Research Article

On Irreducible Linear Groups of Nonprimary Degree

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Received 6 May 2011; Accepted 5 June 2011

Academic Editor: A. Vourdas

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It is established that degree $2|A| + 1$ of irreducible complex linear group with the group A of cosimple automorphisms of odd order is a prime number and proved that if degree $2|H| + 1$ of π -solvable irreducible complex linear group G with a π -Hall TI -subgroup H is not a prime power, then H is Abelian and normal in G .

1. Introduction

Suppose that G is a finite group and A is such a group of its nontrivial automorphisms that $(|G|, |A|) = 1$. Then, A is called a *group of cosimple automorphisms* of the group G and the semidirect product $\Gamma = GA$ of the group G and the group A is a group. If $C_G(a) = C_G(A)$ for each element $a \in A^\#$, then A is said to be a *strong-centralized group of cosimple automorphisms* of the group G .

The theorem, proved in the series of papers [1–3], implies that if G is a finite irreducible complex linear group of degree $n < 2|A|$ with a nontrivial strong-centralized odd-order group A of cosimple automorphisms, then $n = |A| - 1$, $|A| + 1$, $2(|A| - 1)$ or $2|A| - 1$ and n is a degree of a certain prime number.

In this work, it is shown that the degree $n = 2|A| + 1$ of the irreducible complex linear group with a nontrivial strongly centralized group of cosimple automorphisms is a degree of a certain prime number (Theorem 1.1), and, on this basis, it is proved that in the π -solvable irreducible complex linear group G of degree $n = 2|H| + 1$, which is not a prime power, the π -Hall TI -subgroup H is Abelian and normal in G (Theorem 1.2).

Condition B

Let us say that the group $\Gamma = AG$ satisfies B, if $G \triangleleft \Gamma$, $(|A|, |G|) = 1$, $|A|$ is uneven, $C_G(a) = C_G(A) = C$ for each element $a \in A^\#$, and the group G has an extract irreducible complex character of degree $n = 2|A| + 1$, which is a -invariant for at least one element $a \in A^\#$.

Theorem 1.1. *If the group Γ satisfies Condition B, then n is a prime power degree.*

Condition D

G is a π -solvable irreducible complex linear group of degree $n = 2|H| + 1$ with a nonprimary π -Hall TI-subgroup H of odd order.

Theorem 1.2. *If the group G satisfies Condition D and n is not a prime power, the subgroup H is Abelian and normal in G .*

2. Definitions, Notation, and Preliminaries

N is the set of natural numbers; $\mathbf{Z}_+$ is the set of nonnegative integers; notation $i = \overline{1, t}$ stands for $i = 1, 2, \dots, t$; if ψ is the character of a certain group, then $\text{Irr}(\psi)$ is the set of all irreducible components of the character ψ ; $Z(\psi) = \{g \in G \mid |\psi(g)| = \psi(1)\}$ is the center of the character ψ ; $\pi = \pi(A)$; if $M \subseteq G$ and π is the set of primes, then $\pi' = \pi(M) \setminus \pi$; M_π is a Hall π -subgroup of the group M . The rest of the notation and definitions are conventional; they can be found, for example, in [4] or [5]. In what follows, by a group character we always mean a complex character, by a group a finite group.

Suppose that $\Gamma = AB$ is the group, where $B \triangleleft \Gamma$, $(|A|, |B|) = 1$ and $|A|$ is uneven. Then, it satisfies Theorem 13.1 [5]. According to this theorem, there exists one-to-one correspondence $\pi(B, A) : \text{Irr}_A(B) \rightarrow \text{Irr}(C_B(A))$ between the set of all A -invariant characters of the group B and the set of all irreducible characters of the subgroup $C_B(A)$. The set has a range of properties, depending, in particular, on the properties of the subgroup A . Suppose that $\chi \in \text{Irr}_A(B)$. Then, by Lemma 13.3 [5], there is a unique irreducible character $\hat{\chi}$ of the group Γ such that $\hat{\chi}_B = \chi$ and $A \subseteq \ker(\det \hat{\chi})$. It is called a *canonical extension of the character χ to the group Γ* .

Lemma 2.1 (see [6, Lemma 2]). *If χ is an irreducible character of the group G stated in Condition B, then $\chi \in \text{Irr}_A(G)$.*

Lemma 2.2 (see [1, Lemma 10]). *If χ is an irreducible character of the group G stated in Condition B, then $\hat{\chi}_A = k\rho_A + \varepsilon\beta(1)1_A$, where $k \in \mathbf{Z}_+$, ρ_A is a regular character of the subgroup A , $\varepsilon = \pm 1$ and β is an irreducible character of the subgroup C such that $\beta = (\chi)\pi(G, A)$.*

Lemma 2.3. *If χ is an irreducible character of the group G stated in Condition B, then $\chi_C = |A|\theta + \varepsilon\beta$, where θ is a character of the subgroup C or 0 and $\beta = (\chi)\pi(G, A)$.*

Proof. The proof of the lemma is a literal repeat of the proof of equality (3) of Lemma 11 to [1]. $\square$

Lemma 2.4. *If χ is an irreducible character of the group G of degree $2|A| < n < 3|A|$, stated in Condition B, then one of the following assertions holds:*

- (1) $\chi_C = \beta + |A|\beta_1$, $|A| < \beta(1) < 2|A|$, $\beta_1(1) = 1$;
- (2) $\chi_C = \beta + |A|\beta_1$, $\beta(1) < |A|$, $\beta_1(1) = 2$;
- (3) $\chi_C = \beta + 2|A|\beta_1$, $\beta(1) < |A|$, $\beta_1(1) = 1$;
- (4) $\chi_C = \beta + |A|(\beta_1 + \beta_2)$, $\beta(1) < |A|$, $\beta_1(1) = \beta_2(1) = 1$;
- (5) $\chi_C = (|A| + 1)\beta + |A|\beta_1$, $\beta(1) = \beta_1(1) = 1$;
- (6) $\chi_C = (|A| + 1)\beta$, $\beta(1) = 2$;
- (7) $\chi_C = (|A| - 1)\beta + 2|A|\beta_1$, $\beta(1) = \beta_1(1) = 1$;
- (8) $\chi_C = (|A| - 1)\beta + |A|\beta_1$, $\beta(1) = 1$, $\beta_1(1) = 2$;
- (9) $\chi_C = (|A| - 1)\beta + |A|(\beta_1 + \beta_2)$, $\beta(1) = \beta_1(1) = \beta_2(1) = 1$;
- (10) $\chi_C = (|A| - 1)\beta + |A|\beta_1$, $\beta(1) = 2$, $\beta_1(1) = 1$;
- (11) $\chi_C = (|A| - 1)\beta$, $\beta(1) = 3$;
- (12) $\chi_C = (2|A| + 1)\beta$, $\beta(1) = 1$;
- (13) $\chi_C = (2|A| - 1)\beta + |A|\beta_1$, $\beta(1) = \beta_1(1) = 1$;
- (14) $\chi_C = (3|A| - 1)\beta$, $\beta(1) = 1$;
- (15) $\chi_C = \beta$.

Here, $\beta = (\chi)\pi(G, A)$, β_1 and β_2 are distinct irreducible characters of the subgroup C .

Proof. From Lemma 2.3, it follows that

$$(\chi_C, \beta)_C = k_\beta |A| + \varepsilon, \quad (2.1)$$

where $k_\beta = (\theta, \beta)_C$. Suppose that $\beta_1 \in \text{Irr}(\chi_C)$, $\beta_1 \neq \beta$, $k_1 = (\chi_C, \beta_1)_C$ and θ_1 is the character C or 0 such that

$$\theta = k_\beta \beta + k_1 \beta_1 + \theta_1. \quad (2.2)$$

Hence, by Lemma 2.3, it follows that

$$\chi_C = |A|(k_\beta \beta + k_1 \beta_1 + \theta_1) + \varepsilon \beta. \quad (2.3)$$

Hence, by the assumption of the lemma, it follows that

$$|A|k_\beta \beta(1) + \varepsilon \beta(1) \leq \chi(1) < 3|A|. \quad (2.4)$$

Then, $|A|k_\beta \beta(1) < 3|A| - \varepsilon \beta(1)$, that is,

$$k_\beta < \frac{3}{\beta(1)} - \frac{\varepsilon}{|A|}. \quad (2.5)$$

Hence, $k_\beta \leq 3$.

Assume that $\theta \neq 0$ and examine all the possible cases separately.

(1) $k_\beta = 0$.

Since, by Corollary 2.17 [5], $(\chi_C, \beta)_C \in \mathbb{Z}_+$, by equality (2.1), it follows that $\varepsilon = 1$. Since $\chi(1) < 3|A|$, from Lemma 2.3, it follows that $\theta(1) \leq 2$, that is, $k_1\beta_1(1) + \theta_1(1) \leq 2$.

Suppose that $\theta(1) = 1$. Then, we can assume that $\theta_1 = 0$ and $k_1\beta_1(1) = 1$. From equality (2.3), we obtain $\chi_C = \beta + |A|\beta_1$, $|A| < \beta(1) < 2|A|$, $\beta_1(1) = 1$ is assertion (1) of the lemma being proved.

Suppose that $\theta(1) = 2$. Then, $k_1\beta_1(1) + \theta_1(1) = 2$.

If $\theta_1 = 0$, then either $k_1 = 1$ and $\beta_1(1) = 2$, which implies assertion (2) of the lemma, or $k_1 = 2$ and $\beta_1(1) = 1$, which implies assertion (3) of the lemma.

But if $\theta_1 \neq 0$, then it is clear that $k_1\beta_1(1) = 1$ and $\theta_1 = \beta_2 \in \text{Irr}(\theta)$, $\beta_1 \neq \beta_2$, $\beta_2(1) = 1$, $\beta(1) < |A|$. We obtain assertion (4) of the lemma.

(2) $k_\beta = 1$.

Suppose that $\varepsilon = 1$. From (2.3), we obtain $\chi_C = |A|(\beta + k_1\beta_1 + \theta_1) + \beta$. Since $2|A| < \chi(1) < 3|A|$, from the latter equality, it is readily seen that $\beta(1) \leq 2$.

Suppose that $\beta(1) = 1$. Then, $k_1\beta_1(1) = 1$ and $\chi_C = (|A| + 1)\beta + |A|\beta_1$, $\beta(1) = \beta_1(1) = 1$ is assertion (5).

