1}\right)^{1 / 2} D \\
D^{*}\left(I-Q_{1}\right)^{1 / 2} Q_{1}^{1 / 2} & D^{*}\left(I-Q_{1}\right) D & \\
& &
\end{array}\right.
$$

where $Q_{1}$ is a contraction on $\mathscr{H}_{2}, 0$ and 1 are not in $\sigma_{p}\left(Q_{1}\right), D$ is unitary from $\mathscr{H}_{3}$ onto $\mathscr{H}_{2}$, $P_{i 4} \in \mathcal{B}\left(\mathscr{H}_{4}, \mathscr{H}_{i}\right), i=1,2$ are arbitrary, $P_{23} \in B\left(\mathscr{H}_{3}, \mathscr{H}_{2}\right)$ and $P_{23} D^{*} Q_{1}^{1 / 2}=\left(I-Q_{1}\right)^{1 / 2}$. Let

$$
\begin{align*}
& P=I \oplus\left(\begin{array}{cc}
I & P_{23} \\
0 & 0
\end{array}\right) \oplus 0, \\
& Q=\left(\begin{array}{ccc}
I & & P_{14} \\
& Q_{1} & Q_{1}^{1 / 2}\left(I-Q_{1}\right)^{1 / 2} D
\end{array}\right] Q_{1} P_{24} . \tag{3.9}
\end{align*}
$$

Then we can deduce that idempotent operators $P$ and $Q$ satisfy (1.1), $P Q=A$ and $Q P=$ $B$.

Theorem 3.1 shows that the arbitrary pair of idempotent solution $(A, B)$ can be written as $A=P Q, B=Q P$ with idempotent operators $P$ and $Q$ satisfying (1.1). Next, we discuss the uniqueness of the idempotent solution to (1.1).

Theorem 3.2. Let $B$ be given idempotent. Then (1.1) has unique idempotent solution $A$ if and only if $\mathcal{N}\left(\left.B\right|_{\mathcal{R}(A)^{\perp}}\right)=0$. In this case, $A, B$ satisfy $A B=A$ and $B A=B$.

Proof. Suppose that the pair $(A, B)$ being the idempotent solution to (1.1). By the proof of Theorem 3.1, if $\mathscr{H}_{4}=\mathcal{N}\left(\left.B\right|_{\mathcal{R}(A)^{\perp}}\right) \neq 0$, the idempotent solution $A$ is not unique because $P_{14}$ and $P_{24}$ are arbitrary elements; if $\mathscr{H}_{4}=\mathcal{N}\left(\left.B\right|_{\mathcal{R}(A)^{\perp}}\right)=0$, then $A$ and $B$ have the form

$$
A=\left(\begin{array}{ccc}
I & 0 & 0  \tag{3.10}\\
& I & P_{23} \\
& & 0
\end{array}\right), \quad B=\left(\begin{array}{ccc}
I & & \\
& Q_{1} & Q_{1}^{1 / 2}\left(I-Q_{1}\right)^{1 / 2} D \\
D^{*}\left(I-Q_{1}\right)^{1 / 2} Q_{1}^{1 / 2} & D^{*}\left(I-Q_{1}\right) D
\end{array}\right) .
$$

Since $Q_{1}=Q_{1}^{*}$ is injection, $D$ is unitary and $P_{23} D^{*} Q_{1}^{1 / 2}=\left(I-Q_{1}\right)^{1 / 2}$, so $Q_{1}^{1 / 2} D P_{23}^{*}=(I-$ $\left.Q_{1}\right)^{1 / 2}$ and $P_{23}^{*}=D^{*} Q_{1}^{-1 / 2}\left(I-Q_{1}\right)^{1 / 2}$. Hence, we obtain that $P_{23}$ is uniquely determined and $P_{23}=Q_{1}^{-1 / 2}\left(I-Q_{1}\right)^{1 / 2} D$. Therefore the idempotent solution $A$ is unique and $A B=A$ and $B A=B$.