Now, suppose that $\beta(1) = 2$. Since $2|A| < \chi(1) < 3|A|$, we have $k_1\beta_1(1) + \theta_1(1) = 0$ and, therefore, $\chi_C = (|A| + 1)\beta$ is assertion (6).

Further, suppose that $\varepsilon = -1$. By (2.3), it follows that

$$\chi_C(1) = (|A| - 1)\beta(1) + |A|(k_1\beta_1(1) + \theta_1(1)). \quad (2.6)$$

Since $\chi(1) < 3|A|$, we see that $\beta(1) \leq 3$.

Let $\beta(1) = 1$. Then, $k_1\beta_1(1) + \theta_1(1) = 2$.

Let $\theta_1 = 0$. Then, $k_1\beta_1(1) = 2$. If $k_1 = 2$, then $\beta_1(1) = 1$. Hence, $\chi_C = (|A| - 1)\beta + 2|A|\beta_1$ is assertion (7). If $k_1 = 1$, then $\beta_1(1) = 2$. Then $\chi_C = (|A| - 1)\beta + |A|\beta_1$ is assertion (8).

Now suppose that $\theta_1(1) = 1$. Then, $k_1\beta_1(1) = 1$. We obtain $\chi_C = (|A| - 1)\beta + |A|(\beta_1 + \beta_2)$, where $\beta_1(1) = \beta_2(1) = 1$ is assertion (9).

If $\theta_1(1) = 2$, then $k_1\beta_1(1) = 0$ and it is readily seen that (7), (8) or (9) follow.

Now suppose that $\beta(1) = 2$. Hence, $\chi_C = (|A| - 1)\beta + |A|\beta_1$, $\beta_1(1) = 1$ is assertion (10).

Suppose that $\beta(1) = 3$. Then, $k_1\beta_1(1) + \theta_1(1) = 0$. Assertion (11). (3) $k_\beta = 2$.

Let $\varepsilon = 1$. Expression (2.3) implies that $k_1\beta_1(1) + \theta_1(1) = 0$. We obtain $\chi_C = (2|A| + 1)\beta$. We see that $\beta(1) = 1$ is assertion (12).

Let $\varepsilon = -1$. Then, $\chi_C = (2|A| - 1)\beta + |A|\beta_1$, where $\beta(1) = \beta_1(1) = 1$ is assertion (13).

(4) $k_\beta = 3$.

Expression (2.3) implies that $\varepsilon = -1$, $\chi_C = (3|A| - 1)\beta$, where $\beta(1) = 1$ is assertion (14).

Now suppose that in Lemma 2.3 $\theta = 0$. Then, $\varepsilon = 1$ and $\chi_C = \beta$ is assertion (15).

The lemma is proved. $\square$

3. Proof of Theorem 1.1

Suppose that χ is an exact irreducible character of degree $n = 2|A| + 1$ of the group G given in Condition B, a -invariant for at least one element $a \in A^\#$.

Suppose that Γ is a group of the least order that satisfies the assumption of the theorem, but does not satisfy its corollary, that is, $|\pi(n)| > 1$.

Lemma 3.1. *The character $\hat{\chi}$ is exact.*

Proof. Suppose that $\ker \hat{\chi} \neq 1$. Since the character $\hat{\chi}_G = \chi$ is exact, we have $\ker \hat{\chi} \subseteq A$. Since, by Lemma 1 [7], A is a TI -subgroup in Γ , it follows that $A \triangleleft \Gamma$. This contradicts the fact that A is the group of nontrivial automorphisms of the group G .

The lemma is proved. $\square$

Lemma 3.2. *Suppose that $q \in \pi(n)$, Q is an A -invariant Sylow q -subgroup G . Then, $[Q, A] \not\subseteq C$.*

Proof. Suppose that $[Q, A] \subseteq C$. Since, by Lemma 4 [1], $Q = [Q, A]C_Q(A)$, we have $Q \subseteq C$. By Exercise 13.2 [5],

$$\frac{\beta(1)|G : C|}{\chi(1)} \quad (3.1)$$

is an integer, where β is from Lemma 2.2. Since $Q \subseteq C$, we see that q does not divide $|G : C|$. Then q divides $\beta(1)$. From Lemma 2.2, it follows that $k|A| = \hat{\chi}(1) - \varepsilon\beta(1)$. Hence, q divides $k|A|$. Since q divides $|G|$, it follows that q does not divide $|A|$, therefore, q divides k . By using the fact that q is an odd prime, it follows that $k \geq 3$.

Suppose that $\varepsilon = 1$ in Lemma 2.2. Since $n = 2|A| + 1$, we have $k = 2$. This case does not satisfy.

Suppose that $\varepsilon = -1$. Since $\beta(1) \leq \chi(1) < 3|A|$, we obtain

$$k = \frac{\hat{\chi}(1) + \beta(1)}{|A|} < 6, \quad (3.2)$$

that is, $k \leq 5$. Since q divides k and q is an uneven number, it follows that it is enough to consider the cases when $k = 3$ or $k = 5$.

Suppose that $k = 3$. Then, $3|A| = (2|A| + 1) + \beta(1)$. Hence, $|A| - 1 = \beta(1)$. Since $\chi(1) = 2|A| + 1$, by Lemma 2.4, it follows that only such relations satisfy, where $|A| - 1 = \beta(1) = 2$. However, in this case $|A| = 3$. Then, $n = 2|A| + 1 = 7$ is a prime number. A contradiction with the minimality of the group Γ .

Suppose that $k = 5$. Then, $5|A| = (2|A| + 1) + \beta(1)$. Therefore, $\beta(1) = 3|A| - 1 > n$. We have a contradiction.

The lemma is proved. $\square$

Lemma 3.3. *Suppose that R is a proper A -invariant normal subgroup of the group G . Then either R is Abelian, or $R \subseteq C$.*

Proof. Let us consider the character

$$(\hat{\chi})_{AR} = \sum_{i=1}^f \alpha_i \chi_i, \quad (3.3)$$

where $\alpha_i \in \mathbf{N}$, $\chi_i \in \text{Irr}(AR)$, $i = \overline{1, f}$.

Let us note that by the minimality of the group Γ either $f \neq 1$, or $\alpha_1 \neq 1$, that is, $\chi_i(1) \neq \hat{\chi}(1)$ for all $i = \overline{1, f}$.

By the Clifford Theorem

$$(\chi_i)_R = e_i \sum_{j=1}^{t_i} \chi_{ij}, \quad (3.4)$$

where $\chi_{ij} \in \text{Irr}(R)$, e_i are numbers, which divide $|A|$, $i = \overline{1, f}$, $j = \overline{1, t_i}$.

Suppose that $\chi_{i_0 j_0} \in \text{Irr}((\widehat{\chi})_R)$. Since $R \triangleleft \Gamma$, by the Clifford Theorem it follows that $\chi_{ij}(1)$ divides n for each irreducible component χ_{ij} of the character $\widehat{\chi}_R$ and for corresponding elements $x \in \Gamma \setminus I_\Gamma(\chi_{i_0 j_0})$ the equality $\chi_{ij} = (\chi_{i_0 j_0})^x$ is true.

Suppose that $\chi_{i_0 j_0}(1) = 1$ for a certain character $\chi_{i_0 j_0} \in \text{Irr}((\widehat{\chi})_R)$. Since $R' \subseteq \ker \chi_{i_0 j_0}$ and $R' \triangleleft \Gamma$, we see that $R' \subseteq (\ker \chi_{i_0 j_0})^x$ for all $x \in \Gamma \setminus I_\Gamma(\chi_{i_0 j_0})$. Therefore, $R' \subseteq \bigcap_{x \in \Gamma \setminus I_\Gamma(\chi_{i_0 j_0})} (\ker \chi_{i_0 j_0})^x = \ker \widehat{\chi} = 1$. Thus, the subgroup R is Abelian.

Now suppose that $\chi_{ij}(1) \neq 1$ for all $i = \overline{1, f}$ and for all $j = \overline{1, t_i}$. Since n is an uneven number, we have $\chi_{ij}(1) \geq 3$. Therefore, if $I_{AR}(\chi_{i_0 j_0}) = R$ for a certain character $\chi_{i_0 j_0} \in \text{Irr}((\widehat{\chi})_R)$, then $\chi_{i_0}(1) = (\chi_{i_0 j_0})^{AR}(1) = |A|\chi_{i_0 j_0}(1) \geq 3|A|$. Since in this case $\widehat{\chi}(1) \geq \chi_{i_0}(1)$, we obtain a contradiction with $\widehat{\chi}(1) = 2|A| + 1 < 3|A|$. Thus, $I_{AR}(\chi_{ij}) \neq R$ for all $i = \overline{1, f}$ and for all $j = \overline{1, t_i}$. Then, by Lemma 1 [6], $I_{AR}(\chi_{ij}) = AR$, therefore, $t_i = 1$ and $\chi_i(1) = e_i \chi_{ij}(1)$ for all $i = \overline{1, f}$. By Lemma 13.3 [5], for every $i = \overline{1, f}$ there exists a canonical extension $\widehat{\chi}_i$ of the character χ_i to the group AR .

Assume that $A \ker \widehat{\chi}_{i_0} / \ker \widehat{\chi}_{i_0} \not\trianglelefteq AR / \ker \widehat{\chi}_{i_0}$ for a certain i_0 . By Lemma 9 [1], $A \cap \ker \widehat{\chi}_{i_0} = 1$. Then, $A \ker \widehat{\chi}_{i_0} / \ker \widehat{\chi}_{i_0} \cong A / A \cap \ker \widehat{\chi}_{i_0} = A$. Let us apply the theorem [1] to the factor group $AR / \ker \widehat{\chi}_{i_0}$ and its exact irreducible character in terms of Lemma 2.22 [5] $(\widehat{\chi}_{i_0})^*$. By this theorem, $(\widehat{\chi}_{i_0})^*(1) \in \{|A| - 1, |A|, |A| + 1, 2|A| - 2, 2|A| - 1\}$ and $(\widehat{\chi}_{i_0})^*(1)$ is a prime power, probably, with the exception of the case, when $(\widehat{\chi}_{i_0})^*(1) = |A|$. Since $(\widehat{\chi}_{i_0})^*(1) = \widehat{\chi}_{i_0}(1)$ and $\widehat{\chi}_i(1)$ divides n for all $i = \overline{1, f}$, it follows that $\widehat{\chi}_{i_0}(1) = 2|A| - 1$. Thus, $n = 2|A| + 1 = s(2|A| - 1)$ for a certain odd $s \in \mathbb{N}$. It is easily seen that it is not true.

Now suppose that $A \ker \widehat{\chi}_i / \ker \widehat{\chi}_i \triangleleft AR / \widehat{\chi}_i$ for all $i = \overline{1, f}$. Using the fact that $\bigcap_{i=1}^f \ker \widehat{\chi}_i \cap R = \ker \widehat{\chi} \cap R$ and $\ker \widehat{\chi} = 1$ by Lemma 3.1, then $A_0 = \bigcap_{i=1}^f \ker \widehat{\chi}_i \subseteq A$. If $A_0 \neq 1$, then $A \triangleleft AR$, since A is a TI -subgroup. If $A_0 = 1$, then the subgroup AR can be isomorphically embedded in the direct product $\overline{AR} = \prod_{i=1}^f (AR / \ker \widehat{\chi}_i)$. Since the group $\overline{AR}$ contains a normal Hall π -subgroup, we have $A \triangleleft AR$. Thus, $R \subseteq C$.

The lemma is proved. $\square$

Suppose that $q \in \pi(n)$ and Q is a A -invariant Sylow q -subgroup of G that exists by Theorem 6.2.2(i) [4]. By Lemma 3.2, $[Q, A] \not\subseteq C$.

Further, M is an A -invariant proper subgroup of G such that $[Q, A] \subseteq M$.

From the minimality of the group Γ and Lemma 3.1, it follows that the character $\widehat{\chi}_{AM}$ is reducible for every such a subgroup M . Denote $N = N_\Gamma([Q, A])$.

Lemma 3.4. $N \neq \Gamma$.

Proof. Assume that $N = \Gamma$. By Lemma 3.3, the subgroup $[Q, A]$ is either Abelian or $[Q, A] \subseteq C$. Since, by Lemma 3.2, $[Q, A] \not\subseteq C$, it is enough to consider the case, when $[Q, A]$ is Abelian.

Suppose that $[Q, A]$ is Abelian. Let us consider the character $\widehat{\chi}_{AQ}$. It is clear that $AQ \neq \Gamma$ and $A \not\trianglelefteq AQ$. Therefore, there exists at least one irreducible component ξ of the character $\widehat{\chi}_{AQ}$ such that $A \ker \xi / \ker \xi \not\trianglelefteq AQ / \ker \xi$ and since $|AQ|$ is an uneven number, by theorem [1], it

follows that for every such a character it is true that $\xi(1) \in \{|A|, 2|A| - 1\}$ and $2|A| - 1$ is a prime power. Assume that $\xi(1) = 2|A| - 1$. Then, q divides $2|A| - 1$ and q divides $2|A| + 1$. Thus, $q = 2$. A contradiction. Thus, $\xi(1) = |A|$.

Suppose that there exists one more character $\xi_1 \in \text{Irr}(\widehat{\chi}_{AQ})$ such that $A \ker \xi_1 / \ker \xi_1 \not\triangleleft AQ / \ker \xi_1$. Since $\xi_1(1) = |A|$ and $\widehat{\chi}_{AQ} = 2|A| + 1$, we see that $\widehat{\chi}_Q$ is a sum of linear irreducible characters λ . Thus,

$$Q' \subseteq \bigcap_{\lambda \in \text{Irr}(\widehat{\chi}_Q)} \ker \lambda = \ker \widehat{\chi} = 1. \quad (3.5)$$

This yields that the subgroup Q is Abelian. Therefore, $Q \subseteq C_\Gamma([Q, A]) \triangleleft \Gamma$. Suppose that $C_\Gamma([Q, A]) = \Gamma$, that is, $[Q, A] \subseteq Z(\Gamma)$. Then, $[Q, A] \subseteq C$ is a contradiction with Lemma 3.2. Thus, $C_\Gamma([Q, A]) \neq \Gamma$ and since $C_\Gamma([Q, A]) \triangleleft \Gamma$, we see that, by Lemma 3.3, the subgroup $C_\Gamma([Q, A])$ is either Abelian or C contains it. If $C_\Gamma([Q, A])$ is Abelian, then $Q \triangleleft \Gamma$, and we obtain a contradiction with Theorem 6.15 [5], but if $C_\Gamma([Q, A]) \subseteq C$, then we obtain a contradiction with Lemma 3.2.

Assume that $A \ker \psi / \ker \psi \triangleleft AQ / \ker \psi$, where $\widehat{\chi}_{AQ} = \xi + \psi$. By Lemma 3 [1], $[Q, A] \subseteq (\ker \psi)_{\pi'}$. Since $Q' \subseteq \ker \xi$ and $\ker \xi \cap \ker \psi = 1$, we have $Q = [Q, A] \subseteq Z(Q)$. Then, $Q \subseteq C_\Gamma([Q, A])$. This case has been considered above.

The lemma is proved. $\square$

By Lemma 3.4, $N \neq \Gamma$. For each A -invariant proper subgroup M of the group G is such that $[Q, A] \subseteq M$, we have

$$\widehat{\chi}_{AM} = \tau + \psi, \quad (3.6)$$

where τ and ψ are the characters of the subgroup AM such that $A \ker \xi / \ker \xi \triangleleft AM / \ker \xi$ for each $\xi \in \text{Irr}(\tau)$ and $A \ker \xi / \ker \xi \triangleleft AM / \ker \xi$ for each $\xi \in \text{Irr}(\psi)$.

Then, the group $AM / \ker \psi$ can be isomorphically embedded in the direct product $\overline{AM} = \prod_{\xi \in \text{Irr}(\psi)} AM / \ker \xi$. Since each factor contains a normal Hall π -subgroup $A \ker \xi / \ker \xi$, we see that $A \ker \psi / \ker \psi \triangleleft AM / \ker \psi$. It follows that $A \ker \psi \triangleleft AM$. By Lemma 2.2.1 [8], $AM = N_{AM}(A) \ker \psi$. Then,

$$M = (\ker \psi)_{\pi'} C_M(A). \quad (3.7)$$

Since by Lemma 3 [1], $[M, A] \subseteq (\ker \psi)_{\pi'}$, $[Q, A] \subseteq M$ and $[[Q, A], A] = [Q, A]$, by Lemma 6.13 [9], we have

$$[Q, A] \subseteq (\ker \psi)_{\pi'}. \quad (3.8)$$

Assume that $\tau = 0$. Then, $\widehat{\chi}_{AM} = \psi$, that is, $\bigcap_{\xi \in \text{Irr}(\psi)} \ker \psi = 1$. Then, $A \triangleleft AM$, that is, $[Q, A] \subseteq C$. This contradicts Lemma 3.2.

Thus, $\tau \neq 0$. By Theorem [1],

$$\xi(1) \in \{|A| - 1, |A|, |A| + 1, 2|A| - 2, 2|A| - 1\} \quad (3.9)$$

and $\xi(1)$ is a prime power, probably, with the exception of the case, when $\xi(1) = |A|$ or $\xi(1) = 2|A|$.

Lemma 3.5. *The character τ does not contain any irreducible components of degree $|A| - 1$.*

Proof. Assume that there exists $\xi \in \text{Irr}(\tau)$ and $\xi(1) = |A| - 1$. Then, the group AM and its irreducible character ξ satisfy Lemma 11 [1]. Since $A \ker \xi / \ker \xi \not\triangleleft AM / \ker \xi$, by Lemma 9 [1], $A \cap \ker \xi = 1$, and, by Lemma 10 [1], it yields that assertion (7) of Lemma 11 [1] does not satisfy. Since $\xi(1) = |A| - 1$, we see that only assertion (3) of the mentioned Lemma satisfies, that is, $\xi_{C_M(A)} = (|A| - 1)\beta_\xi$, $\beta_\xi(1) = 1$, where $\beta_\xi = (\xi_M)\pi(M, A)$. By Lemma 10 [1],

$$\alpha_\xi = (\xi_A, 1_A)_A = \left((\rho_A - 1_A)\lambda^{-1}, 1_A \right)_A \quad (3.10)$$

for a certain linear irreducible character λ of the subgroup A . We see that $\alpha_\xi = 0$ if $\lambda = 1_A$ and $\alpha_\xi = 1$ if not.

In addition, assume that $\tau(1) \neq \xi(1)$, that is, the character τ contains an irreducible component ξ_1 .

Besides, assume that $\xi_1 = |A| - 1$. In the same way, we establish that

$$\alpha_{\xi_1} = ((\xi_1)_A, 1_A)_A = \left((\rho_A - 1_A)\lambda_1^{-1}, 1_A \right)_A \quad (3.11)$$

for a certain linear irreducible character λ_1 of the subgroup A , moreover, $\alpha_{\xi_1} = 0$, if $\lambda_1 = 1_A$ and $\alpha_{\xi_1} = 1$ in the converse case.

Suppose that $\hat{\chi}_{AM} = \xi + \xi_1 + \eta$ for a certain character η of the subgroup AM . Then,

$$\eta(1) = \hat{\chi}(1) - (\xi(1) + \xi_1(1)) = 2|A| + 1 - (2|A| - 2) = 3. \quad (3.12)$$

Suppose that $|A| = 3$ or 5 . Then, $2|A| + 1 = 7$ or 11 is a prime number. This contradicts the minimality of the group Γ . Thus, $|A| \geq 7$. Therefore, $\eta(1) = 3 < |A| - 1$. By Theorem 1 [10], $A \ker \eta / \ker \eta \triangleleft AM / \ker \eta$. Clearly, $\psi = \eta$ and, as follows from Lemma 11 [1] and Lemma 10 [1], each irreducible component δ of the character ψ is restricted irreducibly to $C_M(A)$, that is, $\delta_{C_M(A)} = \beta_\delta$. By Lemma 10 [1], we see that $\hat{\delta}_A = \beta_\delta(1)1_A$. Then, it is obvious that $\alpha_\psi = (\psi_A, 1_A)_A = \sum_{\delta \in \text{Irr}(\psi)} (\beta_\delta(1)\lambda_\delta^{-1}, 1_A)_A$. Here, $\hat{\delta}$ is a canonic extension of the character δ_M to the group AM and λ_δ are linear irreducible characters of the subgroup A . It is readily seen that $\alpha_\psi \leq \psi(1) = 3$.