The following result was first given by Vidav [1]. We give an alternative short proof.
Theorem 3.3 (see [1, Theorem 2]). The following assertions are equivalent.
(a) A and B are self-adjoint solution to (1.1).
(b) There is an idempotent operator $P$ such that

$$
\begin{equation*}
A=P P^{*}, \quad B=P^{*} P \tag{3.11}
\end{equation*}
$$

Proof. (b) implies (a) is clear. Now, suppose that (a) holds. From

$$
\begin{equation*}
A(B-I)^{2} A=A B^{2} A-2 A B A+A^{2}=A^{2} B A-A^{2}=A^{3}-A^{2} \tag{3.12}
\end{equation*}
$$

we have $A^{3}-A^{2}=A(B-I)(A(B-I))^{*} \geq 0$, so $\sigma\left(A^{3}-A^{2}\right) \subset[0, \infty)$. The spectral mapping theorem gives

$$
\begin{equation*}
\lambda^{3}-\lambda^{2} \geq 0, \quad \forall \lambda \in \sigma(A) \backslash\{0\} \tag{3.13}
\end{equation*}
$$

Thus, for $\forall \lambda \in \sigma(A) \backslash\{0\}$, we have $\lambda \geq 1$ and therefore $A \geq 0 . \mathcal{R}(A)$ is closed since 0 is not the accumulation point of $\sigma(A)$. Hence $A$ has the matrix form $A=A_{1} \oplus 0$ according to the space decomposition $\mathscr{H}=\mathcal{R}(A) \oplus \mathcal{R}^{\perp}(A)$, where $A_{1}$ is invertible. Similarly, we can derive that $B \geq 0$.

By Lemma 2.4, $B$ can be written as $B=\left(\begin{array}{l}B_{1} B_{2} \\ B_{2}^{*} \\ B_{4}\end{array}\right)$ with $B_{1} \geq 0$ and $B_{4} \geq 0$. From $A B A=A^{2}$, we have $B_{1}=I$. From $B A B=B^{2}$, we have

$$
\left.\begin{array}{c}
I+B_{2} B_{2}^{*}=A_{1},  \tag{3.14}\\
B_{2}^{*} B_{2}+B_{4}^{2}=B_{2}^{*} A_{1} B_{2}
\end{array}\right\} \Longrightarrow B_{4}^{2}=\left(B_{2}^{*} B_{2}\right)^{2} \Longrightarrow B_{4}=B_{2}^{*} B_{2},
$$

since the square root of a positive operator is unique. Define $P=\left(\begin{array}{cc}I & B_{2} \\ 0 & 0\end{array}\right)$. Then $P^{2}=P$,

$$
P P^{*}=\left(\begin{array}{cc}
I+B_{2} B_{2}^{*} & 0  \tag{3.15}\\
0 & 0
\end{array}\right)=A, \quad P^{*} P=\left(\begin{array}{cc}
I & B_{2} \\
B_{2}^{*} & B_{2}^{*} B_{2}
\end{array}\right)=B .
$$

The next characterizations of the solutions to (1.1) are clear.
Corollary 3.4. (a) For arbitrary idempotent operators $P$ and $Q, A=P Q, B=Q P$ are the solution to (1.1).
(b) If $A$ is an idempotent operator satisfying $A B=B A$, then $B$ is one solution to (1.1) if and only if there exists a square-zero operator $N_{0}$ such that $B=A+N_{0}$ and $A N_{0}=N_{0} A=0$.
(c) If $A$ is an orthogonal projection, then self-adjoint operator $B$ satisfies (1.1) if and only if $A=B$.

Proof. (a) See Theorem 2.2 in [3].
(b) By simultaneous similarity transformations, $A$ and $B$ can be written as $A=I \oplus 0$ and $B=N_{21} \oplus N_{22}$ since $A B=B A$. From $A B A=A^{2}$, we obtain $N_{21}=\mathrm{I}$. From $B A B=B^{2}$, we obtain $N_{22}^{2}=0$. Select $0 \oplus N_{22}=N_{0}$. Then $N_{0}^{2}=0, B=A+N_{0}$ and $A N_{0}=N_{0} A=0$.
(c) We use the notations from Theorem 3.3. If $A$ is an orthogonal projection, then $A_{1}=$ $I, B_{2}=0$ in the proof of Theorem 3.3, so the result is a direct corollary of Theorem 3.3.