On the other hand, since $C_M(A) \subseteq C$, we have $\beta_\xi \in \text{Irr}(\beta_{C_M(A)})$, $\beta_{\xi_1} \in \text{Irr}(\beta_{C_M(A)})$, and $(\beta_{C_M(A)}, \psi_{C_M(A)})_{C_M(A)} \neq 0$, where β is from Lemma 2.4. Since by Theorem 13.1 [5], $\beta_\xi \neq \beta_{\xi_1}$, we have $\beta(1) \geq 3$. Since $\hat{\chi}(1) = 2|A| + 1$, we see that only assertion (1) or assertion (15) of Lemma 2.4 satisfy, that is, $\beta(1) = |A| + 1$ or $\chi_C = \beta$. Suppose that assertion (1) of Lemma 2.4 satisfies. Then, by Lemma 2.2,

$$\alpha = (\hat{\chi}_A, 1_A)_A = (\rho_A + (|A| + 1)1_A, 1_A)_A = |A| + 2. \quad (3.13)$$

Since $\alpha = \alpha_\xi + \alpha_{\xi_1} + \alpha_\psi \leq 5$, we obtain $|A| + 2 \leq 5$. Hence, $|A| \leq 3$. As shown above $|A| \geq 7$. A contradiction. Suppose that assertion (15) of Lemma 2.4 satisfies. Then, by Lemma 2.2 $\hat{\chi}_A = \beta(1)1_A$, that is, $A \subseteq \ker \hat{\chi}$. We obtain a contradiction with Lemma 3.1.

Now assume that $\xi_1 = |A|$. Then, $(\xi_1)_A = \rho_A$ and $\alpha_{\xi_1} = 1$. Since $\beta_{\xi} \in \text{Irr}(\beta_{C_M(A)})$ and $(\beta_{C_M(A)}, \psi_{C_M(A)})_{C_M(A)} \neq 0$, it follows that $\beta(1) \geq 2$. Again, we see that only assertion (1) of Lemma 2.4 satisfies. Since in this case $\psi(1) = 2$, that is, $\alpha_{\psi} \leq 2$, we have $\alpha = \alpha_{\xi} + \alpha_{\xi_1} + \alpha_{\psi} \leq 4$. Thus, $|A| + 2 \leq 4$, that is, $|A| \leq 2$. A contradiction.

There only remains to consider the case, when $\tau = \xi$. Since $\ker \tau \cap \ker \psi = 1$, then the character $\xi_{\ker \psi}$ is exact. Further, $A \not\trianglelefteq A \ker \psi$, because if not we have $[Q, A] \subseteq C$ by relation (3.8), and we get a contradiction with Lemma 3.2. Then, the character $\tilde{\xi} = \xi_{A \ker \psi}$ is also exact and from Lemma 7 [10] and Lemma 2.27 [5] it yields that $|A \ker \psi : Z(\tilde{\xi})| = |A \ker \psi : Z(A \ker \psi)| = |A|^{2^r}$, $r \in \mathbf{N}$. Hence, $[Q, A] \subseteq Z(A \ker \psi)$, that is, $[Q, A] \subseteq C$. A contradiction with Lemma 3.2.

The lemma is proved. $\square$

Lemma 3.6. *Let M be a solvable A -invariant subgroup of G . Assume that AM has an exact irreducible character γ of degree $|A| + 1 = 2^{\beta}$, $\beta \in \mathbf{N}$. Then, $|M : C_M(A)| = 2^m$, where $m \in \mathbf{Z}_+$.*

Proof. Since $|\pi| = 1$, we see that $|A| = p^f$, $f \in \mathbf{N}$, therefore, $p^f + 1 = 2^{\beta}$, $\beta \in \mathbf{N}$, that is, $p^f = 2^{\beta} - 1$. From Lemma 1 [1], it follows that $f = 1$ and p is a Mersenne prime. Thus, $|H| = p$. By Theorem [11], the assertion of the theorem holds. We obtain a contradiction with the minimality of the group Γ . Therefore, $|\pi| > 1$.

Suppose that $r, s \in \pi$, $r \neq s$. Since, by Lemma 7 [1], the subgroup A is a twiddle factor of a certain Frobenius group, it follows that by Corollary 1 [10], the subgroup A contains a subgroup $H \times B$ of order rs . By assumption B, $C_M(H) = C_M(B) = C_M(A)$. Since M is solvable, by Theorem 6 [9], it follows that $[M, H] \subseteq F(M)$. Since, by Lemma 4 [1], $M = [M, H]C_M(H) = [M, H]C_M(A)$, we obtain $M = F(M)C_M(A)$. Assume that $F = F(M)$. Let us consider the character

$$\gamma_{AF} = \sum_{i=1}^t \alpha_i \gamma_i, \quad (3.14)$$

$\gamma_i \in \text{Irr}(AF)$, $\alpha_i > 0$, $i = 1, 2, \dots, t$.

Suppose that $A \triangleleft AF$. Then, $F \subseteq C_M(A)$. Thus, $M \subseteq C_M(A)$, therefore, $|M : C_M(A)| = 2^0 = 1$. In this case, the lemma is true.

Suppose that $A \not\triangleleft AF$. Then for a certain i , let us say that $i = 1$, $A \ker \gamma_1 \not\triangleleft AF$. From Theorems 1 and 2 [10] and the assumption of the lemma, we obtain $\gamma_1(1) = |A| - 1 = 2^{\alpha}$, $|A|$ or $|A| + 1 = 2^{\beta}$. Here $\alpha, \beta \in \mathbf{N}$.

If $\gamma_1(1) = |A| - 1 = 2^{\alpha}$, then $|A| = 2^{\alpha} + 1 = 2^{\beta} - 1$. Hence, $2^{\beta} = 2^{\alpha} + 2 = 2(2^{\alpha-1} + 1)$, that is, $\alpha = 1$. Thus, $|A| = 3$. This case is impossible.

Suppose that $\gamma_1(1) = |A|$. Then,

$$\bar{\gamma}(1) = \gamma(1) - \gamma_1(1) = |A| + 1 - |A| = 1, \quad (3.15)$$

where $\bar{\gamma} = \gamma_{AF} - \gamma_1$. Therefore, all irreducible components of the character γ_F are linear, that is, the subgroup F is Abelian. Since $F \triangleleft AM$, we have $C_{AM}(F) \triangleleft AM$. Then, it is not hard to see that $C^* = C_M(F) \triangleleft AM$. Since $F \not\subseteq C_M(A)$ and $F \subseteq C^*$, it follows that $C^* \not\subseteq C_M(A)$, that is, $A \not\triangleleft AC^*$.

Arguing as above for the subgroup AF , we can show that in the contingency γ_{AC^*} there is an irreducible component $\tilde{\gamma}_1$ such that $A \ker \tilde{\gamma}_1 \not\triangleleft AC^*$ and $\tilde{\gamma}_1(1) = |A|$ or $|A| + 1 = 2^{\beta}$.

Assume that $\tilde{\gamma}_1(1) = |A| + 1 = 2^\beta$. Since the subgroup A is a TI -set in AC^* , from Lemma 1 [6], it yields that $A \subseteq I_{AC^*}(\tilde{\gamma}_1^*)$, where $\tilde{\gamma}_1^*$ is an irreducible component of the character $(\tilde{\gamma}_1)_{C^*}$. Thus, $(\tilde{\gamma}_1)_{C^*}$ is an irreducible character of the subgroup C^* . Since $F \subseteq Z(C^*)$, we have $F \subseteq Z(\tilde{\gamma}_1)$. However, since the character γ is exact, the character $\tilde{\gamma}_1$ is exact. Therefore, $Z(\tilde{\gamma}_1) = Z(AC^*)$. Then, $F \subseteq Z(AC^*)$, that is, $F \subseteq C_M(A)$. This case has already been examined. Therefore, we may assume that $\tilde{\gamma}_1(1) = |A|$. Again, we see that C^* is Abelian, that is, $C^* \subseteq F$. Therefore, $F = C^*$.

Suppose that $\lambda \in \text{Irr}(\gamma_F)$ and $I = I_{AM}(\lambda)$. Assume that $I = AM$. Since the subgroup F is Abelian, we obtain $\lambda(1) = 1$. Therefore, $\gamma_F = \gamma(1)\lambda$, that is, $F \subseteq Z(\gamma) = Z(AM)$, that is, $F \subseteq C_M(A)$. The lemma is true in this case.

Now suppose that $I \neq AM$. By Theorem 6.11 [5], there exists an irreducible character λ' of the subgroup I such that $(\lambda')^{AM} = \gamma$. Hence, $\gamma(1) = 2^\beta = |AM : I|\lambda'(1)$. Thus, $|AM : I|$ and $\lambda'(1)$ is degree 2. Therefore, $A^x \subseteq I$ for a certain element $x \in AM$. Assume that $A^x \triangleleft I$. Then, $F \subseteq C_M(A^x)$. Hence, $F = F^{x^{-1}} \subseteq C_M(A)$. The lemma is true.

Therefore, $A^x \not\triangleleft I$. Let us show that $\lambda'(1) = 1$.

Let us assume the converse, that is, $\lambda'(1) > 1$. Since by Frobenius reciprocity law for characters

$$1 = \left(\gamma, (\lambda')^{AM} \right)_{AM} = (\gamma_I, \lambda')_I, \quad (3.16)$$

we have $\gamma_I = \lambda' + \lambda''$. Since $\lambda'(1) > 1$, we obtain

$$\lambda''(1) = \gamma(1) - \lambda'(1) = |A| + 1 - \lambda'(1) \leq |A| - 1. \quad (3.17)$$

Since $|A| \neq 3$, by Theorems 1 and 2 [10], we see that $A^x \ker \lambda'' \triangleleft I$. By using the fact that $A^x \not\triangleleft I$, it follows that $A^x \ker \lambda' \not\triangleleft I$. Then, $\lambda'(1) \geq |A| - 1$. Since the numbers $\lambda'(1)$ and $|A| + 1$ are of degree 2 and $|A| \neq 3$, we obtain $\lambda'(1) = |A| + 1$, that is, $\lambda'(1) = \gamma(1)$. Since $\gamma(1) = |AM : I|\lambda'(1)$, we have $|AM : I| = 1$, that is, $AM = I$. In this case, as shown above, the lemma is true.

We may assume that $\lambda'(1) = 1$. Then, $\lambda''(1) = |A| + 1 - 1 = |A|$. Since $A^x \ker \lambda' \triangleleft I$, $A^x \not\triangleleft I$ and $|A| \neq 3$, from Theorem 1 and 2 [10], it easily follows that the character λ'' is irreducible. Therefore, all irreducible components of the character $\gamma_{I_{\pi'}}$ are linear, that is, the subgroup $I_{\pi'}$ is Abelian. Since $F = C^*$ is Abelian and $F \subseteq I_{\pi'}$, we have $F = I_{\pi'}$. Then, $I = A^x I_{\pi'} = A^x F$. Since $M = M^x$, we have

$$M = FC_M(A) = (FC_M(A))^x = F^x(C_M(A))^x = F(C_M(A))^x = FC_M(A^x). \quad (3.18)$$

Further,

$$AM = (AM)^x = A^x M^x = A^x M = A^x FC_M(A^x) = IC_M(A^x). \quad (3.19)$$

Hence, for each $g \in AM$, we obtain that $g = ab$, where $a \in I$ and $b \in C_M(A^x)$. Since $\gamma = (\lambda')^{AM}$, for each $h \in A^x$ the following expression satisfies

$$\begin{aligned} \gamma(h) &= \frac{1}{|I|} \sum_{g \in AM} (\lambda')^0(ghg^{-1}) = \frac{1}{|I|} \sum_{g \in AM} (\lambda')^0(abhb^{-1}a^{-1}) \\ &= \frac{1}{|I|} \sum_{g \in AM} (\lambda')^0(a(h^{b^{-1}})a^{-1}) = \frac{1}{|I|} \sum_{g \in AM} \lambda'(h) = |AM : I| \lambda'(h). \end{aligned} \quad (3.20)$$

Here, $(\lambda')^0(x) = \lambda'(x)$, if $x \in I$ and $(\lambda')^0(x) = 0$, if $x \notin I$. Since $\lambda'(1) = 1$, we obtain $|\lambda'(h)| = 1$. The latter chain of equalities implies that

$$|\gamma(h)| = ||AM : I| \cdot \lambda'(h)| = |AM : I| |\lambda'(h)| = |AM : I|. \quad (3.21)$$

Since $\gamma(1) = |AM : I| \lambda'(1)$, we have $\gamma(1) = |AM : I|$. Then, $|\gamma(h)| = |AM : I| = \gamma(1)$. By Lemma 2.27(e) [5], $A^x \subseteq Z(\gamma) = Z(AM)$. Hence, $A^x = A \triangleleft AM$, that is $M \subseteq C_M(A)$, therefore, $|M : C_M(A)| = 1$. The lemma is true.

There only remains to consider the case, when $\gamma_1(1) = |A| + 1 = 2^\beta$. Since the subgroup A is a TI -set in AF , from Lemma 1 [6], it follows that $A \subseteq I_{AF}(\gamma_1)^*$, where $(\gamma_1)^*$ is an irreducible component of the character $(\gamma_1)_F$. Thus, $(\gamma_1)_F$ is an irreducible character of the subgroup F . Since the subgroup F is nilpotent, by Theorem 4.21 [5], we have $(\gamma_1)_F = (\gamma_1)_{F_2} \times \mu$, where $(\gamma_1)_{F_2}$ is an irreducible character of the Sylow 2-subgroup F_2 from F and μ is an irreducible character of the Hall $2'$ -subgroup O from F . Since $\mu(1) = 1$, we have $O \subseteq Z(\gamma_1) = Z(AF)$, that is, $O \subseteq C_M(A)$. Then,

$$M = FC_M(A) = (F_2 \times O)C_M(A) = F_2C_M(A). \quad (3.22)$$

Therefore,

$$|M : C_M(A)| = |F_2C_M(A) : C_M(A)| = |F_2 : F_2 \cap C_M(A)| = 2^m, \quad (3.23)$$

where $m > 0$.

The lemma is proved. $\square$

Lemma 3.7. $\psi \neq 0$.

Proof. Let us assume the converse, that is, $\psi = 0$. Then, $(\hat{\chi})_{AM} = \tau$. Since, by Lemma 3.5, $\xi(1) \neq |A| - 1$ for $\xi \in \text{Irr}(\tau)$, we have

$$\xi(1) \in \{|A|, |A| + 1, 2|A| - 2, 2|A| - 1, 2|A|\}. \quad (3.24)$$

Suppose that $\xi(1) \in \{2|A| - 2, 2|A| - 1, 2|A|\}$. Then, $\hat{\chi}(1) - \xi(1) = 2|A| + 1 - \xi(1) \leq 3$. As mentioned above $|A| \geq 7$. By Theorem 1 [10], $A \ker(\hat{\chi} - \xi) / \ker(\hat{\chi} - \xi) \triangleleft AM / \ker(\hat{\chi} - \xi)$.

Thus, $\text{Irr}(\hat{\chi} - \xi) \subseteq \text{Irr}(\psi)$. This contradicts the assumption. Hence, $\xi(1) \in \{|A|, |A| + 1\}$ and since by the assumption $\tau(1) = 2|A| + 1$, we obtain

$$\tau = \xi + \xi_1, \quad (3.25)$$

where $\xi(1) = |A|$ and $\xi_1(1) = |A| + 1$ is degree 2.

From the Clifford Theorem, it follows that $M' \subseteq \ker \xi$. Since $\ker \xi \cap \ker \xi_1 = \ker \tau = \ker \hat{\chi}_{AM} = 1$, it follows that the subgroup $\ker \xi_1$ is Abelian. By Theorem [12], the factor group $AM / \ker \xi_1$ is solvable. Then, AM is also solvable. In the course of proving Lemma 3.6, it was established that $|\pi| > 1$. Likewise, it is shown that $[M, A] \subseteq F = F(M)$.

On the other hand, by Lemma 3.6,

$$|\overline{M} : C_{\overline{M}}(A)| = 2^m, \quad m \in \mathbf{Z}_+, \quad (3.26)$$

where $\overline{M} = M/K$. Here, $K = (\ker \xi_1)_{\pi'}$. Since $(|K|, |A|) = 1$ and $C = C_M(a) = C_M(A)$ for all elements $a \in A^\#$, we have $|C_{M/K}(A)| = |C_M(A) : C_K(A)|$. Thus,

$$|\overline{M} : C_{\overline{M}}(A)| = \frac{|M||C_K(A)|}{|C_M(A)||K|} = 2^m, \quad m \in \mathbf{Z}_+. \quad (3.27)$$

Since $[Q, A] \subseteq [M, A]$ and $[Q, A] \not\subseteq C$, we have $F_q \neq 1$. Since $F_s \text{char } F \triangleleft AM$, it follows that $F_s \triangleleft AM$. By using the fact that $\xi(1)$ and $\xi_1(1)$ are q' -numbers and from the Clifford Theorem, it is readily seen that the subgroup F_q is Abelian. Since $[Q, A] \subseteq F_q$, it yields that $[Q, A]$ is also Abelian. Therefore, $[Q, A] = [[Q, A], A] \times C_{[Q, A]}(A)$. Since $[[Q, A], A] = [Q, A]$, we have $C_{[Q, A]}(A) = 1$. Then, $[Q, A] \cap C_M(A) = 1$. Hence and by the equality (3.27), it follows that $[Q, A] \subseteq K$.

Further, we see that

$$\tau_K = (\tau_M)_K = (\xi_M + (\xi_1)_M)_K = \sum_{\lambda \in \text{Irr}(\xi_M)} \lambda_K + \xi_1(1)1_K, \quad \lambda(1) = 1. \quad (3.28)$$

Since $K \subseteq Z(\xi_1)$, it follows that $K \subseteq (\bigcap_{\lambda \in \text{Irr}(\xi_M)} Z(\lambda)) \cap Z(\xi_1)$. By the fact that the character τ_M is exact and Lemma 3.1 [1], it follows that $K \subseteq Z(M)$. Then, $M \subseteq N$. Since $A \subseteq N$ and $Q \subseteq N$, we obtain $L = \langle AM, Q \rangle \subseteq N$. By Lemma 3.4, we have $L \neq \Gamma$.

Let us consider the character $\hat{\chi}_L$. Clearly, it is enough to consider the case, when $A \ker \psi' / \ker \psi' \not\triangleleft L / \ker \psi'$ for a certain character $\psi' \in \text{Irr}(\hat{\chi}_L)$. Then, $\xi \in \text{Irr}(\xi'_{AM})$ or $\xi_1 \in \text{Irr}(\xi'_{AM})$. Thus, we have $\xi'(1) \geq |A|$.

Assume that there exists a character ψ' of the group L such that $\text{Irr}(\psi') \subseteq \text{Irr}(\hat{\chi}_L)$ and $A \ker \psi' / \ker \psi' \triangleleft L / \ker \psi'$. Then, $A \ker \psi' \triangleleft L$, therefore, $(A \ker \psi') \cap AM \triangleleft AM$. Hence, $A \ker \psi'_{AM} \triangleleft AM$. Thus, we have $\psi'_{AM} = \psi \neq 0$. A contradiction with the assumption.

Thus, we have $\hat{\chi}_L = \xi' + \xi'_1$ and can say that $\xi'_{AM} = \xi$ and $(\xi'_1)_{AM} = \xi_1$. Let us apply the reasoning analogous to the reasoning applied to the group AM and its

characters ξ and ξ_1 to the group L and characters ξ' and ξ'_1 . We obtain $[L_{\pi'}, A] \subseteq F(L_{\pi'})$ and

$$[Q, A] \subseteq K_L. \quad (3.29)$$

Here, we have $K_L = (\ker \xi'_1)_{\pi'}$.

Suppose that $q_1 \in \pi([L_{\pi'}, A])$, $q_1 \neq q$ and Q_1 is an invariant Sylow q_1 -subgroup $[L_{\pi'}, A]$. Then, $Q_1 \triangleleft L$. Let $\xi_{q_1} \in \text{Irr}((\xi'_1)_{Q_1})$. By Theorem 6.11 [5], we have $\xi'_1 = \tilde{\xi}_{q_1}^L$ for a certain irreducible character $\tilde{\xi}_{q_1}$ of the group $I_L(\xi_{q_1})$. Thus, $\xi'_1(1) = \tilde{\xi}_{q_1}^L(1) = \tilde{\xi}_{q_1}(1)|L : I_L(\xi_{q_1})|$. Since $\xi'_1(1)$ is degree 2, we see that the numbers $\tilde{\xi}_{q_1}(1)$ and $|L : I_L(\xi_{q_1})|$ are also degrees 2.