## 4. The Perturbation of the Solutions

The operators $A$ and $B$ are said to be c-orthogonal, denoted by $A \perp^{c} B$, whenever $A B=0$ and $B A=0$. The next result is a generalization of Theorems 2.2 and 3.2 in [6], where the same problems have been considered for $\operatorname{ind}(A) \leq 1$ and $\operatorname{ind}(B) \leq 1$.

Theorem 4.1. Let $A$ be generalized Drazin invertible such that $A^{\pi} B\left(I-A^{\pi}\right)=0$. Then

$$
\begin{equation*}
A \text { and } B \text { satisfy (1.1) iff } A=P_{1}+N_{1}, B=P_{2}+N_{2} \tag{4.1}
\end{equation*}
$$

where $N_{1}$ and $N_{2}$ are arbitrary quasinilpotent elements satisfying (1.1), $P_{1}$ and $P_{2}$ are arbitrary idempotent elements satisfying $\mathcal{R}\left(P_{1}\right)=\mathcal{R}\left(P_{2}\right)$ and $P_{i} \perp^{c} N_{j}, i, j=1,2$.

Proof. Let us consider the matrix representation of $A$ and $B$ relative to the $P=I-A^{\pi}$. We have

$$
A=\left(\begin{array}{cc}
A_{1} & 0  \tag{4.2}\\
0 & A_{2}
\end{array}\right)_{P}, \quad B=\left(\begin{array}{ll}
B_{1} & B_{2} \\
B_{3} & B_{4}
\end{array}\right)_{P}
$$

where $A_{1}$ is invertible and $A_{2}$ is quasinilpotent. From $A B A=A^{2}$, we have

$$
\begin{equation*}
B_{1}=I, \quad B_{2} A_{2}=0, \quad A_{2} B_{3}=0, \quad A_{2} B_{4} A_{2}=A_{2}^{2} \tag{4.3}
\end{equation*}
$$

From $B A B=B^{2}$ and (4.3), we have

$$
\begin{gather*}
I+B_{2} B_{3}=A_{1} \\
B_{2}+B_{2} B_{4}=A_{1} B_{2} \\
B_{3}+B_{4} B_{3}=B_{3} A_{1}  \tag{4.4}\\
B_{3} B_{2}+B_{4}^{2}=B_{3} A_{1} B_{2}+B_{4} A_{2} B_{4} .
\end{gather*}
$$

From $A^{\pi} B\left(I-A^{\pi}\right)=0$, we have $B_{3}=0$. Now, it follows from (4.2), (4.3) and (4.4) that

$$
A=\left(\begin{array}{cc}
I & 0  \tag{4.5}\\
0 & A_{2}
\end{array}\right)_{P}, \quad B=\left(\begin{array}{ll}
I & B_{2} \\
0 & B_{4}
\end{array}\right)_{P}
$$

with

$$
\begin{equation*}
A_{2} B_{4} A_{2}=A_{2}^{2}, \quad B_{4} A_{2} B_{4}=B_{4}^{2}, \quad B_{2} A_{2}=0, \quad B_{2} B_{4}=0 \tag{4.6}
\end{equation*}
$$

Hence, $B_{4}$ is quasinilpotent by Lemma 2.6. Select

$$
\begin{equation*}
N_{1}=0 \oplus_{P} A_{2}, \quad N_{2}=0 \oplus_{P} B_{4}, \quad P_{1}=I \oplus_{P} 0, \quad P_{2}=B-N_{2} \tag{4.7}
\end{equation*}
$$

Then $P_{1}$ and $P_{2}$ are idempotent operators and $\mathcal{R}\left(P_{1}\right)=\mathcal{R}\left(P_{2}\right) . N_{1}$ and $N_{2}$ are quasinilpotent operators satisfying (1.1) and $P_{i} \perp^{c} N_{j}, i, j=1,2$.

For the proof of sufficiency observe that $\mathcal{R}\left(P_{1}\right)=\mathcal{R}\left(P_{2}\right)$ leads to $P_{1} P_{2}=P_{2}, P_{2} P_{1}=P_{1}$. Straightforward calculations show that $A B A=A^{2}$ and $B A B=B^{2}$.