Assume that $(\xi'_{Q_1}, \xi_{q_1})_{Q_1} \neq 0$. By the same theorem, we have $\xi' = (\xi'_{q_1})^L$ for a certain irreducible character ξ'_{q_1} of the group $I_L(\xi_{q_1})$. Then we have $|A| = \xi'(1) = \xi'_{q_1}(1)|L : I_L(\xi_{q_1})|$. Since $|L : I_L(\xi_{q_1})|$ is degree 2, we obtain $L = I_L(\xi_{q_1})$. Then, $\xi'_{Q_1} = e'\xi_{q_1}(1)$ for a certain natural number e' . Thus, we have $|A| = e'\xi_{q_1}(1)$. Since $\xi_{q_1}(1)$ is a π' -number, we have $\xi_{q_1}(1) = 1$. Then, $Q_1 \subseteq Z(\xi')$ and, obviously, $Q_1 \subseteq Z(\xi'_1)$. By Lemma 5 [1], $Q_1 \subseteq Z(L)$. This means that $Q_1 \subseteq C_{L_{\pi'}}(A)$.

Now assume that $(\xi'_{Q_1}, \xi_{q_1})_{Q_1} = 0$. By Theorem D [13],

$$t = \frac{|Q_1||\Gamma : L|\xi_1(1)}{\hat{\chi}(1)\xi_{q_1}(1)} \quad (3.30)$$

is an integer. Since $Q \subseteq L$, we see that $|\Gamma : L|$ is a q' -number. Since $\xi_1(1)$ is a q' -number and q divides $\hat{\chi}(1)$, it follows that t is not an integer. We obtain a contradiction. Thus, we have $Q_1 = 1$, therefore, $[L_{\pi'}, A]$ is a q -group.

It is obvious that $[Q, A] \subseteq [L_{\pi'}, A]$. Since $[L_{\pi'}, A] \triangleleft L$, we obtain $[L_{\pi'}, A] \subseteq Q$. Then, we have $[[L_{\pi'}, A], A] \subseteq [Q, A]$. Since $[[L_{\pi'}, A], A] = [L_{\pi'}, A]$, it follows that $[L_{\pi'}, A] \subseteq [Q, A]$. Thus, we have $[L_{\pi'}, A] = [Q, A]$. Then, $[L_{\pi'}, A] \subseteq K_L$. Hence, $AK_L/K_L \triangleleft L/K_L$. Therefore, we have $A \ker \xi'_1 / \ker \xi'_1 \triangleleft L / \ker \xi'_1$. Thus, $\xi'_1 \subseteq \text{Irr}(\psi)$, that is, $\psi \neq 0$. This contradicts the original assumption.

The lemma is proved. $\square$

Lemma 3.8. *The character $\mu = \tau_{A \ker \psi}$ is an exact irreducible character of degree $n_1 = |A|, |A| + 1, 2|A| - 2$ or $2|A| - 1$ and n_1 is a prime power, probably, with the exception of the case, when $\mu(1) = |A|$.*

Proof. By Lemma 3.7, we have $\psi \neq 0$.

As stated above, the character $\hat{\chi}_{AM}$ cannot be irreducible.

Let $\psi(1) = 1$. Then, we have $\Gamma' \subseteq \ker \psi$. Suppose that $G \not\subseteq \Gamma'$. Then, $G' \neq G$. By Lemma 3.3, the subgroup G' is either Abelian or $G' \subseteq C$. If G' is Abelian, G is solvable. By Theorem 1 [6], $\hat{\chi}(1)$ is a prime power. This contradicts the minimality of the group Γ . Now let $G' \subseteq C$. Then, we have $C \triangleleft G$. Since $\chi(1) = 2|A| + 1$, it is readily seen from Lemma 2.4 that the character χ_C contains a linear component β or a linear component β_1 . From the Clifford Theorem, it follows that C is Abelian. Thus, G' is Abelian. The case is examined.

Thus, $G \subseteq \Gamma'$. Then, we have $G \subseteq \ker \psi$. By Corollary 6.7 [5], $G \subseteq \ker \hat{\chi}$. Since the character $\hat{\chi}$ is exact, we have $G = 1$. Since A is a group of nontrivial automorphisms of the group G , it follows that the latter statement is impossible.

Therefore, we obtain $\psi(1) \geq 2$. Then, $\tau(1) \leq 2|A| - 1$. From Lemma 3.5, it follows that the character τ is irreducible. Since $\ker \tau \cap \ker \psi = 1$, we see that the character $\tau_{\ker \psi}$ is exact.

Suppose that the character $\tau_{A \ker \psi}$ is not exact. Then, we have $\ker \tau_{A \ker \psi} \subseteq A$. Since A is a TI -subgroup, it follows that $A \triangleleft A \ker \psi$. Thus we have $(\ker \psi)_{\pi'} \subseteq C_M(A)$. From formula (3.8), it follows that $[Q, A] \subseteq C$. A contradiction with the choice of the subgroup M . Thus, the character $\tau_{A \ker \psi}$ is exact.

Since $(\ker \psi)_{\pi'} \not\subseteq C_M(A)$, we have $A \ker \mu / \ker \mu \not\trianglelefteq A \ker \psi / \ker \mu$ for a certain character $\mu \subseteq \text{Irr}(\tau_{A \ker \psi})$. By Theorem [1], we have

$$\mu(1) \in \{|A| - 1, |A|, |A| + 1, 2|A| - 2, 2|A| - 1\} \quad (3.31)$$

and $\mu(1)$ is a prime power, probably, with the exception of the case when $\mu(1) = |A|$. Besides, it is readily seen that $\mu(1) = \tau(1)$ when $\mu(1) \neq |A| - 1$.

Let us consider this case separately. Let $\mu(1) = |A| - 1$. Since $\tau(1) < 2|A|$, subject to Lemma 3.5, we have $\tau(1) = 2|A| - 2$ and by the Clifford Theorem, we obtain $\tau_{A \ker \psi} = \mu + \mu^g$, where $g \in AM \setminus A \ker \psi$. Hence, by Lemma 7 [10], we have

$$|(\ker \psi)_{\pi'} : Z(\mu)| = |(\ker \psi)_{\pi'} : Z(\mu^g)| = 2^{2\alpha}, \quad (3.32)$$

where $\mu(1) = 2^\alpha$, $\alpha \in \mathbb{N}$. Since $[Q, A] \subseteq (\ker \psi)_{\pi'}$, we see that $[Q, A] \subseteq Z(\mu) \cap Z(\mu^g)$. Since $\ker \mu \cap \ker \mu^g = 1$, by Lemma 5 [1], we have $[Q, A] \subseteq Z(A \ker \psi)$. Then $[Q, A] \subseteq C$. A contradiction with Lemma 3.2. Thus,

$$\mu(1) \in \{|A|, |A| + 1, 2|A| - 2, 2|A| - 1\} \quad (3.33)$$

and $\mu(1)$ is a prime power, probably, with the exception of the case, when $\mu(1) = |A|$. Since $\mu(1) = \tau(1)$ under the value indicated, it follows that $\mu(1)$ and the character $\tau_{A \ker \psi}$ is exact, the lemma is proved. $\square$

Lemma 3.9. $n_1 = |A|$.

Proof. By Lemma 3.8, it is enough to show that $n_1 \neq |A| + 1, 2|A| - 2$ and $2|A| - 1$.

First, let $n_1 = |A| + 1$. By Theorem [12], the group $A \ker \psi$ is solvable. Then by Lemma 3.6, we have $|(\ker \psi)_{\pi'} : C_{(\ker \psi)_{\pi'}}(A)| = 2^m$, where $m \in \mathbb{Z}_+$. From equality (3.8), it follows that $[Q, A] \subseteq C_{(\ker \psi)_{\pi'}}(A) \subseteq C_M(A)$. We obtain a contradiction with the choice of the subgroup M .

Now suppose that $n_1 = 2|A| - 2$ or $2|A| - 1$. By Lemma 3.8, n_1 is a prime power s .

Let S be a A -invariant Sylow s -subgroup from $(\ker \psi)_{\pi'}$ and $Z = Z(S)$. Since, by Lemma 3.8, the character μ is exact and irreducible, by Lemma 2.27 [5], it follows that $Z(\mu) = Z(A \ker \psi)$ is a cyclic group. Therefore, $Z \cap Z(\mu) \subseteq Z(A \ker \psi)$, that is, $Z \cap Z(\mu) \subseteq C_Z(A)$. Since Z is Abelian, it follows that $Z = [Z, A] \times C_Z(A)$. It can be easily seen that $|[Z, A]|$ divides $|Z/Z \cap Z(\mu)|$.

Assume that $Z \not\subseteq Z(\mu)$. Then, by Corollary 1 [14], $|Z/Z \cap Z(\mu)|$ divides n_1 . Thus, $|[Z, A]|$ divides n_1 . Thus, we have $|[Z, A]| = s^r$, $r \in \mathbb{N}$. Since $A[Z, A]$ is a Frobenius group with a kernel $[Z, A]$, the order of its kernel can be compared with 1 modulo $|A|$. Therefore, $s^r - 1 = d|A|$, that is, $s^r = d|A| + 1$, where $d \in \mathbb{N}$. Since $s^r \leq n_1$, we have $d = 1$ and $s^r = |A| + 1$. Therefore, $s = 2$, as stated above, $n_1 \neq |A| + 1$, and we obtain $n_1 = 2|A| - 2$. Since $2^r = |A| + 1$ and $2|A| - 2$ is degree 2, we have $|A| = 3$. As stated above, it is not true. Our assumption that $Z \not\subseteq Z(\mu)$ is false.

Thus, $Z \subseteq Z(\mu)$. Then, it is clear that $Z \triangleleft AM$. Since $Z \subseteq \ker \psi$, $\ker \tau \cap \ker \psi = \ker \hat{\chi} = 1$, and the character τ is irreducible, Corollary 6.7 [5] yields that $(\tau_Z, \varsigma)_Z \neq 0$ for a certain nonprincipal irreducible character ς of the subgroup Z . Then, by Theorem D [13],

$$\frac{|Z||\Gamma : AM|\tau(1)}{\hat{\chi}(1)\varsigma(1)} = \frac{|Z||\Gamma : AM|n_1}{n} \quad (3.34)$$

is an integer. Since q divides n and q does not divide $|Z||\Gamma : AM|n_1$, we obtain a contradiction. The lemma is proved. $\square$

Lemma 3.10. *Let M be a maximal proper A -invariant subgroup G such that $[Q, A] \subseteq M$. Then, one has $M = [Q, A] \times C, [Q, A] \subseteq Z(M)$.*

Proof. By assumption of the lemma and by Lemma 3.4, we have $N \subseteq AM$.

By Lemma 3.9, $n_1 = |A|$. Since the character $\mu = \tau_{A/\ker \psi}$ is irreducible, the character τ is also irreducible.

Since $\tau(1) = |A|$, all irreducible components of the character τ_M are linear. This means that $M' \subseteq \ker \tau$. Since $\ker \tau \triangleleft AM$, $A \ker \psi \triangleleft AM$ and $\ker \tau \cap \ker \psi = 1$, we have $A_0 = \ker \tau \cap A \ker \psi \subseteq A$. But A is a TI -subgroup and $A \not\triangleleft A \ker \psi$ because in the converse case, by equality (3.8), we have $[Q, A] \subseteq C$, which contradicts Lemma 3.2. Then, we obtain $A_0 = 1$. Thus, $\ker \tau \subseteq C_{AM}(A \ker \psi)$, that is, $\ker \tau \subseteq C_M(A)$. Then, $M' \subseteq C_M(A)$, that is, $C_M(A) \triangleleft M$.

By Lemma 3 [1], we have $[M, A] \subseteq M \cap A \ker \psi$. By using the fact that $M' \subseteq \ker \tau$ and $\ker \tau \cap \ker \psi = 1$, it follows that $\ker \psi \cap M$ is Abelian. Thus, the subgroup $[M, A]$ is Abelian. Since,

$$[M, A] = [[M, A], A] \times C_{[M, A]}(A) \quad (3.35)$$

and $[[M, A], A] = [M, A]$, we conclude that $C_{[M, A]}(A) = 1$. Since

$$[M, A] \cap C_M(A) \subseteq C_{[M, A]}(A), \quad (3.36)$$

it follows that $[M, A] \cap C_M(A) = 1$. Since by Lemma 6 [9], $[M, A] \triangleleft M$, $C_M(A) \triangleleft M$ and $M = [M, A]C_M(A)$ by Lemma 4 [1], we have $M = [M, A] \times C_M(A)$. By using the fact that $[M, A]$ is Abelian, it follows that $[M, A] \subseteq Z(M)$.

Since $[Q, A] \subseteq M$, we conclude that $[Q, A] = [[Q, A], A] \subseteq Z(M)$. Then, we have $M \subseteq N$. Thus, $AM \subseteq N$. Since the reverse inclusion holds as well, we have $AM = N$. In view of the fact that $Q \subseteq N$ since $[Q, A] \triangleleft QA$, we obtain $L = \langle AM, Q \rangle \subseteq N = AM$. From the maximality of the subgroup M , it follows that $Q \subseteq M$.

Further, $[M, A] = [M, A]_q \times X$, where X is a Hall q' -subgroup from $[M, A]$.

Assume that $X \neq 1$.

Let $\gamma \in \text{Irr}((\tau)_X)$. Since X is Abelian, it follows that $\gamma(1) = 1$. Since $[M, A] \subseteq \ker \psi$, we obtain $X \subseteq \ker \psi$. Thus, $\psi_X = \psi(1)1_X$.

First assume that $\gamma \in \text{Irr}(\psi_X)$. Then, $\gamma = 1_X$. Further, $X \text{char}[M, A] \triangleleft L$, that is, $X \triangleleft L$. Then, $X \subseteq \ker \tau$. Thus, $X \subseteq \ker \psi \cap \ker \tau = \ker \hat{\chi} = 1$. Hence, we obtain that $X = 1$. We get a contradiction with the assumption.

Now, let $\gamma \notin \text{Irr}(\psi_X)$, that is, $(\psi_X, \gamma)_X = 0$. Since $X \triangleleft L$, we apply Theorem D [13] to the subgroup AM and the character $\hat{\chi}_{AM} = \tau + \psi$. By this Theorem,

$$\frac{|X||\Gamma : AM|\tau(1)}{\hat{\chi}(1)\tau(1)} = \frac{|X||\Gamma : AM||A|}{\hat{\chi}(1)} = \frac{|X||G : M|}{\hat{\chi}(1)} \quad (3.37)$$

is an integer. On the other hand, we have $Q \subseteq M$, that is, q does not divide $|G : M|$. Therefore, $(q, |X||G : M|) = 1$. But q divides $\hat{\chi}(1)$. We obtain a contradiction.

Thus, $X = 1$, that is, $[M, A]$ is a q -group.

Since $[M, A]$ is a normal q -subgroup in M , it follows that $[M, A] \subseteq Q$. It is readily seen that $[M, A] = [[M, A], A] \subseteq [Q, A]$. Therefore, $[M, A] = [Q, A]$.

Thus, we have

$$M = [Q, A] \times C_M(A), [Q, A] \subseteq Z(M). \quad (3.38)$$

Let us ascertain that assertion (1) of Lemma 2.4 holds and $A[Q, A] \subseteq \ker \psi$.

Hence, $n = 2|A| + 1$ and, by Lemma 2.4, it follows that only (1)–(5), (12) or (15) satisfy.

If assertion (15) of Lemma 2.4 satisfies, Lemma 2.3 and Lemma 2.2 imply that $A \subseteq \ker \hat{\chi}$. This contradicts Lemma 3.1. If assertion (12) of Lemma 2.4 satisfies it follows that $C \subseteq Z(G)$. Since $M' \subseteq \ker \tau$, we have $(\hat{\chi}_{M'}, 1_{M'})_{M'} \neq 0$. By Corollary 6.7 [5], $M' \subseteq \ker \hat{\chi} = 1$. Thus, the subgroup M is Abelian. Then, $M \subseteq N^* = N_G(Q)$. Since Q is Abelian, by Theorem 6.15 [5], we see that $N^* \neq G$. Since $N^* \subseteq M$, we obtain $N^* = M$. By the Burnside Theorem, the group G contains a normal q -complement, which is, by Lemma 3.3, either an Abelian group, or is contained in $C \subseteq Z(G)$. Thus, G is solvable. In this case, the theorem holds [6].

Assume that assertion (3)–(5) of Lemma 2.4 satisfy. Since $\beta(1) = \beta_1(1) = 1$ and $\beta_2(1) = 1$ (assertion (4)), it follows that $C' = 1$, that is, the subgroup C is Abelian. Therefore, $C_M(A)$ is Abelian. Then by equality (3.38), the subgroup M is Abelian. This case is examined.

Suppose that assertion (2) of Lemma 2.4 satisfies.

In addition, assume that the character β_1 is imprimitive, that is, $\beta_1 = \lambda^C$ for a certain linear irreducible character λ of the subgroup C_0 of index 2 in C . Then, it is readily seen that C_0 is Abelian. Thus, the subgroup Q is Abelian. Since we have $[Q, A] \subseteq Z(M)$ by equality (3.38), and we can think that $N^* = M$, by Theorem 6.4.4 [4], q divides $|G : G'|$. Thus, $G' \neq G$. This case is also examined.

Now assume that the character β_1 is primitive. By Theorem 14.23 [5], $|\bar{C} : Z(\bar{C})| = 12, 24$ or 60 , where $\bar{C} = C / \ker \beta_1$ and, by Lemma 2.27 [5], $Z(\bar{C}) = Z(\beta_1) / \ker \beta_1$. Thus, $|C : Z(\beta_1)| = 12, 24$ or 60 . Since $\beta(1) = 1$, it follows that $Z(\beta_1) \subseteq Z(\beta)$. Since $\ker \beta_1 \cap \ker \beta = 1$, by Lemma 5 [1], we have $Z(\beta_1) = Z(C)$. Then, $|C : Z(C)| = 12, 24$ or 60 . Let $q \neq 3, 5$. Then, $C_M(A) \subseteq Z(C)$. Thus, Q is Abelian and as above we obtain that q divides $|G : G'|$. This case is examined. Suppose that $q = 3, 5$. Then, the factor group $C_q Z(C) / Z(C)$ is cyclic. Since $Z(C) \subseteq Z(C_q Z(C))$, $C_q Z(C)$ is Abelian. We obtain that the subgroup Q is Abelian as well. The case is examined.

It remains to consider the case when assertion (1) of Lemma 2.4 holds, that is, $\chi_C = \beta + |A|\beta_1$, $\beta_1(1) = 1$. Then, $\beta(1) = |A| + 1$. From Lemma 2.2, it follows that

$\hat{\chi}_A = \rho_A + \beta_1(1)1_A$. Thus, $\hat{\chi}_A = \rho_A + (|A| + 1)1_A$, therefore, we have $(\hat{\chi}_A, 1_A)_A = |A| + 2$. On the other hand,

$$\begin{aligned} (\hat{\chi}_A, 1_A)_A &= ((\hat{\chi}_{AM})_A, 1_A)_A \\ &= ((\tau + \psi)_A, 1_A)_A = (\mu_A, 1_A)_A + (\psi_A, 1_A)_A = 1 + (\psi_A, 1_A)_A. \end{aligned} \quad (3.39)$$

Here, we take into account that $\mu_A = \rho_A$. Thus, $|A|+2 = 1 + (\psi_A, 1_A)_A$, that is, $(\psi_A, 1_A)_A = |A|+1$. Since $\psi(1) = |A| + 1$, we conclude $A \subseteq \ker \psi$. Since, by equality (3.8), $[Q, A] \subseteq \ker \psi$, it follows that $A[Q, A] \subseteq \ker \psi$.

By equality (3.38) for a complete proof of the lemma, it is enough to make sure that $C = C_M(A)$.

Since $C_M(A) \subseteq C$, it remains to show that $C \subseteq C_M(A)$.