We also prove the next result which can be seen as one corollary of Theorem 4.1.
Corollary 4.2. Let $A$ be generalized Drazin invertible such that $A^{\pi} A B=A^{\pi} B A$. Then

$$
\begin{equation*}
A \text { and } B \text { satisfy (1.1) iff } A=P_{1}+N_{1}, B=P_{2}+N_{2} \tag{4.8}
\end{equation*}
$$

where $N_{1}$ and $N_{2}$ are 2-nilpotent operators satisfying $N_{1} N_{2}=N_{2} N_{1}, P_{1}$ and $P_{2}$ are arbitrary idempotent elements satisfying $\mathcal{R}\left(P_{1}\right)=\mathcal{R}\left(P_{2}\right)$ and $P_{i} \perp^{c} N_{j}, i, j=1,2$.

Proof. The sufficiency is clear. For the proof of the necessity, let $A$ and $B$ have the matrix representation as (4.2). From $A B A=A^{2}$ and $B A B=B^{2}$, we know that $A_{i}, i=1,2$ and $B_{i}, i=$ $1,2,3,4$ satisfy (4.3) and (4.4). The condition $A^{\pi} A B=A^{\pi} B A$ implies that

$$
\begin{equation*}
A_{2} B_{3}=B_{3} A_{1}, \quad A_{2} B_{4}=B_{4} A_{2} \tag{4.9}
\end{equation*}
$$

It follows that $B_{3}=0$ because $A_{2} B_{3}=0$ and $A_{1}$ is invertible by (4.3) and (4.4). Also, (4.2), (4.3), and (4.4) imply $A_{2}$ is quasinilpotent and

$$
\begin{equation*}
A_{2}^{2}=A_{2} B_{4} A_{2}, \quad B_{4}^{2}=B_{4} A_{2} B_{4}, \quad B_{4} A_{2}=A_{2} B_{4} \tag{4.10}
\end{equation*}
$$

It follows immediately that $A_{2}^{2}=B_{4}^{2}=0$ by Lemma 2.6. Now, we obtain

$$
A=\left(\begin{array}{cc}
I & 0  \tag{4.11}\\
0 & A_{2}
\end{array}\right)_{P}, \quad B=\left(\begin{array}{ll}
I & B_{2} \\
0 & B_{4}
\end{array}\right)_{P}
$$

with

$$
\begin{equation*}
B_{2} B_{4}=0, \quad B_{2} A_{2}=0, \quad A_{2} B_{4}=B_{4} A_{2}, \quad A_{2}^{2}=B_{4}^{2}=0 \tag{4.12}
\end{equation*}
$$

Select

$$
\begin{equation*}
N_{1}=0 \oplus_{P} A_{2}, \quad N_{2}=0 \oplus_{P} B_{4}, \quad P_{1}=I \oplus_{P} 0, \quad P_{2}=B-N_{2} \tag{4.13}
\end{equation*}
$$

Then $P_{1}$ and $P_{2}$ are idempotent operators and $\mathcal{R}\left(P_{1}\right)=\mathcal{R}\left(P_{2}\right) . N_{1}^{2}=0$ and $N_{2}^{2}=0$ satisfying $P_{i} \perp^{c} N_{j}, i, j=1,2$ and $N_{1} N_{2}=N_{2} N_{1}$.

If we assume that $A B=B A$ instead of the condition $A^{\pi} A B=A^{\pi} B A$, we will get a much simpler expression for $A$ and $B$.

Corollary 4.3. Let $A$ be generalized Drazin invertible such that $A B=B A$. Then

$$
\begin{equation*}
A \text { and } B \text { satisfy (1.1) iff } A=P+N_{1}, B=P+N_{2} \tag{4.14}
\end{equation*}
$$

where $N_{1}$ and $N_{2}$ are 2-nilpotent operators satisfying $N_{1} N_{2}=N_{2} N_{1}, P$ is arbitrary idempotent element satisfying $P \perp^{c} N_{1}$ and $P \perp^{c} N_{2}$.