Suppose that $c \in C$. Then, we have $\hat{\chi}_{AM^c} = \tau^{(c)} + \psi^{(c)}$. Since $A \subseteq \ker \psi$, we have $A \subseteq \ker \psi^{(c)}$. Hence, $[Q^c, A] \subseteq \ker \psi^{(c)}$. Therefore,

$$\begin{aligned} \hat{\chi}_{A[Q, A]} &= (\tau)_{A[Q, A]} + \psi_{A[Q, A]} = (\tau)_{A[Q, A]} + \psi(1)1_{A[Q, A]}, \\ \hat{\chi}_{A[Q^c, A]} &= (\tau_1^{(c)})_{A[Q^c, A]} + \psi_{A[Q^c, A]}^{(c)} = (\tau^{(c)})_{A[Q^c, A]} + \psi^{(c)}(1)1_{A[Q^c, A]}. \end{aligned} \quad (3.40)$$

Thus,

$$\begin{aligned} &(\hat{\chi}_{A[Q, A]}, 1_{A[Q, A]})_{A[Q, A]} + (\hat{\chi}_{A[Q^c, A]}, 1_{A[Q^c, A]})_{A[Q^c, A]} \\ &= ((\tau_{A[Q, A]}, 1_{A[Q, A]})_{A[Q, A]} + \psi(1)) + ((\tau^{(c)})_{A[Q^c, A]}, 1_{A[Q^c, A]})_{A[Q^c, A]} + \psi^{(c)}(1) = 2\psi(1) \\ &= 2\beta(1) > \beta(1) + 1 = (\hat{\chi}_A, 1_A)_A = (\rho_A + \beta(1)1_A, 1_A)_A \\ &\geq (\hat{\chi}_{A[Q, A] \cap A[Q^c, A]}, 1_{A[Q, A] \cap A[Q^c, A]})_{A[Q, A] \cap A[Q^c, A]}. \end{aligned} \quad (3.41)$$

Here we take into account that

$$\psi(1) = \hat{\chi}(1) - \tau(1) = |A| + \beta(1) - |A| = \beta(1), \quad (3.42)$$

and consider that $\beta(1) = |A|+1 > 1$. Therefore, by the Brouwer Theorem 5.19 [5], the subgroup

$$D = \langle A[Q, A], A[Q^c, A] \rangle \quad (3.43)$$

is proper.

Let D_q be a A -invariant Sylow q -subgroup D such that $[Q, A] \subseteq D_q$. Such a subgroup exists by Theorem 6.2.2(iii) [4]. Let M_D be a maximal A -invariant subgroup of the group G , containing $D_{\pi'}$. Let us apply the reasoning concerning the group AM to the character $\widehat{\chi_{AM_D}}$ and the group AM_D . We get $M_D = [M_D, A] \times C_{M_D}(A)$, $[M_D, A] \subseteq Z(M_D)$.

Since $[Q, A] \subseteq M_D$, we have $[[Q, A], A] \subseteq [M_D, A]$. Since $[[Q, A], A] = [Q, A]$, it follows that $[Q, A] \subseteq [M_D, A]$. Therefore, $[Q, A] \subseteq Z(M_D)$. Thus, $M_D \subseteq N_{\pi'}$. Since $[M, A] \subseteq Z(M)$ and $[M, A] = [Q, A]$, as ascertained above, we have $M \subseteq N_{\pi'}$. From the maximality of the subgroup M and Lemma 3.4, it follows that $M = N_{\pi'}$. Then, $M_D \subseteq M$. Thus, $[Q^c, A] \subseteq M$. Since $M = [Q, A] \times C_M(A)$, as ascertained above, we have $[Q, A] = [Q^c, A] = ([Q, A])^c$. Therefore, $c \in N_{\pi'}$. Then, $c \in M$, that is, $C \subseteq C_M(A)$.

The lemma is proved. $\square$

This completes the proof of the theorem.

Let us show that n is a primary number.

Let us assume the converse. Let q_1 and q_2 be distinct odd primes from $\pi(n)$.

Let Q_1 and Q_2 be A -invariant Sylow q_i -subgroups G , $N_i = N_\Gamma(Q_i)$. Assume that $N_1 = \Gamma$. From Lemma 3.3, it follows that Q_1 is Abelian or $Q_1 \subseteq C$. In the first case we obtain a contradiction with Theorem 6.15 [5]; in the second case we obtain a contradiction with Lemma 3.2. Thus, we have $N_i \neq \Gamma$, $i = 1, 2$. Let M_i be maximal proper A -invariant subgroups G such that $N_i \subseteq AM_i$, $i = 1, 2$. By Lemma 3.10, $M_i = [Q_i, A] \times C$, $i = 1, 2$, that is, $M_i \subseteq N_G(C)$, $i = 1, 2$. Therefore, $C \triangleleft L = \langle M_1, M_2 \rangle$. Since M_i are maximal A -invariant subgroups G , we see that either $L = M_1 = M_2$, or $L = G$. Suppose that $L \neq G$. Then, $|M_1| = |M_2|$. Thus, $|[Q_1, A]| = |[Q_2, A]|$. Since $q_1 \neq q_2$, we obtain $|[Q_1, A]| \neq |[Q_2, A]|$. A contradiction.

Now suppose that $L = G$, that is, $C \triangleleft G$. Since, by Lemma 3.10, $M = C \times [Q, A]$, it follows that $M \subseteq C_G(C) \triangleleft G$. By Lemma 3.3, $C_G(C)$ is either Abelian or $C_G(C) \subseteq C$. In the first case, M is Abelian. It was considered above. In the second case, we have $M \subseteq C$. A contradiction with Lemma 3.2.

Thus, our assumption on the fact that n will divide by two distinct odd primes does not satisfy. The Theorem 1.1 is proved.

4. Proof of Theorem 1.2

Let us carry out induction over the order of the group G .

Let us examine the subgroup $O_{\pi', \pi}(G)$. By definition, put

$$\frac{O_{\pi', \pi}(G)}{O_{\pi'}(G)} = O_\pi \left(\frac{G}{O_{\pi'}(G)} \right). \quad (4.1)$$

Since the factor group $G/O_{\pi'}(G)$ is π -solvable and $O_{\pi'}(G/O_{\pi'}(G)) = 1$, we have $O_\pi(G/O_{\pi'}(G)) \neq 1$. Since $HO_{\pi'}(G)/O_{\pi'}(G)$ is a π -Hall TI-subgroup in $G/O_{\pi'}(G)$, we obtain $O_{\pi', \pi}(G) = HO_{\pi'}(G)$. Denote $X = O_{\pi', \pi}(G)$.

Identity substitution of the group G defines an exact irreducible character φ of degree n .

First let us assume that $|G : X| \neq 1$. By the Clifford Theorem

$$\varphi_X = e \sum_{i=1}^t \varphi_i, \quad (4.2)$$

where $\varphi_{i=1, t} \in \text{Irr}(X)$ and e, t are numbers, dividing $|G : X|$.

In addition, assume that $e \neq 1$ or $t \neq 1$. Then, the numbers $\varphi_{i=\overline{1,t}}(1) < 2|H|$. Therefore, the factor groups $X/\ker \varphi_i$, $i = \overline{1,t}$ satisfy the theorem proved in [1–3]. If

$$\frac{H \ker \varphi_i}{\ker \varphi_i} \not\trianglelefteq \frac{X}{\ker \varphi_i}, \quad i = \overline{1,t}, \quad (4.3)$$

from this theorem, it follows that $\varphi_{i=\overline{1,t}}(1) = |H| - 1, |H|, |H| + 1, 2|H| - 2$ or $2|H| - 1$. Since the numbers $\varphi_{i=\overline{1,t}}(1)$ divide $\varphi(1)$, it is readily seen that the numbers $\varphi_{i=\overline{1,t}}(1)$ cannot possess the stated values. Therefore,

$$\frac{H \ker \varphi_i}{\ker \varphi_i} \trianglelefteq \frac{X}{\ker \varphi_i}, \quad i = \overline{1,t}, \quad (4.4)$$

that is, $X/\ker \varphi_i$ contain normal π -Hall subgroups $H \ker \varphi_i / \ker \varphi_i$, $i = \overline{1,t}$. Since $\bigcap_{i=1}^t (\ker \varphi_i) = 1$, we see that X can be isomorphically embedded in the direct product

$$\overline{X} = \frac{X}{\ker \varphi_1} \times \cdots \times \frac{X}{\ker \varphi_t}. \quad (4.5)$$

Since $\overline{X}$ contains a normal π -Hall subgroup, it follows that X also contains a normal π -Hall subgroup H , that is, $H \trianglelefteq G$. Commutativity of the subgroup H results from the faithfulness of the character φ and the fact that $\varphi(1)$ is a π' -number.

Now suppose that $e = t = 1$. Then, the character φ_X is irreducible. Since $|X| < |G|$, by induction we see that either $\varphi_X(1)$ is a prime power or $H \trianglelefteq G$. Since $\varphi_X(1) = n$, in both cases the theorem holds.

Now suppose that $|G : X| = 1$, that is, $G = X = HO_{\pi'}(X)$. Let us apply the notation to Condition B and Theorem 1.1. For this, suppose that $X = \Gamma$, $H = A$, $O_{\pi'}(X) = G$. Then, φ is an irreducible character of the group $\Gamma = AG$. Since $(|A|, \varphi(1)) = 1$, it is readily seen that the character φ_G contains an A -invariant faithful irreducible component φ^* . It is obvious that $\varphi_G = \varphi^* \in \text{Irr}_A(G)$.

Suppose that $\ker \widehat{\varphi^*} \neq 1$. Thus, we have $\ker \widehat{\varphi^*} \subseteq A$. Since A is a TI -subgroup in Γ , it follows that $A \triangleleft \Gamma$. This contradicts the fact that A is a group of nontrivial automorphisms of the group G .

Therefore, $\ker \widehat{\varphi^*} = 1$. Since $\varphi^*(1) = \varphi(1)$ and, by assumption of the theorem being proved, $\varphi(1) = n$ is not a prime power, it follows that $\widehat{\varphi^*}(1)$ is not a prime power either. Taking into account that by Lemma 6 [1] $C_G(a) = C_G(A) = C$ for each element $a \in A^\#$, by Theorem 1.1 we again see that $A \triangleleft \Gamma$. A contradiction.

The theorem is proved.

Acknowledgment

The research is carried out within the Project no. Ph09URB-001 of June 2, 2009.

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