Proof. Similar to the proof of Theorem 4.1, Corollary 4.2. If $A B=B A$, then $A$ and $B$ have the matrix representations

$$
\begin{equation*}
A=I \oplus_{P} A_{2}, \quad B=I \oplus_{P} B_{4} \tag{4.15}
\end{equation*}
$$

with $A_{2}^{2}=B_{4}^{2}=0$ and $B_{4} A_{2}=A_{2} B_{4}$, so, by Corollary 4.2 , the result is clear.

Remark 4.4. (1) Let $\gamma(T)$ denote the spectrum radius of operator $T$. In Corollaries 4.2 and 4.3, since nilpotent operators $A_{2}$ and $B_{4}$ are commutative, we get

$$
\begin{equation*}
r\left(A_{2}-B_{4}\right) \leq r\left(A_{2}\right)+\gamma\left(-B_{4}\right)=0, \quad \gamma\left(B_{4} A_{2}\right) \leq \gamma\left(B_{4}\right) \gamma\left(A_{2}\right)=0 \tag{4.16}
\end{equation*}
$$

that is, $A_{2}-B_{4}$ and $B_{4} A_{2}$ are nilpotent. Hence, $A-B$ is a nilpotent operator and $B A$ can be decomposed as the $c$-orthogonality sum of an idempotent operator and a 2-nilpotent operator.
(2) Let $A$ have the Drazin inverse $A^{d}$. Then, by (4.2), $A$ can be written as $A=A_{1} \oplus A_{2}$, where $A_{1}$ is invertible and $A_{2}$ is quasi-nilpotent (see also $[4,5]$ ). If $A=A^{*}$, then

$$
\begin{equation*}
A^{d}=A^{\#}=A^{+}=A_{1}^{-1} \oplus 0 \tag{4.17}
\end{equation*}
$$

where $A^{+}$is the Moore-Penrose inverse of $A$ and $A^{\#}$ is the group inverse. In fact, if $A=A^{*}$, then $A_{2}=0$ because the self-adjoint quasi-nilpotent operator must be zero. Hence $A=A_{1} \oplus$ $0, A^{d}=A^{\#}=A^{+}=A_{1}^{-1} \oplus 0$.
(3) If self-adjoint operators $A$ and $B$ satisfy (1.1), then $\|A\| \geq 1$,

$$
\begin{equation*}
A B=B A \quad \text { iff }\|A\|=1 \text { iff } A=B=P_{\mathcal{N}(A)^{\perp}} \tag{4.18}
\end{equation*}
$$

where $P_{\mathcal{N}(A)^{\perp}}$ is the orthogonal projection on $\mathcal{N}(A)^{\perp}$. In fact, let $\mathscr{H}=\mathcal{N}(A)^{\perp} \oplus \mathcal{N}(A)$, then $A=A_{1} \oplus 0$. Select

$$
B=\left(\begin{array}{ll}
B_{1} & B_{2}  \tag{4.19}\\
B_{2}^{*} & B_{4}
\end{array}\right)
$$

Similar to the proof of Theorem 4.1, we have

$$
\begin{equation*}
B_{1}=I, \quad A_{1}=I+B_{2} B_{2}^{*}, \quad B_{2}^{*} A_{1} B_{2}=B_{2}^{*} A_{1} B_{2}+B_{4}^{2} \tag{4.20}
\end{equation*}
$$

This shows that $\|A\| \geq 1$ since $A_{1}=I+B_{2} B_{2}^{*} \geq I$. If $\|A\|=1$, then $A_{1}=I, B_{2}=0, B_{4}=0$. Hence $A=B=P_{\mathcal{N}(A)^{\perp}}$. If $A B=B A$, then $B_{2}=0$, so $A=I \oplus 0, B=I \oplus B_{4}$. From $B A B=B^{2}$, we have $B_{4}^{2}=0$, so $B_{4}=0$ since $B_{4}=B_{4}^{*}$. Hence $A=B=P_{\mathcal{N}(A)^{\perp}}$. These results (see Theorem 4.2 in [6]) can be seen as the particular case of Corollary 4.3.

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